N. MUSKHELISHVILI

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FUNDAMENTAL EQUATIONS
PLANE THEORY OF ELASTICITY
TORSION AND BENDING

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BY

N. I. MUSKHELISHVILI

TRANSLATED FROM THE RUSSIAN
BY

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OXFORD UNIVERSITY PRESS
BOMBAY CALCUTTA MADRAS

PREFACE TO THE THIRD EDITION.

The second edition of this book, which was published in 1935 almost immediately after the first (which appeared in 1933), has been out of print for a long time, but, as I was engaged on other work, I have only now been able to start the preparation of a new edition. The warm reception given to my book and the high distinction with which it was favoured made it imperative to treat its reissue with special attention. To this was added the circumstance, highly gratifying to me, that soon after the appearance of the first editions many papers were published in which the methods expounded by me were applied to different concrete problems and also substantially amplified and generalized. It is natural that the new edition should at least reflect the main results of these papers as well as some results obtained by me. I have tried to accomplish this, but I am afraid that some papers may still have escaped my notice for which omission I tender my apologies to their authors.

The general design of exposition in this edition has remained the same as before. However, the text of the book, with the exception of the first two and last Parts *, has been thoroughly rewritten and considerably enlarged. Two new Parts have been added, namely the fourth and the sixth. The contents of the fourth Part are only to a negligible extent taken from the preceding edition; in Part VI, results are given which have been obtained by me and also by other authors since the publication of the preceding edition, ** if no account is taken of the few problems whose solution had been given in the previous edition, but by means of other methods.

* However, Chapter 25 has been considerably enlarged and in this way the theory of extension and flexure of compound bars has attained an aspect of completeness.

** A large part of these results were introduced by me into my book "Singular Integral Equations" (Moscow-Leningrad 1946), but now I find that their natural place is in the present book. They will accordingly be omitted from the following edition of "Singular Integral Equations". Here, the exposition of these results has been rearranged in order to make it independent of the above-mentioned book.

Although I do not think it possible or necessary to indicate all the changes and additions in the text of the earlier editions, I must draw the reader's attention to the new arrangement of Chapters 14 – 16 (Chapter 17 is new). Compared with the previous editions, the results contained in these chapters are not new, but the method by which they have been obtained has been replaced by a different one which seems to me to be more germane to the matter under consideration. However, I should like to mention that the new method (which was influenced by the work of J. Plemelj on the theory of functions of a complex variable, published long before the first edition of my book, but unfortunately unknown to me at the time) leads to the same calculations as the former method. For this reason and also because of the, say, greater complexity of the new method, I am not certain that I acted rightly in introducing this change. Be that as it may, a comparison of the new and the old approach may prove useful.

In conclusion, I should like to add that as far as possible I have carefully quoted the authors of any results which I have used, just as I have done with regard to some of my own results, at times even of minor importance and adduced merely as examples. I have followed this practice, not because I attach exaggerated importance to these results, but only to avoid puzzling the reader who might not be acquainted with the previous editions of my book and who might have encountered material taken from it without clear indication of its original source in some other publications (mostly non-Russian).

To simplify reference, the quoted works have been listed at the end of the book in alphabetical order. In references, the author is named and the number of his publication, according to this list, is given in square brackets. The first edition was greatly assisted to its success by the preface of the late Academician Alexei Nikolaevich Krylov whose outstanding scientific and public merits are well known to all and for whom I shall always entertain feelings of profound gratitude and respect. Krylov's preface is reproduced below without any changes. I have not been able to fulfil in this edition the wish expressed by Krylov at the end of his preface concerning the development of numerical methods of solution. While realizing the importance of his request, I felt that I would be unable to meet it sufficiently well. Nevertheless, Krylov's wish has been fulfilled by other authors, referred to in the text of the book.

Tiflis, November 1948. N. MUSKHELISHVILI

EXTRACT FROM THE AUTHOR'S PREFACE TO THE FIRST EDITION.

This book reproduces, in a considerably revised and enlarged form, the contents of a course of lectures, delivered by me in Spring 1931 at the invitation of the Seismological Institute of the Academy of Sciences of the U.S.S.R. before the scientific workers of the Institute, and of lectures delivered in 1932 before post-graduate students of the Physico-Mathematical Institute of Mathematics and Mechanics at the University of Leningrad. The lectures were intended for persons acquainted with the principles of the theory of elasticity and were to be devoted to separate fundamental questions the choice of which was largely left to me; I naturally dwelt on subject matter in which I had been working myself.

Thus, this book deals only with a few chapters of the theory of elasticity each of which receives fairly complete treatment. I shall not touch here on the subject matter of the book an idea of which may be gained from the list of contents, but I consider it necessary to make the following comments.

Seeing that the problems considered in this book may prove of interest to a wider circle of people, in particular to those whose work requires application of the theory of elasticity, I have tried to make the exposition as far as possible intelligible for readers who are only familiar with the fundamentals of differential and integral calculus and the elementary theory of functions of a complex variable. Thus, for example, problems involving integral equations are relegated to separate sections which may be passed over without impairing the understanding of the remainder; Part I which deals with the foundations of the mathematical theory of elasticity (it contains even more than is required) is intended for readers not specializing in the theory of elasticity. In order to make the text more accessible, I refrained from employing tensor calculus which I used in my lectures at the Seismological Institute; an elementary introduction to tensors is given in Appendix 1. Appendices 2 and 3 are devoted to some aspects of elementary mathematics which are necessary for the understanding of the subject matter of the book and which, as a rule, are insufficiently elucidated in elementary courses on analysis.

Leningrad, Spring 1933. N. MUSKHELISHVILI

pressions; this again helped to reveal mistakes in other papers. The establishment of the connection between thermal stresses and multi-valued displacements also belongs to Muskhelishvili.

All examples in this Part have either been solved by the author for the first time or, if solved before, more complicated methods had been applied for their solution.

Part III belongs entirely to the author in the originality and in the generality of the problems solved in it as well as in the originality of the methods applied. How important this method is may be seen from the fact that in § 68 (§ 82 of the 3rd edition) the author gives the general solution of the second fundamental problem for an infinite plate with an elliptic hole on two pages of large print. A particular case of this problem was solved by L. Föppl in the Zeitschrift für angewandte Mathematik und Mechanik, his solution occupying five large pages of small print which, if set up in our academic type, would fill about twenty pages; in § 69 (§ 82a of the 3rd edition), an example is solved in a few lines the simplest particular case of which agrees with that of Föppl.

In Part IV, as previously mentioned, all matter relating to non-homogeneous bodies, beginning with the very statement of the problem, belongs to Muskhelishvili.

From this short sketch may be seen the rich content as well as the variety and importance of the problems covered in this book and the originality and generality of the methods applied for their solution.

There only remains to express the wish that in future editions, which without doubt will be required, the author illustrate the general deductions and formulae by numerical examples, by diagrams and by indications as to the number of ordinates or subdivisions required for approximate integration in order to ensure accuracy within, say, $\frac{1}{2}\%$. He will thereby render a great service to engineers and make his excellent book more accessible to those people who will apply its deductions to the solution of the purely practical problems of the building industry.

Academician A. Krylov

CONTENTS

PART I

FUNDAMENTAL EQUATIONS OF THE MECHANICS OF AN ELASTIC BODY

In this introductory Part the basic concepts of the mathematical theory of elasticity will be recapitulated. A deduction of the complete system of equations of the mechanics of an elastic isotropic body will be given and some fundamental propositions about these equations proved.

It will be assumed that the reader has some knowledge of the physical foundations of the theory of elasticity and little consideration will be given to this side of the subject. A more detailed account of the physics and also of a number of general theoretical and practical problems, not treated in this book, may be found in the following text books on the theory of elasticity:

A. E. H. Love [1] (This book, first published in 1892—1893, is in many respects obsolete, but nevertheless is very useful for the abundance of material it offers.)

P. F. Papkovicz	[1]	1939
L. S. Leibenson	[1]	1947
S. Timoshenko	[1, 2]	1914, 1916
R. Grammel	[1]	1928
P. Burgatti	[1]	1931
I. S. Sokolnikoff	[1]	1946

Further, the following textbooks on theoretical mechanics should be mentioned:

G. Kirchhoff	[1]	1897
A. G. Webster	[1]	1904

These last two books contain a study of the basic theory of elasticity; the first of these, although its first edition appeared more than 70 years ago, is still of interest at the present time.

A brief, but rather detailed outline of the historical development of the theory of elasticity is given at the beginning of the book by A. E. H. Love [1]. A very detailed history of the theory up to 1893 with a careful analysis of the different papers and books was presented by I. Todhunter and K. Pearson [1].

The first two chapters of this book deal with all types of bodies which may, with sufficient approximation, be called "continuous" (i.e. fluids, elastic and plastic bodies, etc.). It is only at the beginning of the third chapter that assumptions are introduced which characterize the (ideal) elastic body as such. Throughout the first Part orthogonal rectilinear coordinates are used.

CHAPTER 1

ANALYSIS OF STRESS

§ 1. **Body Forces.** In the mechanics of continuous bodies a distinction is made between two types of forces:

1. *Body forces*, acting on the elements of volume (or mass) of the body;
2. *Stresses*, acting on surface elements inside or on the boundary of the body.

In order to explain this distinction in detail, imagine that a volume V of arbitrary shape, bounded by the surface S, has been detached from the continuous body under consideration. It is seen that the sum of the external forces acting on V may be conceived as consisting of *body* forces (e.g. gravity) and *surface* forces (e.g. pressure).

The body forces will be considered first. They act on the volume elements of the body, or actually on the mass contained in these elements. Assume that the forces, acting on the infinitely small volume element dV, have the form $\vec{\Phi}\, dV$ where $\vec{\Phi}$ is some finite vector; any point (x, y, z) of the element dV may be chosen as point of application of the vector $\vec{\Phi}$. The vector $\vec{\Phi}$ is called a *body force*, referred to unit volume. If ρ denotes the density at a given point of a body (i.e., the quantity of mass contained in a unit volume), the vector $\frac{1}{\rho}\vec{\Phi}$ will be the body force per unit mass.

In the case of gravity forces the vector $\vec{\Phi}$ is directed vertically downwards and is in magnitude equal to ρg, where g is the acceleration due to gravity. Speaking generally, the vector $\vec{\Phi}$ depends on the position of the volume element inside the body, or, in other words, on the coordinates x, y, z of a point within the infinitely small volume element. In dynamics the vector $\vec{\Phi}$ depends also on the time.

NOTE. The mathematical statement that a body force, acting on a volume element dV, may be represented by a vector $\vec{\Phi}\, dV$, applied to

some point of the element dV, must be understood in the sense that the resultant force vector $\vec{\Psi}$, acting on any finite volume V of the body, may be represented by a triple integral, i.e., by

$$\vec{\Psi} = \iiint_V \vec{\Phi}\, dV = \iiint_V \vec{\Phi}\, dx\, dy\, dz, \tag{1.1}$$

and similarly the resultant moments of these forces about the axes Ox, Oy, Oz of an orthogonal, rectilinear system by

$$M_x = \iiint_V (yZ - zY)dx\, dy\, dz,\ M_y = \iiint_V (zX - xZ)\, dx\, dy\, dz,$$

$$M_z = \iiint_V (xY - yX)dx\, dy\, dz, \tag{1.2}$$

where X, Y, Z are the components of the vector $\vec{\Phi}$.

Components of a vector will always be *scalar* quantities. Many authors, e.g Love [1], denote by X, Y, Z the components of body forces, referred to unit mass. In that case the components of the vector $\vec{\Phi}$ will be ρX, ρY, ρZ, where ρ is the density.

§ 2. Stress. Surface forces act on the elements of the surface S of a volume V, detached from a body (cf. § 1). It will be assumed that the force acting on the infinitely small surface element dS has the form $\vec{F}\, dS$, where $\vec{F}$ is some finite vector. Any point of the element dS may be assumed as the point of application of the vector $\vec{F}$. The precise mathematical statement of this fact must be understood in the same way as was indicated in the Note at the end of § 1 with regard to body forces. The force $\vec{F}\, dS$ will be called the traction exerted on the element dS, and the vector $\vec{F}$ the traction per unit area or the *stress*. $\vec{F}$ will often also be called the *stress vector*.

The traction $\vec{F}\, dS$ represents the force acting between the parts of the continuous body adjacent to either side of the surface element dS. Thus $\vec{F}\, dS$ is the force with which the part outside V acts on the part within V;

the force with which the part within V acts on the part lying outside is by Newton's third law of motion equal to $-\vec{F}\,dS$.

In general, any area (i.e., surface element), conceived inside a body, is bounded by two parts of the body adjoining the area on either side. In order to distinguish between these two elements of the body draw the normal n to the area in question and give it a definite positive direction (Fig. 1).

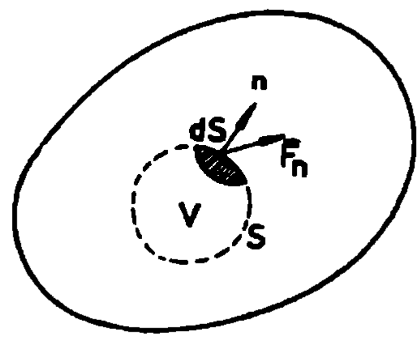


Fig. 1.

The traction, acting on an area, will always be understood to be the force which the part lying on the positive side of a surface element exerts on the part lying on the negative side. (The same is of course true for the stresses, i.e., the tractions per unit area.)

For example, when considering the traction exerted by the sides of the surrounding body on the surface S of a part V imagined detached from the body, one has to use the normal to S which is outward with regard to V. As in the case of body forces, the vector $\vec{F}$ depends on the position of the element S and (in dynamics) on the time. In addition, it depends on the *orientation* of the area in the body, i.e., *on the direction of the normal n*. Therefore, when it is necessary to point out that the stress $\vec{F}$ refers to a plane with the normal n, this will be indicated by writing $\vec{F}_n$. The components of this vector will be denoted by X_n, Y_n, Z_n.

§ 3. Components of stress. Dependence of stress on the orientation of the plane. In order to study the dependence of stress on the orientation of the plane to which it refers, select any orthogonal, rectilinear system of axes $Oxyz$. Let M be a given point contained in that plane. It will be shown that it is sufficient to know the stresses acting on three mutually perpendicular planes passing through M, in order to be able to calculate the stress acting on a plane orientated in any direction whatsoever (and passing through that point).

For the above-mentioned three planes select those, which are perpendicular to the coordinate axes Ox, Oy, Oz respectively, and *as positive directions of the normals* to these planes take the *positive directions of the corresponding axes.*

The following standard notation will be used throughout this book.

Denote by X_x, Y_x, Z_x the components of the stress vector acting on the plane normal to Ox; here the index x indicates that the plane under consideration is normal to Ox. X_x is the *normal stress component* acting on this plane, while Y_x, Z_x are the *tangential or shear stress components.* Similarly denote the components of the stress vector acting on the plane normal to Oy by X_y, Y_y, Z_y and the stress components acting on the plane normal to Oz by X_z, Y_z, Z_z.

It will be shown that the quantities

$$\begin{matrix} X_x, & Y_x, & Z_x, \\ X_y, & Y_y, & Z_y, \\ X_z, & Y_z, & Z_z, \end{matrix} \tag{3.1}$$

characterize completely the state of stress in the neighbourhood of the point considered. Therefore they are called *stress components* (at a given point, at a given instant of time).

These components are shown in Fig. 2. However, it must not be forgotten that they are, by definition, scalar quantities. For example, in Fig. 2. the actual quantity X_x is not depicted, but rather the vector whose x-wise component equals X_x.

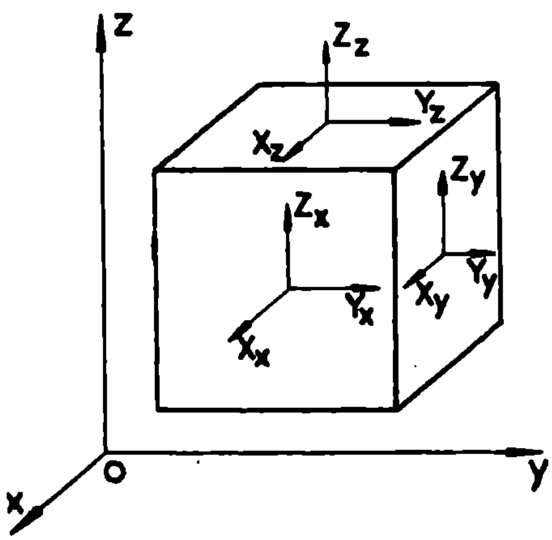


Fig. 2.

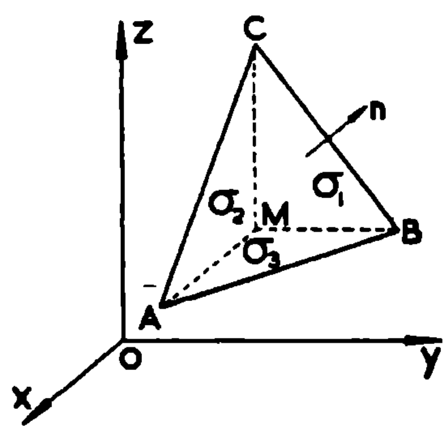


Fig. 3.

In order to find the relations between the quantities (3.1) and the components of the stress vector $\vec{F}_n$ acting on the plane with the normal n, passing through the given point M, consider the following approach.

Through the point M draw three planes, parallel to the coordinate planes, and in addition another plane having the normal n and lying a

distance h from M. These four planes form a tetrahedron, three faces of which are parallel to the coordinate planes, while the fourth ABC is the face to be considered (Fig. 3).

Here and in the sequel it will be assumed (unless stated otherwise) that the body forces and stresses change continuously with the position of the point to which they refer. Further, it will be assumed that they maintain *equilibrium*. This means, by a known principle of statics, that the sum of the external forces, acting on the considered tetrahedron, has a resultant vector equal to zero. Having in mind the transition to the limit $h \to 0$ the size of the tetrahedron will be assumed infinitely small.

Consider the projection on the x axis of the resultant vector of all external forces acting on the tetrahedron.

The arguments will be based upon the supposition that the segments $\overrightarrow{MA}$, $\overrightarrow{MB}$, $\overrightarrow{MC}$ have the same directions as the axes Ox, Oy, Oz. The reader will easily convince himself that the results will hold true in all other cases.

The projection of the body force equals $(X + \varepsilon)dV$, where dV is the volume of the tetrahedron. The value X refers to the point M and ε is an infinitely small quantity (on account of the continuity of X).

Further, the projection of the tractions, acting on the face ABC, is $(X_n + \varepsilon')\sigma$ where σ denotes the area of the triangle ABC and ε' is again infinitely small; X_n, Y_n, Z_n, as will be remembered, are the components of the stress vector acting on the plane through M with normal n.

Finally, the projection of the external forces acting on MBC, normal to Ox, is $(-X_x + \varepsilon_1)\sigma_1$ where σ_1 is the area of MBC. Here $-X_x$ has been taken instead of $+X_x$, since one is dealing with a force acting on an area from that side of the body which lies on the negative side of the surface element MBC (remembering that, by definition, X_x was to be positive when the normal has the same direction as the axis Ox). For the sides MCA and MAB one obtains similarly $(-X_y + \varepsilon_2)\sigma_2$ and $(-X_z + \varepsilon_3)\sigma_3$ respectively. Here ε_1, ε_2 and ε_3 denote again infinitesimal quantities.

Thus, noting that

$$dV = \tfrac{1}{3}h\sigma, \quad \sigma_1 = \sigma\cos(n, x), \quad \sigma_2 = \sigma\cos(n, y), \quad \sigma_3 = \sigma\cos(n, z),$$

one has

$$(X + \varepsilon)\tfrac{1}{3}h\sigma + (X_n + \varepsilon')\sigma + (-X_x + \varepsilon_1)\sigma\cos(n, x) + \\ + (-X_y + \varepsilon_2)\sigma\cos(n, y) + (-X_z + \varepsilon_3)\sigma\cos(n, z) = 0.$$

Dividing by σ and taking the limit $h \to 0$ one obtains the following for-

mulae the last two of which have been written by analogy with the first:

$$\begin{aligned} X_n &= X_x \cos(n, x) + X_y \cos(n, y) + X_z \cos(n, z), \\ Y_n &= Y_x \cos(n, x) + Y_y \cos(n, y) + Y_z \cos(n, z), \\ Z_n &= Z_x \cos(n, x) + Z_y \cos(n, y) + Z_z \cos(n, z). \end{aligned} \tag{3.2}$$

The relations (3.2), as well as those to be deduced in § 4, were first found by A. L. Cauchy (1789—1857) in a memoir, presented to the Paris Academy in 1822 the results of this memoir were published in parts in the years 1823—1828.

§ 4. Equations, relating components of stress. It is known from elementary theoretical mechanics that the resultant force and moment of all external forces, acting on any body in equilibrium, are equal to zero. In the case of absolutely *rigid* bodies (i.e., bodies which do not deform) this condition leads to a system of six equations completely specifying the state of equilibrium. In the case of a deformable body, however, the above condition, when applied to the body as a whole, does not, by any means, completely define the state of equilibrium.

However, in this last case as well, equations may be derived from the above condition which (together with a law, expressing the relationship between the stresses and deformations, to be discussed later) will give all the necessary relations. For this purpose it is necessary to apply the above condition not only to the body as a whole, but to each part which may be imagined detached from it.

In the sequel, unless stated otherwise, it will be assumed that the components of stress are not only continuous, but also have continuous partial derivatives of the first order in the entire region occupied by the body.

Let V be an arbitrary part of the body under consideration (which, by assumption, is in equilibrium), bounded by a closed surface S. The condition of equilibrium will again be expressed by saying that the resultant vector of all *external* forces, acting on V, is zero.

The projection of the resultant vector of the body forces on the Ox axis is equal to

$$\iiint_V X\,dV,$$

and the projection of the resultant vector of the tractions, exerted on the surface S, is equal to

$$\iint X_n\,dS.$$

Replacing in the last formula X_n by the expression given by (3.2) and equating to zero the sum of the projections on the Ox axis of the body and surface forces, one obtains

$$\iiint\limits_V X dV + \iint\limits_S [X_x \cos(n, x) + X_y \cos(n, y) + X_z \cos(n, z)]\, dS = 0,$$

where n denotes the outward normal.

But by Green's Theorem

$$\iint\limits_S [X_x \cos(n, x) + X_y \cos(n, y) + X_z \cos(n, z)]\, dS =$$
$$= \iiint\limits_V \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} \right) dV.$$

Introducing this expression into the preceding formula one obtains finally

$$\iiint\limits_V \left(X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} \right) dV = 0.$$

Remember now that this equation must hold true for *any* region V in the body. This can only be so, however, if the function under the integral signs is zero at each point of the body. Thus one obtains the equations

$$\begin{aligned}
\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + X = 0, \\
\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + Y = 0, \\
\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + Z = 0.
\end{aligned} \tag{4.1}$$

These equations, to which reference will often be made, will be called *equilibrium equations.*

The last step leading to (4.1) is based on the following reasoning. If $F(x, y, z)$ is a function continuous in a given region and

$$\iiint F(x, y, z)\, dV = 0$$

for *any* part V, contained in that region, then $F(x, y, z) = 0$ in the entire region.

In fact, let, for example, $F(x, y, z) > 0$ at some point (x_0, y_0, z_0). Then, on the basis of the continuity of F, one will have around the point (x_0, y_0, z_0) some region V, where $F(x, y, z) > \varepsilon$, ε being a positive constant. Hence

$$\iiint_V F\,dV > \varepsilon V > 0$$

which contradicts the original condition.

Next, use will be made of the condition that the moment of the external forces about the origin of the coordinate system must be zero, or, what is the same thing, that the resultant moments about the coordinate axes must be zero.

Writing that the resultant moment about the Ox axis of the body forces and stresses acting on the surface S containing the volume V is equal to zero, one obtains

$$\iiint_V (yZ - zY)dV + \iint_S (yZ_n - zY_n)dS = 0. \tag{a}$$

But by (3.2)

$$\iint_S (yZ_n - zY_n)dS = \iint_S \{(yZ_x - zY_x)\cos(n, x) + \\ + (yZ_y - zY_y)\cos(n, y) + (yZ_z - zY_z)\cos(n, z)\}\,dS,$$

or transforming, using Green's Theorem,

$$\iint_S (yZ_n - zY_n)dS = \iiint_V \Big\{y\Big(\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}\Big) - \\ - z\Big(\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}\Big) + Z_y - Y_z\Big\}dV.$$

Introducing this expression into (a) and using (4.1) one finally finds

$$\iiint_V (Z_y - Y_z)dV = 0.$$

Since the region V is arbitrary, it follows by the same reasoning used to obtain (4.1) that

$$Y_z = Z_y, \quad Z_x = X_z, \quad X_y = Y_x. \tag{4.2}$$

The two last formulae may be obtained from the first by cyclic permutation of the symbols (or by applying the above reasoning to the axes Oy and Oz).

Thus it is seen that in the table of the stress components

$$\begin{matrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{matrix} \tag{A}$$

the terms, symmetrical with respect to the principal diagonal (running from the upper left-hand to the lower right-hand corner), are equal in pairs; in other words, table (A) is *symmetric*. Thus only six of the nine terms of the table are distinct, i.e.,

$$X_x, Y_y, Z_z, Y_z = Z_y, Z_x = X_z, X_y = Y_x.$$

Hence it may be said that the state of stress at a given point is characterized by *six* of the quantities (A).

The formulae (4.2) may be presented in the form of a proposition. Let there be two planes, passing through one and the same point; then *the projection of the stress, acting on the first plane, on the normal to the second plane is equal to the projection of the stress, acting on the second plane, on the normal to the first plane.* Actually, the formulae (4.2) prove this proposition immediately only in the case when the planes are perpendicular to one another, (i.e., parallel to two coordinate planes). But it is easy to generalize this result to the case of two arbitrary planes and thus to obtain the proposition formulated above.

In fact, let α', β', γ' be the direction cosines of the normal n' to the first plane, and α'', β'', γ'' those of the normal n'' to the second plane. Then the components of the stress vector $\vec{F}_{n'}$, acting on the first plane, are by (3.2)

$$X_{n'} = X_x\alpha' + X_y\beta' + X_z\gamma', \; Y_{n'} = Y_x\alpha' + Y_y\beta' + Y_z\gamma',$$
$$Z_{n'} = Z_x\alpha' + Z_y\beta' + Z_z\gamma'.$$

Using now the relations (4.2), the projection of this stress on the normal to the second plane will be given by

$$(\vec{F}_{n'})_{n''} = X_{n'}\alpha'' + Y_{n'}\beta'' + Z_{n'}\gamma'' = X_x\alpha'\alpha'' + Y_y\beta'\beta'' + Z_z\gamma'\gamma'' + \\ + Y_z(\beta'\gamma'' + \beta''\gamma') + Z_x(\gamma'\alpha'' + \gamma''\alpha') + X_y(\alpha'\beta'' + \alpha''\beta'), \tag{4.3}$$

where $(\;)_{n''}$ indicates projection on the direction n''.

It will be seen that the above expression is quite symmetrical in the quantities α', β', γ' and α'', β'', γ'' and that hence the parts played by the two planes may be interchanged; but this proves the proposition.

NOTE ON NOTATIONS. The notation X_x, Y_y etc., used here for the

stress components, was first introduced by F. Neumann (1841) and has been widely used, e.g. in books by G. Kirchhoff [1], A. E. H. Love [1], S. Timoshenko [1, 2] and others. Besides this notation certain others have been used, but only the following will be mentioned:

$$\tau_{xx} = X_x, \ \tau_{yy} = Y_y, \ \tau_{zz} = Z_z, \ \tau_{yz} = \tau_{zy} = Y_z = Z_y,$$
$$\tau_{zx} = \tau_{xz} = Z_x = X_z, \ \tau_{xy} = \tau_{yx} = X_y = Y_x,$$

which is as widely used (with one or the other unimportant modification) in contemporary literature as the notation used here. It is very convenient from many points of view, especially as it agrees with the modern tensor notation. In many places one finds σ_x, σ_y, σ_z written instead of τ_{xx}, τ_{yy}, τ_{zz}.

§ 5. Transformation of coordinates. Invariant quadratic form. Stress tensor. Formula (4.3) allows the calculation of the projection in any direction of the stress vector, acting on a given plane. In particular, this formula may be used to deduce the transformation formulae for the transition from one rectangular system of axes $Oxyz$ to another $Ox'y'z'$.

Let the direction cosines of the axes of the "new" system $Ox'y'z'$ with regard to the axes of the "old" system $Oxyz$ be given by the following table:

	x	y	z
x'	α_1	β_1	γ_1
y'	α_2	β_2	γ_2
z'	α_3	β_3	γ_3

In this table, for example, α_1, β_1, γ_1 denote the direction cosines of the axis Ox' with regard to the old axes, i.e.,

$$\alpha_1 = \cos(x', x), \ \beta_1 = \cos(x', y), \ \gamma_1 = \cos(x', z).$$

The stress components in the new system of axes will now be denoted by $X'_{x'}$, $Y'_{y'}$, $Z'_{z'}$, $Y'_{z'}$, $Z'_{x'}$, $X'_{y'}$ and the formulae will be found which express these "new" components in terms of the old X_x, Y_y, ..., X_y. Formula (4.3) immediately gives the required expression. For example, for $X'_{x'}$ one obtains

$$X'_{x'} = (\vec{F}_{x'})_{x'}$$

where $F_{x'}$ denotes the stress vector acting on the plane, normal to the new axis Ox'. Consequently one has to put in (4.3)

$$\alpha' = \alpha'' = \alpha_1, \ \beta' = \beta'' = \beta_1, \ \gamma' = \gamma'' = \gamma_1$$

which leads to the first of the following formulae, the others being obtained in an analogous manner:

$$\begin{aligned}
X'_{x'} &= X_x\alpha_1^2 + Y_y\beta_1^2 + Z_z\gamma_1^2 + 2Y_z\beta_1\gamma_1 + 2Z_x\gamma_1\alpha_1 + 2X_y\alpha_1\beta_1,\\
Y'_{y'} &= X_x\alpha_2^2 + Y_y\beta_2^2 + Z_z\gamma_2^2 + 2Y_z\beta_2\gamma_2 + 2Z_x\gamma_2\alpha_2 + 2X_y\alpha_2\beta_2,\\
Z'_{z'} &= X_x\alpha_3^2 + Y_y\beta_3^2 + Z_z\gamma_3^2 + 2Y_z\beta_3\gamma_3 + 2Z_x\gamma_3\alpha_3 + 2X_y\alpha_3\beta_3,\\
Y'_{z'} &= X_x\alpha_2\alpha_3 + Y_y\beta_2\beta_3 + Z_z\gamma_2\gamma_3 + Y_z(\beta_2\gamma_3 + \beta_3\gamma_2) +\\
&\qquad + Z_x(\gamma_2\alpha_3 + \gamma_3\alpha_2) + X_y(\alpha_2\beta_3 + \alpha_3\beta_2), \qquad (5.1)\\
Z'_{x'} &= X_x\alpha_3\alpha_1 + Y_y\beta_3\beta_1 + Z_z\gamma_3\gamma_1 + Y_z(\beta_3\gamma_1 + \beta_1\gamma_3) +\\
&\qquad + Z_x(\gamma_3\alpha_1 + \gamma_1\alpha_3) + X_y(\alpha_3\beta_1 + \alpha_1\beta_3),\\
X'_{y'} &= X_x\alpha_1\alpha_2 + Y_y\beta_1\beta_2 + Z_z\gamma_1\gamma_2 + Y_z(\beta_1\gamma_2 + \beta_2\gamma_1) +\\
&\qquad + Z_x(\gamma_1\alpha_2 + \gamma_2\alpha_1) + X_y(\alpha_1\beta_2 + \alpha_2\beta_1).
\end{aligned}$$

One important result follows from these formulae. Adding the first three and using the well-known relations

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \beta_1^2 + \beta_2^2 + \beta_3^2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,$$

$$\beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3 = \gamma_1\alpha_1 + \gamma_2\alpha_2 + \gamma_3\alpha_3 = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0,$$

one finds

$$X'_{x'} + Y'_{y'} + Z'_{z'} = X_x + Y_y + Z_z.$$

This formula may be interpreted as follows. *The expression*

$$\Theta = X_x + Y_y + Z_z$$

is invariant with regard to transformation of (orthogonal, rectilinear) coordinates, or, in other words, the sum of the normal stress components, acting on three mutually perpendicular planes, does not depend on the orientation of these planes.

Next (3.4) will be used to calculate the normal component of the stress $\vec{F}_n$, acting on a plane with normal n. Let N denote the unknown normal component, i.e., $N = (\vec{F}_n)_n$. For $N > 0$, the normal stress will be *tensile*, for $N < 0$, *compressive.*

If α, β, γ are the direction cosines of the normal n, then one obtains by (3.4) the simple and important result

$$N = X_x\alpha^2 + Y_y\beta^2 + Z_z\gamma^2 + 2Y_z\beta\gamma + 2Z_x\gamma\alpha + 2X_y\alpha\beta. \qquad (5.2)$$

Introduce the notation

$$2\Omega(\xi, \eta, \zeta) = X_x\xi^2 + Y_y\eta^2 + Z_z\zeta^2 + 2Y_z\eta\zeta + 2Z_x\zeta\xi + 2X_y\xi\eta. \quad (5.3)$$

The function $\Omega(\xi, \eta, \zeta)$ is a homogeneous rational function of the second degree in ξ, η, ζ, i.e., in other words, a *quadratic form* in the variables ξ, η, ζ. It has a very simple geometric meaning. Thus, let $\vec{P} = (\xi, \eta, \zeta)$ denote a vector, normal to the considered plane and acting in the same direction as the positive normal n. [In general, (ξ, η, ζ) will denote a vector with components ξ, η, ζ, but it may also at times refer to the point with coordinates ξ, η, ζ.] Then

$$\alpha = \frac{\xi}{P}, \quad \beta = \frac{\eta}{P}, \quad \gamma = \frac{\zeta}{P},$$

where P is the length of the vector $\vec{P}$, and, by (5.2),

$$N.P^2 = 2\Omega(\xi, \eta, \zeta). \quad (5.4)$$

Now the following will be noted. The quantity N, by definition, has physical meaning and hence cannot depend on the particular choice of coordinate axes. In the same way the quantity P^2 (i.e., the square of the length of the vector) does not depend on this choice. Consequently the quadratic form $\Omega(\xi, \eta, \zeta)$ cannot depend on it, i.e., it must be invariant to transformation of (orthogonal, rectilinear) coordinates. In other words, if ξ', η', ζ' denote the components of the vector $\vec{P}$ relative to new axes and $\Omega'(\xi', \eta', \zeta')$ is the quadratic form, involving ξ', η', ζ' and $X'_{x'}, Y'_{y'}, \ldots, X'_{y'}$ in the same manner as $\Omega(\xi, \eta, \zeta)$ involves $\xi, \eta, \zeta, X_x, Y_y, \ldots, X_y$, then

$$\Omega'(\xi', \eta', \zeta') = \Omega(\xi, \eta, \zeta), \quad (5.5)$$

i.e.,

$$X'_{x'}\xi'^2 + Y'_{y'}\eta'^2 + Z'_{z'}\zeta'^2 + 2Y'_{z'}\eta'\zeta' + 2Z'_{x'}\zeta'\xi' + 2X'_{y'}\xi'\eta' =$$
$$= X_x\xi^2 + Y_y\eta^2 + Z_z\zeta^2 + 2Y_z\eta\zeta + 2Z_x\zeta\xi + 2X_y\xi\eta. \quad (5.5')$$

This equality must become an identity, if on the left-hand side one substitutes for $X'_{x'}, \ldots, X'_{y'}$ from (5.1) and on the right-hand side expresses ξ, η, ζ in terms of the new coordinates, using the following formulae known from analytic geometry:

$$\begin{aligned} \xi &= \alpha_1\xi' + \alpha_2\eta' + \alpha_3\zeta', \\ \eta &= \beta_1\xi' + \beta_2\eta' + \beta_3\zeta', \\ \zeta &= \gamma_1\xi' + \gamma_2\eta' + \gamma_3\zeta'. \end{aligned} \quad (5.6)$$

That this is so, is easily checked directly. For this it is sufficient to in-

troduce on the right-hand side of (5.5′) the expressions (5.6) and to compare the coefficients of ξ'^2, η'^2, ζ'^2, $\eta'\zeta'$, $\zeta'\xi'$, $\xi'\eta'$ on both sides. It is then seen that one finds for $X'_{x'}$, $Y'_{y'}$ etc. the expressions (5.1).

Thus, to deduce the formulae (5.1), one may use the above stated rule which is very convenient in practice. Namely, it is sufficient to write down (5.5′), to transform on the right-hand side (or on the left-hand side, if one wants to obtain the transformation formulae from the new to the old components of stress) the variables ξ, η, ζ into ξ', η', ζ' (or ξ', η', ζ' into ξ, η, ζ) and to compare the coefficients of the squares and products of ξ', η', ζ' (or ξ, η, ζ).

The property of invariance of the quadratic form $\Omega(\xi, \eta, \zeta)$ proves that the stress components $X_x, \ldots, X_y$ are components of a (symmetric) *tensor* of second order which will be called the *stress tensor*.

In the main part of this book the reader will not be assumed to be conversant with tensor calculus. For the understanding of certain remarks it will be sufficient to study Appendix 1 at the end of this book. The following will help to elucidate the final paragraph of this section.

Let there be given a quadratic form

$$2\Omega(\xi, \eta, \zeta) = \tau_{xx}\xi^2 + \tau_{yy}\eta^2 + \tau_{zz}\zeta^2 + 2\tau_{yz}\eta\zeta + 2\tau_{zx}\zeta\xi + 2\tau_{xy}\xi\eta,$$

where ξ, η, ζ are the components of some (arbitrary) vector and the coefficients $\tau_{xx}, \ldots, \tau_{xy}$ are quantities independent of ξ, η, ζ, but depending on the direction of the axes of the orthogonal rectilinear coordinate system. If, for transition from one system of axes to another, the coefficients $\tau_{xx}, \ldots, \tau_{xy}$ change in such a way that the quadratic form Ω remains invariant, one says that the set of quantities τ_{xy} (involving *two* subscripts) represents a *symmetric second order tensor*. The quantities τ_{xx} etc. are called the *components* of the tensor.

In the notation of § 5, $\tau_{xx} = X_x$ etc. (cf. Note at the end of § 4). With regard to the definition of non-symmetric tensors of second order see Appendix 1.

§ 6. Stress Surface. Principal stresses.

Consideration of the quadratic form $\Omega(\xi, \eta, \zeta)$, introduced in § 5, admits of a very simple and clear geometric representation of the dependence of the stress vector on the orientation of the plane to which it refers. This representation is concerned with planes, passing through any definite point of a body.

In order to save space let the origin of the coordinate system coincide with the point under consideration. Formula (5.4), viz.,

$$N.P^2 = 2\Omega(\xi, \eta, \zeta),$$

allows calculation of the normal stress component acting on the plane the normal to which has the direction of the vector $\vec{P} = (\xi, \eta, \zeta)$; the length P of this vector may be fixed quite arbitrarily.

In the sequel $\Omega(\xi, \eta, \zeta)$ will be assumed to be not identically zero, because in that case there are no stresses at the given point.

Use of an arbitrary length for P will be introduced by putting $N \cdot P^2 = \pm c^2$, where c is arbitrary, but constant and different from zero (note that c^2 has the dimension of a force). The case has not been excluded, when for some orientation of the plane: $N = 0$. When $N = 0$ it will be assumed that $P = \infty$.

Thus

$$P = \sqrt{\frac{\pm c^2}{N}}, \quad N = \frac{\pm c^2}{P^2}, \tag{6.1}$$

where the sign with c^2 will be chosen such that $\pm c^2$ and N have the same sign (or, in other words, $+ c^2$ will be used when dealing with tensile, and $- c^2$ when dealing with compressive normal stresses).

Let one end of the vector $\vec{P} = \overrightarrow{OH}$ be at the origin O of the coordinate system. Then the end $H(\xi, \eta, \zeta)$ of the vector $\vec{P}$ will lie on the surface

$$2\Omega(\xi, \eta, \zeta) = \pm c^2, \tag{6.2}$$

i.e.,

$$X_x\xi^2 + Y_y\eta^2 + \ldots + 2X_y\xi\eta = \pm c^2. \tag{6.3}$$

The sign on the right-hand side must be chosen in the manner stated above, depending on the sign of N.

The surface (6.2) or (6.3) is obviously a quadric with the centre at the origin. It is called the *stress surface* (stress quadric of Cauchy) referring to a given point of the body. It will be seen later that two cases may occur: in the one, the sign on the right-hand side of (6.2) or (6.3) remains the same for all possible orientations of the planes; in the other, the sign will change depending on the orientation of the planes. Thus, in the second case, one will, in actual fact, be dealing not with one but with two second order surfaces

$$2\Omega = + c^2 \text{ and } 2\Omega = - c^2$$

which obviously have a common axis (cf. below). (One may also fix the sign of c^2 once for all and hence always deal with only *one* surface. But in that case one has to give consideration to imaginary surfaces.)

Once the stress surface has been constructed, the normal stress acting on a given plane (passing through the origin of the coordinate system) may be found without difficulty; it is sufficient to find the intersection H of the normal n to the plane with the surface (6.3). [It will be seen later that such an intersection always exists, provided a definite choice has

been made for the sign on the right-hand side.] Then the normal stress is given by (6.1) with $P = |OH|$.

Further, it is likewise easy to obtain the *direction* of the stress vector, acting on the plane. In fact, equation (3.2) may be written

$$X_n \quad \frac{1}{P}(X_x\xi + X_y\eta + X_z\zeta) = \frac{1}{P}\frac{\partial\Omega}{\partial\xi},$$

$$Y_n = \frac{1}{P}\frac{\partial\Omega}{\partial\eta}, \quad Z_n = \frac{1}{P}\frac{\partial\Omega}{\partial\zeta}, \tag{6.4}$$

remembering that $\cos(n, x) = \frac{\xi}{r}$ etc.

These formulae show that the vector F_n is parallel to the normal to the surface (6.2) at the point $H(\xi, \eta, \zeta)$. Thus, in order to find the direction of $\vec{F}_n$, it is sufficient to construct the tangent plane to the stress surface at the point H and to draw the perpendicular to this plane from the origin. The vector $\vec{F}_n$ then lies along this perpendicular (Fig. 4). Further, since the projection N of $\vec{F}_n$ on to the normal n to the plane under consideration is already known, the construction of $\vec{F}_n$ offers no difficulty.

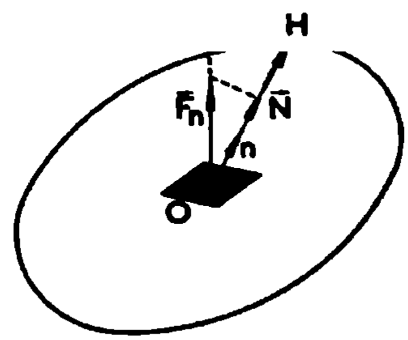


Fig. 4.

The vector $\vec{F}_n$ will have the direction of the normal n to the considered plane only in the case when the radius vector OH is perpendicular to the tangent plane at H. In that case only a normal stress will act on the plane, and no shear stress.

As is known, the radius vector OH will be perpendicular to the tangent plane at H only when OH, and hence the normal n to the plane, has the direction of one of the principal axes of the surface (6.3); in that case the plane will coincide with the principal plane, normal to this axis.

In the general case there are known to be three such principal axes which are mutually perpendicular. Only when the stress surface is a surface of rotation will there be an infinity of such axes: one of these coinciding with the axis of rotation, while all the others are perpendicular to it. Finally, if the stress surface is a sphere, each diameter will be a principal axis.

A direction with the property that only a normal stress acts on the plane normal to it will be called a *principal direction of stress* or *a principal axis* of stress, while the corresponding normal stress will be referred to as a *principal stress.*

As has just been seen, there are always three such directions (and in the general case only three) which are mutually perpendicular; in special cases there may be infinitely many, of which, however, one may always select three perpendicular to one another.

If one selects the coordinate axes along the three principal axes of stress, i.e., along the axes of the surface (6.3), then its equation is known to have the form

$$N_1\xi^2 + N_2\eta^2 + N_3\zeta^2 = \pm\, c^2 \tag{6.5}$$

(i.e., the products of the coordinates disappear), where N_1, N_2, N_3 denote the values of the quantities X_x, Y_y, Z_z for the new coordinate axes.

It is seen from this equation (as likewise on the basis of the definition of principal axes of stress) that relative to the new axes the components Y_z, Z_x, X_y become zero, i.e., no shearing stresses act on the planes coinciding with the coordinate planes. It should again be noted that all the time consideration is being given to planes passing through a given point (i.e., in the present case the origin of coordinates). In general, when passing from one point of the body to another, the principal directions will alter.

By definition, the quantities N_1, N_2, N_3 are the principal stresses. The stress distribution around the point O depends on the signs of these quantities; for the time being they will be assumed to be different from zero.

First the case will be considered when all the principal stresses are positive

$$N_1 > 0,\; N_2 > 0,\; N_3 > 0.$$

In that case one has obviously to take the positive sign on the right-hand side of (6.5) which takes the form

$$N_1\xi^2 + N_2\eta^2 + N_3\zeta^2 = +\, c^2. \tag{6.5a}$$

The corresponding surface is an ellipsoid. By (6.1)

$$N = \frac{+\, c^2}{|\,OH\,|^2},$$

whence it is seen that the normal stress components acting on any plane through O are *tensile.*

Next consider the case when all principal stresses are negative

($N_1 < 0$, $N_2 < 0$, $N_3 < 0$). Then the negative sign has to be taken in (6.5) which becomes

$$N_1\xi^2 + N_2\eta^2 + N_3\zeta^2 = - c^2. \tag{6.5b}$$

The stress surface is again an ellipsoid, but the normal stresses are now given by $N = \frac{-c^2}{|OH|^2}$, indicating that, in contrast to the preceding case, the stresses on all planes are *compressive.*

Finally consider the case when the principal stresses differ in sign, e.g.

$$N_1 > 0,\ N_2 > 0,\ N_3 < 0.$$

Then (6.5) takes the form

$$N_1\xi^2 + N_2\eta^2 - |N_3| \zeta^2 = + c^2, \tag{6.5c}$$

or

$$N_1\xi^2 + N_2\eta^2 \qquad N_3 | \zeta^2 = - c^2. \tag{6.5d}$$

The surface (6.5*c*) is a hyperboloid of one sheet and the surface (6.5*d*) a hyperboloid of two sheets. Both surfaces are separated by the common asymptotic cones

$$N_1\xi^2 + N_2\eta^2 - |N_3| \zeta^2 = 0 \tag{6.6}$$

(see Fig. 5). If the normal to the plane lies outside the asymptotic cone, it intersects the surface (6.5*c*); hence the normal stress is given by

$$N = \frac{+c^2}{|OH|^2}$$

and it will be tensile. If the normal is inside the cone, it intersects (6.5*d*), so that the normal stress which is now compressive is given by

$$N = \frac{}{OH}$$

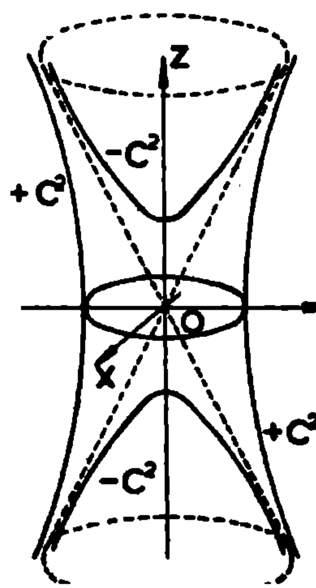


Fig. 5.

Finally, if the normal to the plane is directed along one of the generators of the asymptotic cone, $|OH| = \infty$ and $N = 0$, i.e., the corresponding plane is only subject to shear.

The case $N_1 < 0$, $N_2 < 0$, $N_3 > 0$ differs from the preceding one only in that the regions of tension and compression are interchanged. All other cases differ from those considered above in the way that the parts, played by the coordinate axes, are interchanged.

Previously the cases, when one or two of the quantities N_1, N_2, N_3 are

zero, have been excluded. (When $N_1 = N_2 = N_3 = 0$, no stress whatsoever occurs.) In the case when one of these quantities is zero, the stress surface degenerates into a cylinder and the *state of stress* at that point is then called *plane*. This case will be studied in detail in § 8. When two of the quantities N_1, N_2, N_3 vanish, the stress surface evidently degenerates into two parallel planes.

§ 7. Determination of principal stresses and axes. The problem of finding the principal stresses and the corresponding principal axes has been seen to be linked with the problem of determining a system of coordinates for which the quadratic $\Omega(\xi, \eta, \zeta)$ reduces to its "canonical" form

$$N_1\xi^2 + N_2\eta^2 + N_3\zeta^2.$$

This problem is equivalent to finding the principal planes of stress, i.e., to reducing the equation to the form

$$N_1\xi^2 + N_2\eta^2 + N_3\zeta^2 = \pm c^2. \tag{7.1}$$

Its solution may be found in any textbook on Analytic Geometry or Higher Algebra. It is likewise given in Appendix 1 at the end of this book. It will be solved in § 8 for the case of plane stress.

It will be remembered that the values of the coefficients N_1, N_2, N_3 of (7.1), i.e., the values of the principal stresses, are given by the roots of the third order equation in N (cf. Appendix 1)

$$\begin{vmatrix} X_x - N & X_y & X_z \\ Z_x & Z_y & Z_z - N \end{vmatrix} = -N^3 + \Theta N^2 + AN + B = 0, \tag{7.2}$$

where

$$\Theta = X_x + Y_y + Z_z,$$

$$A = Y_z^2 + Z_x^2 + X_y^2 - Y_yZ_z - Z_zX_x - X_xY_y, \tag{7.3}$$

$$B \quad \begin{matrix} X_x & X_y & X_z \\ Y_x & Y_y & Y \\ Z_x & Z_v & Z \end{matrix} \quad X_xY_yZ_z + 2Y_zZ_xX_y - X_xY_z^2 - Y_yZ_x^2 - Z_zX_y^2.$$

Since the roots N_1, N_2, N_3 do not depend on the choice of the coordinate system, the coefficients of (7.2), i.e., Θ, A, B, likewise cannot depend on it. In other words, these quantities are invariant with respect to trans-

formation of orthogonal rectilinear systems of axes. The invariance of the expression

$$\Theta = X_x + Y_y + Z_z$$

has already been proved above by independent reasoning. This result is likewise obvious on the basis of the fact that the sum of the roots of (7.2) must be equal to Θ, from which follows

$$N_1 + N_2 + N_3 = X_x + Y_y + Z_z. \tag{7.4}$$

§ 8. Plane stress. The state of stress of a body is called *plane*, parallel to the plane Π, if, taking for Π the plane Oxy, one has for all points of the body

$$X_z = Y_z = Z_z = 0. \tag{8.1}$$

Thus there will be only three non-zero components of stress

$$X_x, \ Y_y, \ X_y.$$

If (8.1) does not hold true throughout the body, but only at some given point, one speaks of a plane state of stress at a *given point*.

The formulae (3.2) indicate that the vector components of stress acting on any plane, passing through a given point, will in the present case be given by:

$$\begin{aligned} Z_n &= 0, \\ X_n &= X_x \cos(n, x) + X_y \cos(n, y), \\ Y_n &= Y_x \cos(n, x) + Y_y \cos(n, y). \end{aligned} \tag{8.2}$$

It follows from $Z_n = 0$ that for any orientation of the plane the stress acting on it will be parallel to the plane Oxy.

In the present case the quadratic form $2\Omega(\xi, \eta, \zeta)$ becomes

$$2\Omega(\xi, \eta) = X_x\xi^2 + 2X_y\xi\eta + Y_y\eta^2 \tag{8.3}$$

and the equation of the stress surface

$$X_x\xi^2 + 2X_y\xi\eta + Y_y\eta^2 = \pm c^2. \tag{8.4}$$

This is a cylindrical surface the intersection of which with the plane Oxy is the second order curve (8.4) with the origin as centre.

Limiting consideration to planes, parallel to Oxy, it is sufficient for an investigation, similar to that in § 7, to deal only with the above curve instead of with the entire surface.

Now the transformation formulae will be found for the transition from the stress components

$$X_x, Y_y, X_y$$

to the components

$$X'_{x'}, Y'_{y'}, X'_{y'}$$

referring to a new system of axes, obtained from the old system Oxy by rotation through an angle α in its own plane. The angle α will be measured from the old axis Ox to the new Ox' in the positive direction of rotation in the plane Oxy (i.e., anti-clockwise; see Fig. 6.)

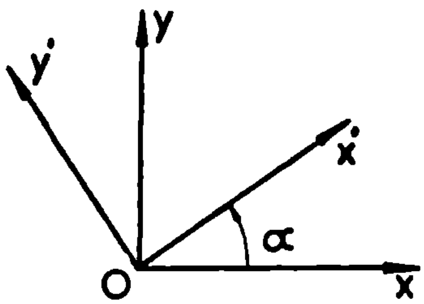


Fig. 6.

These transformation formulae may be obtained from (5.1), but they will be deduced here anew using the property of invariance of the quadratic form $\Omega(\xi, \eta)$ (cf. end of § 5). Using the known formulae for the transformation of a vector (ξ, η) into (ξ', η'), i.e.,

$$\begin{aligned} \xi &= \xi' \cos\alpha - \eta' \sin\alpha, \\ \eta &= \xi' \sin\alpha + \eta' \cos\alpha, \end{aligned} \tag{8.5}$$

and introducing them on the right-hand side of

$$X'_{x'}\xi'^2 + 2X'_{y'}\xi'\eta' + Y'_{y'}\eta'^2 = X_x\xi^2 + 2X_y\xi\eta + Y_y\eta^2 \tag{a}$$

one obtains

$$X'_{x'}\xi'^2 + 2X'_{y'}\xi'\eta' + Y'_{y'}\eta'^2 = X_x(\xi'\cos\alpha - \eta'\sin\alpha)^2 +$$
$$+ 2X_y(\xi'\cos\alpha - \eta'\sin\alpha)(\xi'\sin\alpha + \eta'\cos\alpha) + Y_y(\xi'\sin\alpha + \eta'\cos\alpha)^2,$$

whence follows by comparison of the coefficients of ξ'^2, η'^2 and $\xi'\eta'$

$$\begin{aligned} X'_{x'} &= X_x\cos^2\alpha + Y_y\sin^2\alpha + 2X_y\sin\alpha\cos\alpha, \\ Y'_{y'} &= X_x\sin^2\alpha + Y_y\cos^2\alpha - 2X_y\sin\alpha\cos\alpha, \\ X'_{y'} &= (-X_x + Y_y)\sin\alpha\cos\alpha + X_y(\cos^2\alpha - \sin^2\alpha). \end{aligned} \tag{8.6}$$

After obvious transformations these formulae become

$$\begin{aligned} X'_{x'} &= \frac{X_x + Y_y}{2} + \frac{X_x - Y_y}{} \cos 2\alpha + X_y \sin 2\alpha, \\ Y'_{y'} &= \frac{X_x + Y_y}{2} \quad X_x - Y_y \cos 2\alpha - X_y \sin 2\alpha, \\ X'_{y'} &= \quad X_x - \quad \sin 2\alpha + X_y \cos 2\alpha. \end{aligned} \tag{8.7}$$

A direct check shows that from (8.7) follows

$$\begin{aligned} X'_{x'} + Y'_{x'} &= X_x + Y_y, \\ Y'_{y'} \quad X'_{x'} + 2iX'_{y'} &= (Y_y - X_x + 2iX_y)e^{2i\alpha}. \end{aligned} \tag{8.8}$$

The first of these formulae has been known for a long time and it was proved above for the more general case [cf. (5.2)]. The second, very important and convenient formula was stated by J. H. Michell [3] and it was found independently by G. V. Kolosov [1].

Introducing in this formula $e^{2i\alpha} = \cos 2\alpha + i \sin 2\alpha$ and separating real and imaginary parts, one obtains expressions for $Y'_{y'} - X'_{x'}$ and $X'_{y'}$ in terms of the old stress components. Combining these with the first equation of (8.8) one obtains expressions for $X'_{x'}, Y'_{y'}, X'_{y'}$ which, as is easily verified, agree with those given in (8.7). Finally, note yet another formula obtained by subtracting the equations (8.8) from each other:

$$2(X'_{x'} - iX'_{y'}) = X_x + Y_y - (Y_y - X_x + 2iX_y)e^{2i} \tag{8.8'}$$

Returning to (8.7) it will be shown that these formulae offer a very simple way of determining the principal axes of stress and the principal stresses. In fact, if Ox', Oy' be the unknown principal axes (the third principal axis obviously being the axis Oz), then $X'_{y'} = 0$, whence by the last equation of (8.7)

$$\tan 2\alpha \qquad \frac{2X_y}{\qquad Y_y} \tag{8.9}$$

Here α denotes the angle, measured in the sense stated earlier, which the principal axis Ox' makes with Ox. Formula (8.9) gives two values for α; if one of these is denoted by α_0, the second will be $\alpha_0 + \frac{\pi}{2}$. All other possible values differ from these two by multiples of π, and obviously α may take any of these values. Substituting this value in the first two formulae of (8.7) one obtains the principal stresses N_1, N_2, the first formula giving N_1 corresponding to the angle α, the second N_2 corresponding to $\alpha + \frac{\pi}{2}$.

Next, if one takes for the original coordinate axes the principal axes, then

$$X_x = N_1, \; Y_y = N_2, \; X_y = 0$$

and the formulae (8.7) become even simpler:

$$X'_{x'} = \frac{N_1 + N_2}{2} + \frac{N_1 - N_2}{2} \cos 2\alpha, \quad Y'_{y'} = \frac{N_1 + N_2}{2} - \frac{N_1 - N_2}{2} \cos 2\alpha,$$

$$X'_{y'} = -\frac{N_1 - N_2}{2} \sin 2\alpha. \tag{8.10}$$

These formulae show that the maximum absolute value of the shear stress is given by

$$|X'_{y'}|_{max} = \left|\frac{N_1 - N_2}{2}\right|$$

i.e., it is equal to half the absolute value of *the difference of the principal stresses*. This value is attained on two mutually perpendicular planes, bisecting the angle between the principal directions Ox, Oy.

Finally, the formulae will be written down which give X_x, Y_y, X_y, if the principal stresses N_1, N_2 and the angle α between the principal axis corresponding to N_1 and the Ox axis are known. They are obtained from (8.10) by interchanging the parts played by the old and the new systems and by replacing the angle α by $-\alpha$. In this way

$$X_x = \frac{N_1 + N_2}{2} + \frac{N_1 - N_2}{2} \cos 2\alpha, \quad Y_y = \frac{N_1 + N_2}{2} - \frac{N_1 - N_2}{2} \cos 2\alpha,$$

$$X_y = \frac{N_1 - N_2}{2} \sin 2\alpha. \tag{8.11}$$

The formulae (8.11) are equivalent to the following which likewise result directly from (8.8.):

$$X_x + Y_y = N_1 + N_2, \quad Y_y - X_x + 2iX_y = -(N_1 - N_2)e^{-2i\alpha}. \tag{8.12}$$

NOTE. It is easily seen that the transformation formulae for the stress components X_x, Y_y, X_y into $X'_{x'}$, $Y'_{y'}$, $X'_{y'}$, for rotation of the system Oxy in its own plane, remain the same as those deduced above, in the case of a more general state of stress (and not a plane one), provided that the axis Oz is *one of the principal axes* at the point considered. In fact, in that case

$$X_z = Y_z = X'_{z'} = Y'_{z'} = 0$$

there. The identity (5.5') then takes the form

$$X_x\xi^2 + 2X_y\xi\eta + Y_y\eta^2 + N_3\zeta^2 = X'_{x'}\xi'^2 + 2X'_{y'}\xi'\eta' + Y'_{y'}\eta'^2 + N_3\zeta^2,$$

because, by assumption, the Oz axis remains unchanged and hence $\zeta' = \zeta$; N_3 denotes the principal stress corresponding to Oz, i.e.,

$$N_3 = Z_z = Z'_{z'} .$$

The earlier equation (a) follows from the preceding relation and the transformation formulae could have been deduced from it.

CHAPTER 2

ANALYSIS OF STRAIN

§ 9. General remarks. The term *deformation*, when applied to a continuous body, will refer to changes in the position of the points of this body such that their relative distances are altered.

Refer such a body to an orthogonal coordinate system $Oxyz$ and denote by x, y, z the coordinates of a point of the body before deformation and by x^*, y^*, z^* the coordinates of the same point afterwards. Let V be the region occupied by the body before deformation. Each point of the body, occupying before deformation the position (x, y, z) of the region V, will afterwards occupy a unique position (x^*, y^*, z^*). This is the basic assumption of the present chapter. Thus the coordinates x^*, y^*, z^* must be definite functions of the coordinates x, y, z of the same point before deformation of the body:

$$x^* = f_1(x, y, z), y^* = f_2(x, y, z), z^* = f_3(x, y, z). \tag{9.1}$$

The functions f_1, f_2, f_3 will be assumed to be continuous in the region V (i.e., the deformation causes no cleavage of the body). The points (x^*, y^*, z^*), corresponding to the points (x, y, z) of V, cover some region V* occupied by the body after deformation. Conversely, it will be assumed that the coordinates x, y, z are definite functions of x^*, y^*, z^* [in other words, that the equations (9.1) can be solved uniquely for x, y, z] and that these functions are likewise continuous for x^*, y^*, z^* in V*.

From a geometrical point of view the formulae (9.1) represent a certain *transformation* of V into V*. It will be noted that not each such transformation, i.e., not all relations of the form (9.1), represent a deformation of the body in the above sense. In fact, if one displaces the considered body as a rigid unit (such a displacement will be called *rigid body motion*), then the coordinates x^*, y^*, z^* of the new positions of the points of the body will be definite functions of x, y, z; however, this is not a *deformation*, i.e., a displacement of the points of the body *with respect* to each other. For the sequel it will be very important, once the equations (9.1) are given. to separate the actual deformation from the rigid body motion; in other

words, it will be important to find the quantities characterizing *deformation as such.*

§ 10. Affine Transformation. A transformation of the form (9.1) is called *affine*, if the coordinates x^*, y^*, z^* are linear functions of the coordinates x, y, z, i.e., if (9.1) has the form

$$\begin{aligned} x^* &= (1 + a_{11})x + a_{12}y + a_{13}z + a, \\ y^* &= a_{21}x + (1 + a_{22})y + a_{23}z + b, \\ & \quad a_{31}x + a_{32}y + (1 + a_{33})z + c, \end{aligned} \tag{10.1}$$

where $a_{11}, a_{12}, \ldots, a, b, c$ are constants (for reasons, which will become clear in the sequel, the diagonal terms have been denoted by $1 + a_{11}$, $1 + a_{22}$, $1 + a_{33}$ instead of by a_{11}, a_{22}, a_{33}). With reference to § 9 it must be assumed that these equations are soluble with regard to x, y, z, i.e., that

$$D \quad \begin{matrix} 1 + a_{11} & a_{12} & a_{13} \\ a_{21} & 1 + a_{22} & a_{23} \\ a_{31} & a_{32} & 1 + a_{33} \end{matrix} \tag{10.2}$$

is different from zero.

The affine transformation possesses many simple important properties of which only the following will be noted. First of all, it is obvious that the inverse transformation will be affine, since, solving (10.1) for x, y, z, one clearly obtains linear expressions in terms of x^*, y^*, z^*, i.e.,

$$\begin{aligned} x &= (1 + b_{11})x^* + b_{12}y^* + b_{13}z^* + a', \\ y &= b_{21}x^* + (1 + b_{22})y^* + b_{23}z^* + b', \\ z &= b_{31}x^* + b_{32}y^* + (1 + b_{33})z^* + c', \end{aligned} \tag{10.3}$$

where $b_{11}, b_{12}, \ldots, a', b', c'$ are constants.

Further, it is easily shown that points, lying before the transformation in some plane Π, will after the transformation lie in some plane Π^*. In fact, let $Ax + By + Cz + D = 0$ be the equation of the plane Π. Substituting for x, y, z from (10.3) one sees that this equation is transformed into one which is again linear in x^*, y^*, z^*, i.e., into an equation of the form $A^*x^* + B^*y^* + C^*z^* + D^* = 0$ which is, of course, the equation of the plane Π^*. The points which were previously in the plane Π will now lie in the plane Π^*.

It may also be shown that the above, in combination with the property of continuity of the transformation (i.e., that points at a finite distance correspond to points at a finite distance, and points infinitely close together correspond to points infinitely close together), characterizes the transformation, so that every transformation with these properties will be affine.

It follows from the above property that points lying before the transformation on some straight line Δ will move to points likewise on a straight line Δ^*. In fact, the straight line Δ may be considered as the intersection of some planes Π_1, Π_2. After the transformation, the points of the straight line Δ, i.e., the points common to the planes Π_1 and Π_2, become points common to two planes Π_1^* and Π_2^* which are the transformed planes Π_1 and Π_2, and this proves the assertion.

It follows from this that any straight segment is transformed into a straight segment, and any vector into a vector. Let the vector $\vec{P} = (\xi, \eta, \zeta)$, as the result of the transformation, become a vector

$$\vec{P}^* = (\xi^*, \eta^*, \zeta^*).$$

Further, let (x_0, y_0, z_0) and (x, y, z) be respectively the starting and end points of $\vec{P}$, so that

$$\xi = x - x_0, \quad \eta = y - y_0, \quad \zeta = z - z_0.$$

The vector P^* will similarly have the components

$$\xi^* = x^* - x_0^*, \quad \eta^* = y^* - y_0^*, \quad \zeta^* = z^* - z_0^*,$$

where, for example, by (10.1)

$$x^* = (1 + a_{11})x + a_{12}y + a_{13}z + a, \quad x_0^* = (1 + a_{11})x_0 + a_{12}y_0 + a_{13}z_0 + a.$$

Subtracting these two equations one finds the first of the following formulae; the others can be obtained in an analogous manner:

$$\begin{aligned} \xi^* &= (1 + a_{11})\xi + a_{12}\eta + a_{13}\zeta, \\ \eta^* &= a_{21}\xi + (1 + a_{22})\eta + a_{23}\zeta, \\ \zeta^* &= a_{31}\xi + a_{32}\eta + a_{33}\zeta. \end{aligned} \tag{10.4}$$

The formulae (10.4) simply express (cf. Appendix 1.2) that the vector (ξ^*, η^*, ζ^*) is a linear vector function of the vector (ξ, η, ζ). Consequently the quantities $1 + a_{11}, a_{12}, \ldots, a_{33}$, or more briefly $a_{ij} + \delta_{ij}$, are components of a certain tensor. But since (δ_{ij}) is a tensor, then also (a_{ij}) is a tensor obtained from the former by subtraction of the tensor (δ_{ij}).

It follows directly from (10.4) that two equal vectors (i.e., vectors having identical components ξ, η, ζ) become after transformation two equal vectors, and that two parallel vectors become two parallel vectors, the ratio of their lengths remaining unchanged. (The ratio of the moduli of non-parallel vectors, generally speaking, is altered by the transformation. Cf. Appendix 1.2). It follows also from this first property that two identical and identically orientated polygons (lying in different parts of space) are also transformed into identical and identically orientated polygons.

But since every geometric figure may be considered the limit of polygonal figures, it follows that the above property is valid for all figures. This means that all parts of a body, independent of their position, will deform in an identical manner. Therefore, the deformation arising from an affine transformation is often called *homogeneous*.

NOTE. It will always be assumed that the coordinates are not only rectilinear, but also orthogonal. However, all the above will also be true for an oblique coordinate system.

It is almost obvious that the character, i.e., the linearity of the relations (10.1) or (10.4) remains unchanged, if one rectilinear systems of coordinates is replaced by another. This follows directly from the linearity of the transformation formulae.

§ 11. Infinitesimal affine transformation. A transformation of the form (10.1) will be called *infinitesimal*, if the a_{ij}, a, b, c are infinitesimal quantities, the squares and products of which may be neglected in comparison with these same quantities. It follows then from (10.1) that by this assumption the differences

$$x^* - x = a_{11}x + a_{12}y + a_{13}z + a,\quad y^* - y = a_{21}x + a_{22}y + a_{23}z + b,$$
$$z^* - z = a_{31}x + a_{32}y + a_{33}z + c$$

between the coordinates of one and the same point before and after the transformation will be infinitesimal quantities.

Consider the result of two consecutive infinitesimal transformations. Let the first of these be the affine infinitesimal transformation

$$\begin{aligned} x^* &= (1 + a_{11})x + a_{12}y + a_{13}z + a,\\ y^* &= a_{21}x + (1 + a_{22})y + a_{23}z + b,\\ z^* &= a_{31}x + a_{32}y + (1 + a_{33})z + c, \end{aligned} \tag{11.1}$$

and apply to x^*, y^*, z^* another infinitesimal transformation

$$\begin{aligned} x^{**} &= (1 + b_{11})x^* + b_{12}y^* + b_{13}z^* + a',\\ y^{**} &= b_{21}x^* + (1 + b_{22})y^* + b_{23}z^* + b',\\ z^{**} &= b_{31}x^* + b_{32}y^* + b_{33}z^* + c'. \end{aligned} \tag{11.2}$$

These two infinitesimal transformations transform the point (x, y, z) into the point (x^{**}, y^{**}, z^{**}). One obtains the relations between the coordinates of these points by substituting the expressions (11.1) in (11.2). Neglecting products of the quantities b_{ij}, a_{ij}, a, b, c one finds

without great difficulty

$$\begin{aligned} x^{**} &= (1 + c_{11})x + c_{12}y + c_{13}z + a'' \\ y^{**} &= c_{21}x + (1 + c_{22})y + c_{23}z + b'' \\ & \qquad\quad + c_{32}y + (1 + c_{33})z + c'' \end{aligned} \tag{11.3}$$

where

$$c_{ij} = a_{ij} + b_{ij}\ (i, j = 1, 2, 3), a'' = a + a', b'' = b + b', c'' = c + c'. \tag{11.4}$$

These formulae prove that the result of two affine transformations is again an affine transformation. This property, as the reader will easily verify for himself, is true for any affine transformation whatsoever (and not only for infinitesimal transformations).

But *the two following properties, deduced directly from* (11.3) *and* (11.4), *are, generally speaking, only true for infinitesimal transformations.* They are as follows: the resulting transformation does not depend on the order in which the two transformations were applied; the coefficients c_{ij}, a'', b'', c'' are the sums of the corresponding coefficients of these transformations.

It will be said that the resulting transformation was obtained by *composition* of two transformations. All the above may be directly generalized to the case, when an arbitrary number of transformations is to be composed.

§ 12. Decomposition of infinitesimal transformations into pure deformation and rigid body motion. Since in the sequel interest will be concentrated on the problem of *deformations*, one may limit consideration to the transformation formulae (10.4) for the components of a *vector*. If these formulae are given, i.e., if the quantities $a_{11}, \ldots, a_{33}$ are given, the formulae (10.1) for the transformation of the coordinates of a *point* will not actually be completely defined, i.e., the quantities a, b, c still remain undetermined. But then these quantities obviously do not influence the deformations, but only the rigid *translatory* displacement of the body.

The formulae (10.4) may be written

$$\begin{aligned} \delta\xi &= a_{11}\xi + a_{12}\eta + a_{13}\zeta, \\ \delta\eta &= a_{21}\xi + a_{22}\eta + a_{23}\zeta, \\ \delta\zeta &= a_{31}\xi + a_{32}\eta + a_{33}\zeta, \end{aligned} \tag{12.1}$$

where

$$\delta\xi = \xi^* - \xi, \ \delta\eta = \eta^* - \eta, \ \delta\zeta = \zeta^* - \zeta \tag{12.2}$$

denote the components of the vector difference $\vec{P}^* - \vec{P} = \delta\vec{P}$, i.e., of the *increment* of the vector $\vec{P}$, caused by the transformation.

Next consider what conditions must be satisfied by the quantities

$$\begin{matrix} a_{11}, & a_{12}, & a_{13}, \\ a_{21}, & a_{22}, & a_{23}, \\ a_{31}, & a_{32}, & a_{33}, \end{matrix} \tag{12.3}$$

which will be called the *coefficients* of the *transformation* under consideration (as stated in § 10 these coefficients represent the components of a second order tensor), in order that (12.1) does not involve deformation, i.e., that it expresses only rigid body motion.

A necessary and sufficient condition for this is that the length P of any vector $\vec{P}$, or what is the same thing, that the square of its modulus

$$P^2 = \xi^2 + \eta^2 + \zeta^2$$

remains unaltered by the transformation.

In the following, consideration will be restricted to infinitesimal transformations. Calculate the increase δP of the length P. The preceding formula together with (21.1) gives, neglecting higher order quantities,

$$P\delta P = \xi\delta\xi + \eta\delta\eta + \zeta\delta\zeta = a_{11}\xi^2 + a_{22}\eta^2 + a_{33}\zeta^2 + \\ + (a_{23} + a_{32})\eta\zeta + (a_{31} + a_{13})\zeta\xi + (a_{12} + a_{21})\xi\eta. \tag{12.4}$$

In order that $\delta P = 0$ for all possible values of ξ, η, ζ, it is obviously necessary and sufficient that

$$a_{11} = a_{22} = a_{33} = 0, \quad a_{23} + a_{32} = a_{31} + a_{13} = a_{12} + a_{21} = 0. \tag{12.5}$$

This is the required condition that the transformation (12.1) represents rigid body motion. It may be written briefly as

$$a_{ij} = - a_{ij} \ (i, j = 1, 2, 3) \tag{12.5'}$$

[which expresses the fact that the tensor (a_{ij}) is antisymmetric (cf. Appendix 1.2)]; in fact, for i $\neq j$ one obtains the second group of the formulae (12.5), while for $i = j$ one finds $a_{ii} = - a_{ii}$, i.e., $a_{ii} = 0$ which agrees with the first of the formulae (12.5).

Thus (12.1) may in the present case be written

$$\delta\xi = q\zeta - r\eta, \quad \delta\eta = r\xi - p\zeta, \quad \delta\zeta = p\eta - q\xi, \tag{12.6}$$

where

$$p = a_{32} = - a_{23}, \quad q = a_{13} = - a_{31}, \quad r = a_{21} = - a_{12}. \tag{12.7}$$

These are the well-known kinematical formulae expressing rigid (infinitesimal) body motion. The quantities p, q, r, are known to be the infinitesimal angles of rotation about the coordinate axes and will be called *components of rotation*. [The set of the quantities (p, q, r) may be considered as a vector (cf. Appendix 1.3); in fact, it is the (infinitesimal) rotation vector commonly used in kinematics]. The terms which refer to the translatory displacement are missing from these formulae, because consideration is being given to the components of a *vector* which is not altered by the translatory displacements.

In order to obtain the formulae of transformation for the coordinates of a *point*, occupying before displacement the position $M(x, y, z)$, it is sufficient to apply the preceding formulae to the vector

$$\overrightarrow{M_0M} = (x - x_0, y - y_0, z - z_0),$$

where $M_0(x_0, y_0, z_0)$ is an arbitrary, but once and for all fixed point of the body. Substituting in (12.6) $x - x_0$, $y - y_0$, $z - z_0$ for ξ, η, ζ one obtains the well known formulae of kinematics

$$\begin{aligned} \delta x &= a + q(z - z_0) - r(y - y_0), \\ \delta y &= b + r(x - x_0) - p(z - z_0), \\ \delta z &= c + p(y - y_0) - q(x - x_0), \end{aligned} \tag{12.8}$$

where

$$a = \delta x_0, \ b = \delta y_0, \ c = \delta z_0;$$

in other words, the vector (a, b, c) describes the displacement of the point (x_0, y_0, z_0). If one uses the origin of the coordinate system for the point M_0, then (12.8) is somewhat simplified; in fact, it becomes

$$\delta x = a + qz - ry, \ \delta y = b + rx - bz, \ \delta z = c + py - qx, \tag{12.8'}$$

where the vector (a, b, c) refers to the displacement of the point which before the transformation coincided with the origin.

Next consider (12.4). It indicates that the change in length of the vector $\vec{P}$ is characterized by the quantities

$$a_{11}, \ a_{22}, \ a_{33}, \ a_{32} + a_{23}, \ a_{13} + a_{31}, \ a_{21} + a_{12},$$

for which the following notation will now be introduced:

$$\begin{gathered} a_{11} = e_{xx}, \quad a_{22} = e_{yy}, \quad a_{33} = e_{zz}, \quad \tfrac{1}{2}(a_{32} + a_{23}) = e_{yz} = e_{zy}, \\ \tfrac{1}{2}(a_{13} + a_{31}) = e_{zx} = e_{xz}, \quad \tfrac{1}{2}(a_{21} + a_{12}) = e_{xy} = e_{yx}. \end{gathered} \tag{12.9}$$

Actually, deformation is characterized by variations in the distances between points, i.e., by changes of length of vectors; it is determined by the six quantities $e_{xx}, e_{yy}, e_{zz}, e_{yz}, e_{zx}, e_{xy}$ which will be called *components of strain.*

Since the a_{ij} are the components of some second order tensor, the quantities $e_{xx}, \ldots, e_{xy}$ are the components of a symmetric second order tensor, as may be seen from Appendix 1.3; a direct proof of this fact will be given below.

Similarly, the quantities $\frac{1}{2}(a_{ij} - a_{ji})$ are components of an anti-symmetric second order tensor which may be represented by means of the vector (p, q, r) (cf. Appendix 1.)

Further, in agreement with what has just been stated, introduce the notation

$$p = \tfrac{1}{2}(a_{32} - a_{23}), \quad q = \tfrac{1}{2}(a_{13} - a_{31}), \quad r = \tfrac{1}{2}(a_{21} - a_{12}). \tag{12.10}$$

In the above notation, obviously,

$$\begin{aligned} a_{32} &= e_{yz} + p, & a_{13} &= e_{zx} + q, & a_{21} &= e_{xy} + r, \\ a_{23} &= e_{yz} - p, & a_{31} &= e_{zx} - q, & a_{12} &= e_{xy} - r, \end{aligned} \tag{12.11}$$

which demonstrates the division of the tensor (a_{ij}) into the sum of symmetric and anti-symmetric parts.

The formulae (12.1) may now be written

$$\begin{aligned} \delta\xi &= e_{xx}\xi + e_{xy}\eta + e_{xz}\zeta + q\zeta - r\eta, \\ \delta\eta &= e_{yx}\xi + e_{yy}\eta + e_{yz}\zeta + r\xi - p\zeta, \\ \delta\zeta &= e_{zx}\xi + e_{zy}\eta + e_{zz}\zeta + p\eta - q\xi. \end{aligned} \tag{12.12}$$

These formulae show that the original affine transformation may be divided into two transformations: one of the form

$$\begin{aligned} \delta\xi &= e_{xx}\xi + e_{xy}\eta + e_{xz}\zeta, \\ \delta\eta &= e_{yx}\xi + e_{yy}\eta + e_{yz}\zeta, \\ \delta\zeta &= e_{zx}\xi + e_{zy}\eta + e_{zz}\zeta, \end{aligned} \tag{12.13}$$

and another of the form (12.6) representing rigid body motion. The transformation (12.13) which contains only components of deformation will be called ***actual*** or ***pure homogeneous deformation*** (see later).

It is characteristic of the formulae (12.13) that ***the array of coefficients***

$$\begin{matrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{matrix}$$

is symmetric.

Each of the components of strain has a very simple geometric meaning. The latter may be deduced directly from the formula (12.4) which in the new notation becomes

$$P\delta P = e_{xx}\xi^2 + e_{yy}\eta^2 + e_{zz}\zeta^2 + 2e_{yz}\eta\zeta + 2e_{zx}\zeta\xi + 2e_{xy}\xi\eta. \quad (12.4')$$

Consider some vector $\vec{P}(\xi, 0, 0)$ which before deformation is parallel to the Ox axis. For this vector

$$P\delta P = e_{xx}\xi^2,$$

or, taking into account that $\xi^2 = P^2$,

$$e_{xx} = \frac{\delta P}{P}. \quad (12.14)$$

Thus e_{xx} represents the relative increase of the vector (or segment), originally parallel to the axis Ox. The components e_{yy} and e_{zz} have an analogous meaning.

If all the components of deformation, except e_{xx}, are zero and if one considers pure deformation, i.e., if

$$p = q = r = 0,$$

then (12.13) gives

$$\delta\xi = e_{xx}\xi, \qquad \delta\eta = \delta\zeta = 0.$$

Hence, in this case, all vectors parallel to the axis Ox are stretched in one and the same manner (the proportional increase being $\frac{\delta\xi}{\xi} = e_{xx}$); however, vectors perpendicular to this axis do not change their direction nor their length. Thus this case represents a *simple* and *homogeneous extension* in the direction Ox. Similar results will be obtained in cases when either e_{yy} or e_{zz} are the only non-zero components.

In order to explain the meaning of e_{yz}, one has to determine the change of the originally right angle between the two vectors $\vec{P}_1(0, \eta_1, 0)$ and $\vec{P}_2(0, 0, \zeta_2)$ which before deformation were directed along Oy and Oz. Let the angle between these vectors after deformation be denoted by $\frac{\pi}{2} - \varepsilon_{yz}$ (i.e., $\varepsilon_{yz} > 0$ if the angle decreases and $e_{yz} < 0$ if it increases). By a known formula the cosine of the angle between two vectors

$$(\delta\xi_1, \eta_1 + \delta\eta_1, \delta\zeta_1) \text{ and } (\delta\xi_2, \delta\eta_2, \zeta_2 + \delta\zeta_2)$$

is given by

$$\cos\left(\frac{\pi}{2} - \varepsilon_{yz}\right) = \frac{\delta\xi_1\delta\xi_2 + (\eta_1 + \delta\eta_1)\delta\eta_2 + \delta\zeta_1(\zeta_2 + \delta\zeta_2)}{\sqrt{\delta\xi_1^2 + (\eta_1 + \delta\eta_1)^2 + \delta\xi_1^2}\,.\,\sqrt{\delta\xi_2^2 + \delta\eta_2^2 + (\zeta_2 + \delta\zeta_2)^2}}$$

But when ε_{yz} is small,

$$\cos\left(\frac{\pi}{2} - \varepsilon_{yz}\right) = \varepsilon_{yz},$$

neglecting infinitely small higher order terms. Omitting higher order terms on the right-hand side of the above equation one obtains

$$\varepsilon_{yz} = \frac{\eta_1 \delta\eta_2 + \zeta_2 \delta\zeta_1}{\eta_1 \zeta_2} \quad \frac{\delta\zeta_1}{\eta_1} + \frac{\delta\eta_2}{\zeta_2}$$

But by (12.12), applying it to $P_1(0, \eta_1, 0)$ and $P_2(0, 0, \zeta_2)$,

$$\delta\zeta_1 = e_{zy}\eta_1 + p\eta_1, \; \delta\eta_2 = e_{yz}\zeta_2 - p\zeta_2;$$

introducing these values in the preceding formula one finds

$$\varepsilon_{yz} = e_{zy} + e_{yz} = 2e_{yz}. \tag{12.15}$$

Thus the quantity $2e_{yz}$ represents the decrease of the angle between two vectors having originally the (positive) directions of the axes Oy and Oz. Similar interpretations may be found for $2e_{zx}$ and $2e_{xy}$.

Now consider pure deformation for which all components but e_{yz} are equal to zero. Let $\overrightarrow{OB}$ and $\overrightarrow{OC}$ be two vectors, starting for simplicity from the origin and directed along the axes Oy and Oz, and let $OBCK$ be a rectangle constructed on these two vectors (Fig. 7). After deformation the rectangle becomes the parallelogram $OB'C'K'$ (where it is assumed that the origin is not displaced; if this assumption does not hold, one may bring the origin back to its old position by means of a translation).

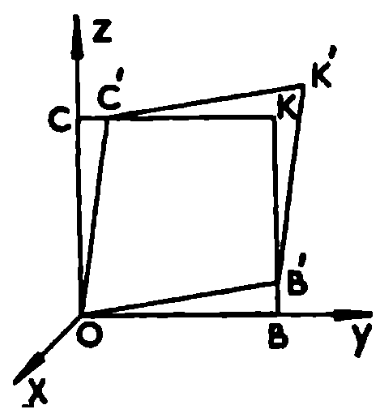


Fig. 7.

By (12.13) the point B is transformed into the point B' on the straight line BK and the point C into C' on CK; further,

$$BB' = e_{zy}.OB, \; CC' = e_{yz}.OC.$$

Since, neglecting infinitely small higher order terms,

$$\frac{BB'}{OB} \quad \tan \widehat{BOB'} = \widehat{BOB'}, \; \frac{CC'}{OC} = \tan \widehat{COC'} = \widehat{COC'},$$

the preceding formulae give

$$\widehat{BOB'} = \widehat{COC'} = e_{yz},$$

whence one obtains again

$$\varepsilon_{yz} = \widehat{BOB'} + \widehat{COC'} = 2e_{yz}.$$

If, by means of rigid rotation about Ox, one causes the segment OB' to coincide with OB (the difference in their lengths obviously being a higher order quantity), the parallelogram $OB'K'C'$ takes the position $OBK''C''$ (Fig. 8) and the angle $\widehat{COC''}$ is again equal to the angle ε_{yz} (where it has been assumed that C'' lies on the straight line CK, since obviously this will be so, neglecting higher order terms).

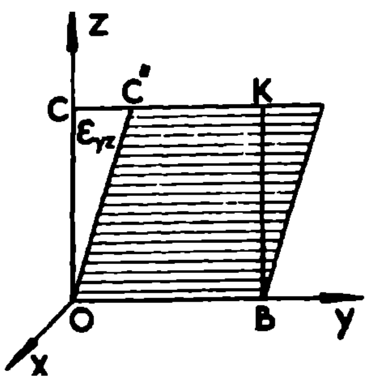


Fig. 8.

Thus the deformation represents a *shearing* of planes, parallel to the plane Oxy in the direction of the axis Oy, and the displacement of each layer is proportional to its distance from the plane Oxy. The quantity CC'' measures the "absolute shear", and

$$\frac{CC''}{OC} \quad \tan \varepsilon_{u} \qquad \varepsilon_{yz} = 2e_{yz}$$

the "relative shear" or the *angle of shear*. The considered deformation is called *simple* (homogeneous) shear.

§ 13. The invariant quadratic form, connected with deformation. The strain surface, principal axes. Transformation of coordinates. The formula (12.4′) may be written

$$P\delta P = 2F(\xi, \eta, \zeta), \tag{13.1}$$

where now

$$2F(\xi, \eta, \zeta) = e_{xx}\xi^2 + e_{yy}\eta^2 + e_{zz}\zeta^2 + 2e_{yz}\eta\zeta + 2e_{zx}\zeta\xi + 2e_{xy}\xi\eta, \tag{13.2}$$

i.e., F is a quadratic form in the variables ξ, η, ζ. Since the left-hand side of (13.1), i.e., $P\delta P$, has a definite meaning, independent of the choice of coordinate axes, it follows that the quadratic form $F(\xi, \eta, \zeta)$ is invariant with regard to transformation of coordinates. In other words, if $e_{x'x'}$, $\ldots, e_{x'y'}$ are the components of strain in a new coordinate system and ξ', η', ζ' are the components of the vector $\vec{P}$ in the new system, then

$$e_{x'x'}\xi'^2 + e_{y'y'}\eta'^2 + \ldots + 2e_{x'y'}\xi'\eta' = e_{xx}\xi^2 + \ldots + 2e_{xy}\xi\eta, \tag{13.3}$$

which becomes an identity in ξ', η', ζ', if on the right-hand side ξ, η, ζ are expressed in terms of ξ', η', ζ'. This proves that the array of the quantities

$$\begin{matrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{matrix}$$

represents a symmetric second order tensor (cf. end of § 5). In particular, as in § 5, it follows that the components of strain in the new coordinate system are related to the old ones by the same formulae (5.1) as the new components of stress were related to the old ones (one only has to replace in those formulae X_x by e_{xx}, Y_z by e_{yz} etc.).

Just as in § 6 the stress surface

$$2\Omega(\xi, \eta, \zeta) = \pm c^2$$

was introduced for the study of stresses, one may here consider an analogous surface.

The formula (13.1) may be written

$$P^2 \frac{\delta P}{P} = 2F(\xi, \eta, \zeta)$$

or

$$P^2 e = 2F(\xi, \eta, \zeta),$$

where $e = \dfrac{\delta P}{P}$ denotes the relative increase of the vector $\vec{P} = (\xi, \eta, \zeta)$. As is known, this quantity does not depend on the length of the vector $\vec{P}$, but only on its direction. Therefore one may for every direction choose the length P so that $P^2 e = \pm c^2$, where c is an arbitrary fixed positive constant which has the dimension of a length.

If one takes as the starting point of the vector $\vec{P}$ the origin of the coordinate system, then the end point H of this vector will lie on the surface

$$2F(\xi, \eta, \zeta) = \pm c^2, \quad \text{or} \quad e_{xx}\xi^2 + \dots + 2e_{xy}\xi\eta = \pm c^2 \qquad (13.4)$$

which is called the *strain surface* (*Cauchy's strain quadric*). Once this surface has been constructed, one can immediately find the relative increase in length e of any vector. For this purpose it is sufficient to draw, parallel to the vector, from the origin the semi-axis OH to its intersection H with the surface; in order that such a point of intersection will exist (i.e., that it is real) it is necessary to choose the sign of c^2 on the

right-hand side in a definite manner. The relative change in length of the considered vector will be

$$e = \pm \frac{}{|OH|} \tag{13.5}$$

All the above is quite analogous to what has been said in § 6 with regard to the determination of the normal component of stress N and therefore it is not necessary to repeat those details here.

If the coordinate axes are chosen in such a way that they coincide with the principal axes of the surface (13.4), its equation takes the form

$$e_1\xi^2 + e_2\eta^2 + e_3\zeta^2 = \pm c^2, \tag{13.4'}$$

where e_1, e_2, e_3 denote the values of e_{xx}, e_{yy}, e_{zz} for the new system; naturally the components e_{yz}, e_{zx}, e_{xy} will be zero in that system. Consequently the new system of axes has the property that the angles between the axes after deformation remain right angles. This means, as there are always three such mutually perpendicular axes, that the angles between them remain unchanged by deformation. Those three axes are called *principal axes of strain.* The quantities e_1, e_2, e_3 are referred to as *principal strains.*

In the general case there exists only one such set of three axes. But if the surface (13.4) is a surface of revolution (i.e., when two of the quantities e_i are equal), there will be an infinity of such sets.

If one chooses the principal axes of strain as coordinate axes, the formulae (12.13), expressing pure deformation, take the form

$$\delta\xi = e_1\xi, \quad \delta\eta = e_2\eta, \quad \delta\zeta = e_3\zeta.$$

Consequently every *pure* deformation may be represented as the result of three simple extensions in three mutually perpendicular directions which are the directions of the principal axes of strain.

Finally note that the principal strains e_1, e_2, e_3 are the roots of a cubic equation in e (cf. § 7)

$$\begin{vmatrix} e_{xx} - e & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} - e & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} - e \end{vmatrix} = -e^3 + \theta e^2 + be + c = 0, \tag{13.6}$$

where, in particular,

$$\theta = e_{xx} + e_{yy} + e_{zz}. \tag{13.7}$$

Since the coefficients of (13.6) must be invariant (cf. § 7), it is clear

that θ must be so. Obviously θ represents the sum of the roots of (13.6), i.e.,

$$\theta = e_{xx} + e_{yy} + e_{zz} = e_1 + e_2 + e_3. \tag{13.8}$$

The quantity θ has a very simple geometrical meaning. In fact, consider a right parallelepiped, constructed on segments OA, OB and OC of the principal axes and having the volume

$$V = l_1 l_2 l_3,$$

where

$$l_1 = OA, \quad l_2 = OB, \quad l_3 = OC.$$

After deformation the considered parallelepiped will still be a right parallelepiped with sides

$$l_1(1 + e_1), \quad l_2(1 + e_2), \quad l_3(1 + e_3),$$

and its volume will be

$$V' = l_1 l_2 l_3 (1 + e_1)(1 + e_2)(1 + e_3) = V(1 + e_1 + e_2 + e_3),$$

neglecting higher order terms. Consequently

$$\frac{V' - V}{V} = e_1 + e_2 + e_3. \tag{13.9}$$

This formula shows that θ is the *relative expansion* of the volume V or the *cubical dilatation.*

§ 14. General deformation. Consider now the most general deformation of a continuous body. Let the point M, having initially the coordinates x, y, z, move as a consequence of deformation to the position

$$M^*(x^*, y^*, z^*).$$

Write

$$x^* = x + u, \quad y^* = y + v, \quad z^* = z + w; \tag{14.1}$$

u, v, w are the components of the vector $\overrightarrow{MM^*}$ which expresses the displacement of the point M as the result of deformation. This vector will be called displacement vector or simply *displacement,* and u, v, w *displacement components.* Since different points of a body, generally speaking, will be displaced in a different manner, u, v, w will be functions of the coordinates x, y, z of the original position of the point under consideration

$$u = u(x, y, z), \quad v = v(x, y, z), \quad w = w(x, y, z). \tag{14.2}$$

(Sometimes the displacement may also be a function of the time; in that case one considers the state of deformation at some definite instant of time.)

In the sequel, unless stated otherwise, it will be assumed that the functions u, v, w are not only single-valued and continuous, but also that they have continuous derivatives up to and including the third order.

Select at some point $M(x, y, z)$ of the body an infinitely small neighbouring volume and investigate its changes as a result of deformation. For this it is sufficient to study the variations of infinitesimal vectors having (before deformation) the point M as their starting point. Let

$$\overrightarrow{MN} = \vec{P} = (\xi, \eta, \zeta)$$

be such a vector. After deformation M will have moved to M^*, and N to N^*, so that the vector $\vec{P}$ becomes the vector $\vec{P^*} = \overrightarrow{M^*N^*}$. Calculate the vectorial increment $\overrightarrow{\delta P}$ of the vector $\vec{P^*}$, i.e., $\overrightarrow{\delta P} = \vec{P^*} - \vec{P}$. The coordinates of M^* are

$$x + u(x, y, z), \quad y + v(x, y, z), \quad z + w(x, y, z),$$

while those of N^*, having before deformation the coordinates

$$x + \xi, \quad y + \eta, \quad z + \zeta,$$

will be

$$x + \xi + u(x + \xi, y + \eta, z + \zeta), \quad y + \eta + v(x + \xi, y + \eta, z + \zeta),$$
$$z + \zeta + w(x + \xi, y + \eta, z + \zeta).$$

Therefore the components of the vector $\vec{P^*}$ will be

$$\xi + u(x + \xi, y + \eta, z + \zeta) - u(x, y, z),$$
$$\eta + v(x + \xi, y + \eta, z + \zeta) - v(x, y, z),$$
$$\zeta + w(x + \xi, y + \eta, z + \zeta) - w(x, y, z).$$

Finally, the components $\delta\xi$, $\delta\eta$, $\delta\zeta$ of the vector $\overrightarrow{\delta P}$ will be

$$u(x + \xi, y + \eta, z + \zeta) - u(x, y, z), v(x + \xi, y + \eta, z + \zeta) - v(x, y, z),$$
$$w(x + \xi, y + \eta, z + \zeta) - w(x, y, z).$$

But by Taylor's Theorem

$$u(x + \xi, y + \eta, z + \zeta) - u(x, y, z) = \frac{\partial u}{\partial x}\xi + \frac{\partial u}{\partial y}\eta + \frac{\partial u}{\partial z}\zeta + \varepsilon,$$

where ε is an infinitely small term of higher order than ξ, η, ζ. Neglecting ε and proceeding analogously in the case of the other components, one finds

$$\begin{aligned}\delta\xi &= \frac{\partial u}{\partial x}\xi + \frac{\partial u}{\partial y}\eta + \frac{\partial u}{\partial z}\zeta,\\ \delta\eta &= \frac{\partial v}{\partial x}\xi + \frac{\partial v}{\partial y}\eta + \frac{\partial v}{\partial z}\zeta,\\ \delta\zeta &= \frac{\partial w}{\partial x}\xi + \frac{\partial w}{\partial y}\eta + \frac{\partial w}{\partial z}\zeta;\end{aligned} \tag{14.3}$$

in these formulae the values of $\frac{\partial u}{\partial x}$ etc. refer to the point $(x, y\ z)$ and do not depend on ξ, η, ζ. These formulae show that, apart from higher order terms involving the linear dimensions of the considered body element, the change of this element may be expressed by means of an affine transformation with the coefficients $a_{11} = \frac{\partial u}{\partial x}$, $a_{12} = \frac{\partial u}{\partial y}$ etc.

Hitherto no limiting assumptions have been introduced with regard to the order of smallness of the displacement components u, v, w. It will now be assumed (*and this condition will always apply*) that the components of displacement u, v, w and also their derivatives with respect to x, y, z are infinitely small quantities the squares and products of which may be neglected in comparison with these quantities. Then (14.3) will be an infinitesimal transformation and everything said in the preceding sections will apply.

It was seen that *pure deformation* of the element under consideration was expressed by the formulae (§ 12)

$$\begin{aligned}\delta\xi &= e_{xx}\xi + e_{xy}\eta + e_{xz}\zeta,\\ \delta\eta &= e_{yx}\xi + e_{yy}\eta + e_{yz}\zeta,\\ \delta\zeta &= e_{zx}\xi + e_{zy}\eta + e_{zz}\zeta,\end{aligned} \tag{14.4}$$

where $e_{xx}, \ldots, e_{xy}$ are the strain components, determined by the formulae

$$\begin{gathered}\frac{\partial u}{\partial x},\quad e_{yy} = \frac{\partial v}{\partial y},\quad e_{zz} = \frac{\partial w}{\partial z}\\ = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right),\quad e_{zx} = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),\quad e_{xy} = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).\end{gathered} \tag{14.5}$$

Generally speaking, the pure deformation should still be combined with the rigid displacement of the considered element with the infinitesimal components of rotation

$$p = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad q = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad r = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{14.6}$$

and the translatory displacement which is equal to the displacement of the point $M(x, y, z)$, i.e., its components will be the values of u, v, w at $M(x, y, z)$.

The essential difference between the present deformation and the homogeneous deformation of § 10 arises from the fact that here the components of strain $e_{xx}, \ldots$ etc. depend on the location of the considered body element, i.e., on the coordinates x, y, z. In particular, the directions of the principal axes of strain will now change from point to point. Similarly, of course, the components of rotation will depend on x, y, z.

Finally, it will be noted that the quantity

$$\theta = e_{xx} + e_{yy} + e_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{14.7}$$

is invariant with regard to transformation of orthogonal coordinates and represents the cubical dilatation. But since one is now dealing with non-homogeneous deformation, it is of course clear that one can only talk of the dilatation of a volume element in the neighbourhood of a given point.

Most of the properties of deformation, studied above, were first deduced by Cauchy in his memoir of 1822 (cf. § 3).

§ 15. Determination of displacements from components of strain. Saint-Venant's conditions of compatibility. In § 14 formulae have been deduced by which the components of deformation can be calculated from the displacement components, given as functions of x, y, z. Now the *inverse problem* will be considered: *to determine the components of displacement u, v, w, if the strain components $e_{xx}, \ldots, e_{xy}$ are given as functions of x, y, z.* Before solving this problem, several preliminary remarks will be made which will make it possible to predict some properties of the solution.

The values of the strain components have been seen to determine the change in shape of an infinitesimal element of the body near a given point. Thus the strain components as functions of x, y, z determine

the change in shape of every infinitesimal element of the body. As a result it is obvious that the deformation of the body as a whole is effectively determined, i.e., the values of the displacements u, v, w as functions of x, y, z. It is likewise clear that u, v, w may not be determined uniquely. In fact, if displacements, corresponding to given strain components, have been found, then, by adding an arbitrary (infinitesimal) displacement of the body as a rigid unit, one will obtain different values for the displacements which will still correspond to the same components of strain, because the rigid body motion has no effect on the deformation. In order to make the problem unique, one may, for example, assume in addition that the displacement of any arbitrary point M_0 of the body and also the components of rotation at this point are given.

The following may also be noted. By an earlier assumption, the components u, v, w are single-valued and have continuous derivatives up to and including the third order. Hence the given components of strain $e_{xx}, \ldots, e_{xy}$ must likewise be single-valued and have continuous derivatives of the second order; this condition will be assumed to be satisfied. However, it is easily seen beforehand that the quantities $e_{xx}, \ldots, e_{xy}$ must still satisfy definite relations, in order that the problem will have a solution. This follows already from the following rough considerations. Let an infinitesimal element, e.g. a cube (which is not adjacent to the boundary), be separated from the body. If one subjects every such cube to a deformation with given components and then tries again to join all the infinitesimal parallelepipeds obtained in this way so that their boundaries, which were adjoining before deformation, again touch, then, generally speaking, this will turn out to be impossible; in the attempt of joining the separate elements there may either appear gaps between several of them, or boundaries of elements which should match may be found to differ from each other in size, or finally some elements may be too large for the space available. All this shows that the components of strain must satisfy certain relations, in order to allow deformation without discontinuities. This will now be proved strictly by actually solving the original problem.

Thus let it be required to find functions u, v, w satisfying the conditions

$$\begin{gathered}\frac{\partial u}{\partial x} = e_{xx}, \quad \frac{\partial v}{\partial y} = e_{yy}, \quad \frac{\partial w}{\partial z} = e_{zz}, \\ \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 2e_{yz}, \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 2e_{zx}, \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad 2e_{x}\end{gathered} \tag{15.1}$$

where $e_{xx}, \ldots, e_{xy}$ are given single-valued functions of x, y, z having continuous second order derivatives.

One has *six* equations for the determination of *three* unknown functions. This again shows that the problem may not have a solution, if the given functions $e_{xx}, \ldots, e_{xy}$ are not subject to certain additional conditions; these conditions will be found while solving the above problem.

Let V be the region originally occupied by the body; this is the domain of values of x, y, z for which the functions $e_{xx}, \ldots, e_{xy}$ are given and for which the functions u, v, w must be found. For the present, V will be assumed to be *simply connected*. It will be remembered that a region is called simply connected, if it has the following property: every closed contour, lying inside the region, may be shrunk into one point by means of continuous changes which do not take the contour outside the region. Such regions are, for example, represented by a sphere, a cube etc. (for more details see Appendix 2.)

Let $M_0(x_0, y_0, z_0)$ be any point of V, u_0, v_0, w_0 the values of the components of displacement there and p_0, q_0, r_0 the corresponding values of the components of rotation. Let $M_1(x_1, y_1, z_1)$ be any other point of V. Consider the problem of determining the components of displacement at the latter point.

Let M_0M_1 denote any curve which joins M_0 and M_1 and lies in V. If the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ were known throughout V, one could find the value u_1 of the function u at the point M_1 from the formula

$$u_1 = u_0 + \int\limits_{M_0M_1} \left(\frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy + \frac{\partial u}{\partial z}\, dz\right), \tag{15.2}$$

where the integral must be taken over the curve M_0M_1. But

$$\frac{\partial u}{\partial x} = e_{xx}, \quad \frac{\partial u}{\partial y} = e_{xy} - r, \quad \frac{\partial u}{\partial z} = e_{zx} + q, \tag{15.3}$$

where p, q, r are determined by the formulae (14.6).
Hence

$$u_1 = u_0 + \int\limits_{M_0M_1} (e_{xx}dx + e_{xy}dy + e_{xz}dz) + \int\limits_{M_0M_1} (q\, dz - r\, dy). \tag{a}$$

The first integrand involves only given functions. Consider now the

second integral. One has

$$\int_{M_0M_1}(q\,dz - r\,dy) = \int_{M_0M_1}\{r\,d(y_1 - y) - q\,d(z_1 - z)\},$$

whence, integrating by parts,

$$\int_{M_0M_1}(q\,dz - r\,dy) = q_0(z_1 - z_0) - r_0(y_1 - y_0) - \int_{M_0M_1}\{(y_1 - y)dr - (z_1 - z)dq\}. \quad (b)$$

In order to evaluate the last integral, one requires the values of dr, dq or, what is the same thing, the values of the first order partial derivatives of the functions r and q. But it may be verified directly that

$$\frac{\partial r}{\partial x} = \frac{\partial e_{xy}}{\partial x} - \frac{\partial e_{xx}}{\partial y}, \quad \frac{\partial r}{\partial y} = \frac{\partial e_{yy}}{\partial x} - \frac{\partial e_{xy}}{\partial y}, \quad \frac{\partial r}{\partial z} = \frac{\partial e_{yz}}{\partial x} - \frac{\partial e_{zx}}{\partial y},$$

$$\frac{\partial q}{\partial x} = \frac{\partial e_{xx}}{\partial z} - \frac{\partial e_{zx}}{\partial x}, \quad \frac{\partial q}{\partial y} = \frac{\partial e_{yx}}{\partial z} - \frac{\partial e_{yz}}{\partial x}, \quad \frac{\partial q}{\partial z} = \frac{\partial e_{zx}}{\partial z} - \frac{\partial e_{zz}}{\partial x}.$$

Substituting these expressions in

$$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy + \frac{\partial r}{\partial z}dz,$$

$$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy + \frac{\partial q}{\partial z}dz,$$

one obtains, using (a) and (b), the first of the three formulae below (the other two may be obtained from the first by cyclic interchange of symbols)

$$\begin{aligned} u(x_1, y_1, z_1) &= u_0 + q_0(z_1 - z_0) - r_0(y_1 - y_0) + \\ &\qquad + \int_{M_0M_1}(U_x\,dx + U_y\,dy + U_z\,dz), \\ v(x_1, y_1, z_1) &= v_0 + r_0(x_1 - x_0) - p_0(z_1 - z_0) + \\ &\qquad + \int_{M_0M_1}(V_x\,dx + V_y\,dy + V_z\,dz), \\ w(x_1, y_1, z_1) &= w_0 + p_0(y_1 - y_0) - q_0(x_1 - x_0) + \\ &\qquad + \int_{M_0M_1}(W_x\,dx + W_y\,dy + W_z\,dz), \end{aligned} \quad (15.4)$$

where, for convenience,

$$
\begin{aligned}
U_x &= e_{xx} + (y_1 - y)\left(\frac{\partial e_{xx}}{\partial y} - \frac{\partial e_{yx}}{\partial x}\right) + (z_1 - z)\left(\frac{\partial e_{xx}}{\partial z} - \frac{\partial e_{zx}}{\partial x}\right),\\
U_y &= e_{xy} + (y_1 - y)\left(\frac{\partial e_{xy}}{\partial y} - \frac{\partial e_{yy}}{\partial x}\right) + (z_1 - z)\left(\frac{\partial e_{xy}}{\partial z} - \frac{\partial e_{yz}}{\partial x}\right),\\
U_z &= e_{xz} + (y_1 - y)\left(\frac{\partial e_{xz}}{\partial y} - \frac{\partial e_{yz}}{\partial x}\right) + (z_1 - z)\left(\frac{\partial e_{xz}}{\partial z} - \frac{\partial e_{zz}}{\partial x}\right).
\end{aligned}
\tag{15.5}
$$

The formulae for V_x, V_y, V_z and W_x, W_y, W_z are obtained from the above by cyclic interchange of symbols (by simultaneously transposing the symbols U, V, W and x, y, z).

The formulae (15.4) essentially agree with those found by V. Volterra [1], p. 406, using transformation formulae given by G. Kirchhoff [1], Vorles. XXVII, § 4. The deduction presented here is due to E. Cesaro (Rendiconti d. R., Academia di Napoli, 1906; it is also quoted in V. Volterra [1], where it is reproduced on pp. 416—417, as due to Cesaro) who gave Volterra's formulae a more symmetrical form.

The formulae (15.4) determine the displacement components u_1, v_1, w_1 at any point $M_1(x_1, y_1, z_1)$ of the body, if the displacement (u_0, v_0, w_0) and the rotation (p_0, q_0, r_0) are given at some other point $M_0(x_0, y_0, z_0)$ which has been chosen once for all. The formulae for the displacements contain integrals taken over some curve connecting the points M_0 and M_1. But u, v, w must be functions of x_1, y_1, z_1 and should not depend on the path of integration M_0M_1. This means, in order that the problem may have a solution, it is necessary that the integrals in (15.4) are independent of the path of integration.

It is easily seen that the necessary and sufficient conditions for the integral

$$\int_{M_0M_1} (U_x dx + U_y dy + U_z dz)$$

to be independent of the path M_0M_1 are (cf. Appendix .2.)

$$\frac{\partial U_z}{\partial y} = \frac{\partial U_y}{\partial z}, \quad \frac{\partial U_x}{\partial z} = \frac{\partial U_z}{\partial x}, \quad \frac{\partial U_y}{\partial x} = \frac{\partial U_x}{\partial y}.$$

For the two other integrals one obtains analogous conditions by cyclic rotation of symbols. These conditions must be satisfied at all points (x, y, z) of V and for all values (x_1, y_1, z_1) in that region.

Performing the differentiations, it will be seen that these conditions may be reduced to the following six:

$$
\begin{aligned}
&\frac{\partial^2 e_{yy}}{\partial z^2}+\frac{\partial^2 e_{zz}}{\partial y^2}=2\frac{\partial^2 e_{yz}}{\partial y\partial z}, \quad \frac{\partial^2 e_{xx}}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}\right),\\
&\frac{\partial^2 e_{zz}}{\partial x^2}+\frac{\partial^2 e_{xx}}{\partial z^2}=2\frac{\partial^2 e_{zx}}{\partial z\partial x}, \quad \frac{\partial^2 e_{yy}}{\partial z\partial x}=\frac{\partial}{\partial y}\left(-\frac{\partial e_{zx}}{\partial y}+\frac{\partial e_{xy}}{\partial z}+\frac{\partial e_{yz}}{\partial x}\right), \quad (15.6)\\
&\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}=2\frac{\partial^2 e_{xy}}{\partial x\partial y}, \quad \frac{\partial^2 e_{zz}}{\partial z\partial y}=\frac{\partial}{\partial z}\left(-\frac{\partial e_{xy}}{\partial z}+\frac{\partial e_{yz}}{\partial x}+\frac{\partial e_{zx}}{\partial y}\right).
\end{aligned}
$$

For example, the condition

$$\frac{\partial U_y}{\partial z}=\frac{\partial U_z}{\partial y}$$

gives, by (15.5),

$$
\begin{aligned}
(y_1-y)\left(\frac{\partial^2 e_{xy}}{\partial y\partial z}-\frac{\partial^2 e_{yy}}{\partial x\partial z}\right)&+(z_1-z)\left(\frac{\partial^2 e_{xy}}{\partial z^2}-\frac{\partial^2 e_{yz}}{\partial x\partial z}\right)=\\
&=(y_1-y)\left(\frac{\partial^2 e_{zx}}{\partial y^2}-\frac{\partial^2 e_{yz}}{\partial x\partial y}\right)+(z_1-z)\left(\frac{\partial^2 e_{zx}}{\partial z\partial y}-\frac{\partial^2 e_{zz}}{\partial x\partial y}\right).
\end{aligned}
$$

Since these relations must hold true for all y_1, z_1 in a given region, one must have

$$\frac{\partial^2 e_{xy}}{\partial y\partial z}-\frac{\partial^2 e_{yy}}{\partial x\partial z}=\frac{\partial^2 e_{zx}}{\partial y^2}-\frac{\partial^2 e_{yz}}{\partial x\partial y}, \quad \frac{\partial^2 e_{xy}}{\partial z^2}-\frac{\partial^2 e_{yz}}{\partial x\partial z}=\frac{\partial^2 e_{zx}}{\partial z\partial y}-\frac{\partial^2 e_{zz}}{\partial x\partial y}.$$

These relations agree with the last two of the right-hand column of (15.6).* The others may be obtained by the same procedure. It should be noted that the formulae in the second and third row of (15.6) may be deduced from those in the first row by cyclic interchange of symbols.

The equations (15.6) are called *conditions of compatibility* of Barré de Saint-Venant (1797—1886), since they were first discovered by him (in fact, he lectured about them to the Société Philomathique in 1860 and published the relations in 1861).

These conditions are the mathematical form of those relations which must be satisfied by the components of strain in order that deformation may take place without discontinuities (cf. the earlier part of this section), and for this reason they are also sometimes called *conditions of continuity*.

Provided these conditions are fulfilled, the formulae (15.4) give completely defined expressions for u, v, w which do not depend on the choice of the path of integration, and it is easily verified directly that displacements found in this way actually satisfy the equations (15.1). Further, the constants

$$u_0, v_0, w_0, p_0, q_0, r_0$$

remain quite arbitrary, as had been anticipated previously. As can be seen from (12.8), variations in these constants will only cause rigid displacement of the body as a whole. In particular, if

$$e_{xx} = e_{yy} = \dots = e_{xy} = 0$$

throughout a region, one obtains, putting for simplicity $x_0 = y_0 = z_0 = 0$ and omitting the subscripts of x_1, y_1, z_1,

$$u = u_0 + q_0 z - r_0 y, \quad v = v_0 + r_0 x - p_0 z, \quad w = w_0 + p_0 y - q_0 x,$$

i.e., only rigid body displacement.

Hitherto it had been assumed that the region V was simply-connected. Consider now cases of multiply-connected regions, i.e., of regions inside which there exist closed contours which cannot be shrunk into one point without cutting them apart or taking them outside V. As an example for a multiply-connected region one may consider a torus, i.e., a body obtained by revolving a circle about an axis lying in its plane but not intersecting it (Fig. 9).

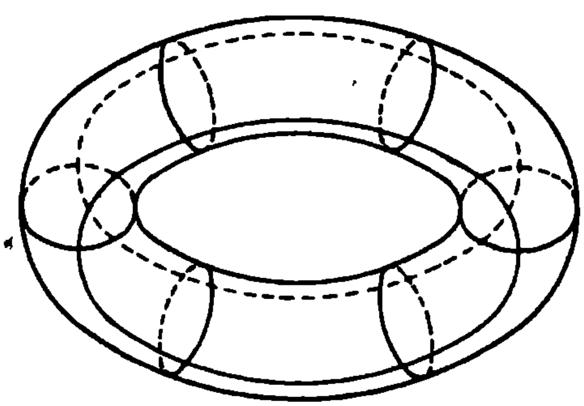


Fig. 9.

A multiply-connected body becomes a simply-connected one, if one introduces suitable cuts (for more detail cf. Appendix 2.); for example, in the case of the tore it is sufficient to cut it at one of its meridional circles, shown in Fig. 9. Everything said above will apply to the region cut in this manner. In fact, provided the compatibility conditions are satisfied, the components u, v, w, determined by (15.4), will be single-valued functions of the coordinates of the point $M_1(x_1, y_1, z_1)$; in addition, it must, of course, be assumed that the path of integration M_0M_1 does not leave the *cut* region, i.e., that it does not intersect the cut. Further, when connecting the point M_1 to any point of the cut, the quantities u, v, w will,

generally speaking, have different values depending on the side from which the point on the cut is approached.

Let u^+, v^+, w^+ and u^-, v^-, w^- be the values of u, v, w, when a point on the cut is reached from one or the other side respectively. The condition of compatibility of deformation *for the body as a whole* will only be satisfied, if in addition to (15.6) the following conditions are satisfied on all cuts, introduced in the body to make it simply-connected:

$$u^+ = u^-, \quad v^+ = v^-, \quad w^+ = w^-. \tag{15.7}$$

When (15.7) is not satisfied, discontinuities will occur at the above-mentioned cuts and even parts of the body may penetrate each other in these places.

It is clear from what has been said that, if (15.7) is not satisfied and if the functions u, v, w are still determined by use of (15.4) in uncut regions, i.e., if one admits intersection of the cuts by the path of integration, then u, v, w will be multi-valued functions of x_1, y_1, z_1, i.e., after travelling once around certain closed contours the functions u, v, w will not revert to their original values; it is easily seen that this may only happen in the case of contours which cannot be shrunk to a point by a continuous process (cf. Appendix 2).

The first to comment on the above was J. H. Michell [1]. A. Timpe [1] indicated for the case of the plane problem of the theory of elasticity the possibility of a physical interpretation of multi-valued displacements. For the general case of three dimensional problems the question of the meaning of multi-valued displacements was studied in detail by V. Volterra in a number of publications; a summary of this work has been given by him in his paper [1], and a short study of Volterra's results is also contained in A. E. H. Love [1] (appendices to chapters VIII and IX) and in P. Burgatti [1]. For the case of plane elasticity this problem will be studied in detail in Chapter 6.

CHAPTER 3

THE FUNDAMENTAL LAW OF THE THEORY OF ELASTICITY. THE BASIC EQUATIONS.

Everything said in the previous chapters may be applied to any continuous body. In order to obtain equations characterizing a body which will be called *elastic* (or more correctly *ideally elastic*), it is still necessary to have a law expressing the connection between the stressed state of the body and the corresponding deformation.

§ 16. The fundamental law of the theory of elasticity (Generalized Hooke's Law). The first, very incomplete formulation of the law relating stresses to strains was due to Robert Hooke (1635—1702). In 1660 Hooke discovered this law which has been named after him; he published it in the form of an anagram in 1676 and gave the solution of the latter in 1678. Expressing the essentials, which Hooke stated in his law, in contemporary language one may say: "The deformation of an elastic body is proportional to the forces acting on it". This formulation may only be given a definite interpretation in the case when the "force" acting on the body and the deformation connected with it can be characterized by one quantity each.

For example, if one has a long thin cylindrical rod, stretched by longitudinal forces applied to its ends, one may assume that the force acting on the body is characterized by the given value F of the applied traction and the deformation by the extension Δl of the rod. In this case Hooke's Law gives $\Delta l = C.F$, where C is a constant depending only on the original length l, the form of the cross-section and the material of the rod. Actually, it will be shown later that $C = l/ES$, where S is the area of the cross-section and E is a constant depending only on the material of the rod. Many similar examples could be quoted here.

Experiments have verified that Hooke's Law agrees well with the behaviour of many solid bodies, provided the deformations are sufficiently small. For finite deformations the law of proportionality fails to give even approximately correct results.

However, also in the case of small deformations, when the law of proportionality may be assumed to be valid, Hooke's Law as introduced above may not give the complete picture of what actually takes place in the deformed body. Indeed, it has been seen that the state of stress and strain is characterized by *six* quantities each, and that these quantities change from one point of the body to another, so that in actual fact one is dealing with an infinite number of quantities characterizing the state of the body as a whole.

For example, in the case quoted above "only" the tensile forces F acting on the ends of the cylindrical rod have been considered. In actual fact, the "force" F expresses only the resultant effect of the external stresses applied near the ends of the rod. These stresses may be distributed in any manner whatsoever, for example they may be spread over the end-sections or over parts of the side surface in the neighbourhood of the ends; the distribution may be uniform or non-uniform, etc.

It is clear that the distribution of stresses and strains inside the rod depends largely on the distribution of those external stresses. It is only in the case, when the dimensions of the cross-section of the rod are small compared with its length, that the manner in which the external forces are distributed near the ends has no great effect on the state of the rod (and then only in parts away from the ends). Under these circumstances consideration may be limited to the resultant "force" F (cf. also § 23).

Thus it is obvious that, if one does not want to limit oneself to a crude and superficial investigation, one has to generalize Hooke's Law. The most natural generalization of a law of simple proportionality of two quantities will be a law of *linear dependence* between several quantities. Hence consider as the generalization of the original law the following *fundamental law of the theory of elasticity* or *generalized Hooke's Law*:

The components of stress at a given point of a body are linear and homogeneous functions of the components of strain at the same point (and vice versa).

Of course the above statement refers to small deformations. (As regards the limits of applicability of Hooke's Law, cf., for example, R. Grammel [1]). The generalized Hooke's Law in this form was first stated by A. L. Cauchy in his memoir of 1822. In subsequent work, published in 1828, Cauchy deduced this law, basing it on molecular theory, under a simple supposition referring to the interaction of forces between molecules considered as material points. The same result was obtained by S. D. Pois-

son (1781—1840) by an analogous method in a memoir delivered to the Paris Academy in 1828 and published in 1829.

It is not proposed to present here the deduction due to Cauchy and Poisson, the more so because it has been found to be insufficient (cf. below), but the generalized Hooke's Law will be accepted as the foundation of the present theory, based on the fact that for small deformations this law agrees sufficiently well with reality for very many materials.

Before going further the following remark should be made. Since generally stresses and strains are different in different parts of a body, it is only possible to discuss their components at a given *point.* However, the expression "at a given point" will be interpreted in a different manner according to whether it is applied to components of strain or stress. For example, when stating that e_{xx} is a function of the coordinates x, y, z, this will always refer to the position (x, y, z) of the point *before deformation.* The same will be true with regard to the components of displacement u, v, w. On the other hand, when it is said that X_x is a function of x, y, z, this will refer to the position (x, y, z) of the point in the final (i.e., stressed and hence deformed) state of the body.

However, for the small deformations considered here this distinction is not essential, since, for example, the values of X_x at (x_1, y_1, z_1) and (x, y, z), where (x, y, z) is the position of the point (x_1, y_1, z_1) before deformation, differ by an amount which is small compared with X_x. Thus the value of X_x at a given point (x_1, y_1, z_1) of the deformed body may be replaced by its value at (x, y, z). In the sequel the values of all functions considered will be taken at (geometric) points representing the original positions of the points of the deformed body.

Now consider the generalization of Hooke's Law. It may be written in the following manner. If $X_x, Y_y, Z_z, Y_z, Z_x, X_y$ are the components of stress at a given point of the body and $e_{xx}, e_{yy}, e_{zz}, e_{yz}, e_{zx}, e_{xy}$ the components of strain, then

$$\begin{aligned}
X_x &= c_{11}e_{xx} + c_{12}e_{yy} + c_{13}e_{zz} + c_{14}e_{yz} + c_{15}e_{zx} + c_{16}e_{xy}, \\
Y_y &= c_{21}e_{xx} + \dots \qquad\qquad\qquad\qquad\qquad\qquad + c_{26}e_{xy}, \\
Z_z &= c_{31}e_{xx} + \dots \qquad\qquad\qquad\qquad\qquad\qquad + c_{36}e_{xy}, \\
Y_z &= c_{41}e_{xx} + \dots \qquad\qquad\qquad\qquad\qquad\qquad + c_{46}e_{xy}, \\
Z_x &= c_{51}e_{xx} + \dots \qquad\qquad\qquad\qquad\qquad\qquad + c_{56}e_{xy}, \\
X_y &= c_{61}e_{xx} + \dots \qquad\qquad\qquad\qquad\qquad\qquad + c_{66}e_{xy}.
\end{aligned} \tag{16.1}$$

Since on the basis of the adopted fundamental law the components of strain must likewise be definite linear functions of the components of

stress, the preceding equations must be soluble with respect to $e_{xx}, \ldots, e_{xy}$, i.e., the determinant of the coefficients c_{ij} must be different from zero.

The quantities c_{ij} are constants characterizing the elastic properties of the body at a given point. They are called *elastic constants*. The term "constant" must be understood in the sense that these quantities do not depend on the values of the components of strain and the corresponding stresses at a given point. However, they may vary from point to point of the body. If that is so, the body will be said to be *non-homogeneous* (as regards its elastic properties). On the other hand, if the elastic constants are the same for all points of the body, it will be called *homogeneous*.

The formulae (16.1) are seen to contain 36 elastic constants. But by considerations based on the law of conservation of energy and on a study of the potential energy of deformation, it may be shown that the following relations must hold between these constants:

$$c_{ij} = c_{ji} \quad (i, j = 1, 2, \ldots, 6),$$

i.e., that the array of coefficients in (16.1) is symmetric. Thus in the most general case the number of elastic constants may be reduced to 21. Application of these considerations and deduction of the stated result was first given by G. Green in 1837 whose paper on the subject was published in 1839. A more complete foundation for this result, based upon the first and second law of thermodynamics, was presented by Lord Kelvin (W. Thomson) in 1855. (For more detail see A. E. H. Love [1]).

It will be seen in the next section that in the case of the isotropic body the number of elastic constants may be reduced to two.

By the old theory of Cauchy, based on the consideration of molecular forces, the number of elastic constants in the most general case is equal to 15, and not 21; in the case of the isotropic body one has by this theory only one elastic constant (in the first of his memoirs, where Cauchy did not rely on molecular theory, he obtained two constants for the isotropic body). Poisson arrived at the same results. However, this was not confirmed by experiments. But it should not be thought that the molecular theory led to the wrong results and that it is impossible to obtain from it the correct number of constants. The point is only that Cauchy and Poisson applied molecular theory in an oversimplified form. Using modern concepts of the structure of materials one can obtain the correct result, i.e., 21 constants. This has been done recently by M. Born [1] (cf. also A. E. H. Love [1], Note B at the end of his book).

It is not proposed to give here further details of these problems, since in the sequel only isotropic bodies will be considered. In that case definite formulae may be deduced by means of very simple considerations.

§ 17. Isotropic bodies. As mentioned earlier a body will be called isotropic, if its properties are the same in all directions. In other words, if one cuts a volume element of a definite shape (say a cube) from an isotropic body, this element will not differ from any other element of the same form (cut from the same part of the body) but orientated differently from the first. For example, wood is not isotropic, since a beam cut in the longitudinal direction (along the fibres) differs very much from a beam cut across the grain. Likewise all crystalline bodies are anisotropic. In nature there are no ideally isotropic bodies, but many materials, important in industry, may with sufficient approximation be assumed to be isotropic. Many such materials (e.g. metals) consist of small anisotropic parts (crystals) arbitrarily placed with respect to each other. It is this random distribution which is the reason that bodies of not too small dimensions made from these materials may be considered to be isotropic.

A body will not only be called isotropic, but also homogeneous, if the properties of volume elements cut from different parts of it are the same. It should still be noted that a body which is isotropic and homogeneous with regard to one property may be anisotropic or non-homogeneous with regard to others.

In the following only isotropic and homogeneous bodies will be considered, where it must be understood that this *isotropy and homogeneity refers only to its elastic behaviour.*

In mathematical language this fact may obviously be expressed in the following manner: *the coefficients* $c_{11}, \ldots, c_{66}$ *in* (16.1) *do not depend on the orientation of the coordinate axes with respect to the body nor on the position of the point under consideration in the body.* Owing to this property the above-mentioned formulae take a very simple form, as will now be shown.

First of all it is easily proved that *at every point of an isotropic body the principal axes of strain and stress coincide.* In fact, let the principal axes of strain at a given point lie along the coordinate axes. Then

$$e_{yz} = e_{zx} = e_{xy} = 0.$$

By the generalized Hooke's Law one has, in particular,

$$Y_z = Ae_{xx} + Be_{yy} + Ce_{zz}, \tag{a}$$

where A, B, C are constants. Introduce now a new coordinate system $Ox'y'z'$, obtained from the old system by a simple rotation of 180° about the axis Oz. The axis Oz' of the new system will coincide with Oz, while Ox', Oy' will be in the opposite directions to Ox, Oy. Since the coefficients A, B, C are not to depend on the choice of axes, one will have in the new system

$$Y'_{z'} = Ae_{x'x'} + Be_{y'y'} + Ce_{z'z'}. \tag{b}$$

But obviously

$$e_{x'x'} = e_{xx},\ e_{y'y'} = e_{yy},\ e_{z'z'} = e_{zz},\ Y'_{z'} = -Y_z.$$

Comparing (a) with (b) one sees that one must have

$$Ae_{xx} + Be_{yy} + Ce_{zz} \qquad Ae_{x} \quad \cdot Be_{y} \quad \cdot Ce_{zz},$$

and hence

$$A = B = C = 0.$$

But this means that $Y_z = 0$. In the same way it can be proved that

$$Z_x = X_y = 0.$$

However, this shows that the coordinate axes are principal axes of stress and the above statement is proved. Thus, in future, it will be unnecessary to distinguish between principal axes of strain and stress; they will simply be called *principal* axes.

Let it still be assumed that the coordinate axes coincide with the principal axes. By the generalized Hooke's Law one may, in particular, write

$$X_x \quad ae_{xx} + be_{yy} + ce_{zz},$$

where a, b, c are constants. Let $Ox'y'z'$ be a new system of axes obtained from $Oxyz$ by a rotation of 90° about the axis Ox. In the new system one must again have

$$X'_{x'} = ae_{x'x'} + be_{y'y'} + ce_{z'z'}.$$

But obviously in the present case

$$X'_{x'} = X_x,\ e_{x'x'} = e_{xx},\ e_{y'y'} = e_{zz},\ e_{z'z'} = e_{yy},$$

and hence

$$X_x = ae_{xx} + be_{zz} + ce_{yy}.$$

Comparing this formula with the earlier one for X_x one sees that $b = c$. Thus

$$X_x = ae_{xx} + b(e_{yy} + e_{zz}) = b(e_{xx} + e_{yy} + e_{zz}) + (a - b)e_{xx}.$$

Finally introduce the notation

$$b = \lambda, \quad a - b = 2\mu,$$

so that the preceding formula becomes

$$X_x = \lambda(e_{xx} + e_{yy} + e_{zz}) + 2\mu e_{xx} = \lambda\theta + 2\mu e_{xx},$$

where

$$\theta = e_{xx} + e_{yy} + e_{zz}.$$

Because of isotropy one can obtain from the above formula for X_x those for Y_y, Z_z simply by replacing the letter x by y or by z. Consequently, one finally finds

$$N_1 = \lambda\theta + 2\mu e_1, \; N_2 = \lambda\theta + 2\mu e_2, \; N_3 = \lambda 0 + 2\mu e_3. \qquad (c)$$

In these formulae N_1, N_2, N_3 and e_1, e_2, e_3 denote the principal stresses and strains. The corresponding coordinate axes will now be denoted by Ox', Oy', Oz', where it should not be forgotten that they are principal axes.

In order to find now the formulae relating the stress components $X_x, \ldots, X_y$ to the strain components $e_{xx}, \ldots, e_{xy}$ in any coordinate system $Oxyz$, it is sufficient to express the quantities $X_x, \ldots, X_y$ by the known transformation formulae for the transition from one system of axes to another in terms of N_1, N_2, N_3. Using (c), this will give expressions for $X_x, \ldots, X_y$ in terms of e_1, e_2, e_3. Expressing, finally, e_1, e_2, e_3 in terms of $e_{xx}, \ldots, e_{xy}$ one finds the required formulae. Actual execution of this process leads to unwieldy calculations which may be avoided in the following way.

One can replace the set of formulae (c) by a single equation which is obtained by multiplying the equations (c) by $\xi'^2, \eta'^2, \zeta'^2$ respectively, where ξ', η', ζ' are the components of some arbitrary vector $\vec{P}$ in the system $Ox'y'z'$, and by adding them:

$$N_1\xi'^2 + N_2\eta'^2 + N_3\zeta'^2 =$$
$$= \lambda\theta(\xi'^2 + \eta'^2 + \zeta'^2) + 2\mu(e_1\xi'^2 + e_2\eta'^2 + e_3\zeta'^2). \qquad (d)$$

Now transform from the axes $Ox'y'z'$ to the axes $Oxyz$. It is known that the quadratic form

$$N_1\xi'^2 + N_2\eta'^2 + N_3\zeta'^2$$

will then become the quadratic form (cf. § 5)

$$X_x\xi^2 + \ldots + 2X_y\xi\eta,$$

and

$$e_1\xi'^2 + e_2\eta'^2 + e_3\zeta'^2$$

the form (cf. § 13)

$$e_{xx}\xi^2 + e_{yy}\eta^2 + e_{zz}\zeta^2 + 2e_{yz}\eta\zeta + 2e_{zx}\zeta\xi + 2e_{xy}\xi\eta.$$

Here ξ, η, ζ are the components of the vector $\vec{P}$ in the system $Oxyz$. But obviously

$$\xi'^2 + \eta'^2 + \zeta'^2 = \xi^2 + \eta^2 + \zeta^2.$$

As regards the quantity

$$\theta = e_1 + e_2 + e_3,$$

its value in terms of the components for the new axes will be

$$\theta = e_{xx} + e_{yy} + e_{zz}$$

(cf. end of § 14). Hence in the new coordinate system equation (d) becomes

$$X_x\xi^2 + Y_y\eta^2 + Z_z\zeta^2 + 2Y_z\eta\zeta + 2Z_x\zeta\xi + 2X_y\xi\eta =$$
$$= \lambda\theta(\xi^2 + \eta^2 + \zeta^2) + 2\mu(e_{xx}\xi^2 + e_{yy}\eta^2 + e_{zz}\zeta^2 + 2e_{yz}\eta\zeta + 2e_{zx}\zeta\xi + 2e_{xy}\xi\eta).$$

But since this equation will be true for the components of any vector $\vec{P}$, i.e., for all values of ξ, η, ζ, the coefficients of $\xi^2, \ldots, \xi\eta$ on either side of the equation must be equal, and hence

$$\begin{gathered} X_x = \lambda\theta + 2\mu e_{xx}, \quad Y_y = \lambda\theta + 2\mu e_{yy}, \quad Z_z = \lambda\theta + 2\mu e_{zz}, \\ Y_z = 2\mu e_{yz}, \quad Z_x = 2\mu e_{zx}, \quad X_y = 2\mu e_{xy}, \end{gathered} \tag{17.1}$$

where

$$\theta = e_{xx} + e_{yy} + e_{zz} \tag{17.2}$$

is the cubical dilatation.

Formulae (17.1) give the unknown relations between the components of stress and strain in an isotropic body. The quantities λ, μ are constants characterizing the elastic behaviour of a given body. This notation was introduced by G. Lamé [1] (1795—1870) and for this reason they are called the *constants of Lamé*. They have to be determined for every material by experiment, but in actual fact other quantities, in terms of which these constants are easily expressed, are more suitable for direct measurements, and that is the procedure normally adopted.

By a condition, stated during the formulation of the generalized Hooke's Law, the equations (17.1) must be soluble for $e_{xx}, \ldots, e_{xy}$. Consider what conditions must be satisfied by λ and μ, so that the above demand is satisfied. For this purpose (17.1) will now be solved for the components of strain. Adding the first three equations, one gets

$$X_x + Y_y + Z_z = (3\lambda + 2\mu)\theta = (3\lambda + 2\mu)\,(e_{xx} + e_{yy} + e_{zz}). \tag{17.3}$$

This equation can be solved for $e_{xx} + e_{yy} + e_{zz}$ only if $3\lambda + 2\mu \neq 0$. Further, solving the last three equations of (17.1) for e_{yz}, e_{zx}, e_{xy}, one finds that one must have $\mu \neq 0$. It will be seen in § 19 that for all actual bodies $\lambda > 0$, $\mu > 0$. Assume now that these conditions are satisfied. Substituting the value for θ, obtained from (17.3), in (17.1) one finds the formulae

$$\begin{aligned} e_{xx} &= \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} X_x - \frac{\lambda}{2\mu(3\lambda + 2\mu)} (Y_y + Z_z), \\ e_{yy} &= \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} Y_y \qquad \frac{}{2\mu(3\lambda + 2\mu)} (Z_z + X_x), \\ e_{zz} &= \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} Z_z - \frac{\lambda}{2\mu(3\lambda + 2\mu)} (X_x + Y_y), \\ e_{yz} &= \frac{1}{2\mu} Y_z, \quad e_{zx} = \frac{1}{2\mu} Z_x, \qquad \frac{1}{2\mu} X_y, \end{aligned} \tag{17.4}$$

expressing the components of strain in terms of the stress components.

§ 18. The basic equations for the statics of an elastic isotropic body. It is now possible to write down the complete system of equations for the statics of an elastic body. This system consists of the "equilibrium equations", relating the stress components (§ 4), and of the equations (17.1), relating stresses to strains. It will be shown in § 20 that the following equations constitute a complete system:

$$\begin{aligned} \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + X = 0, \\ \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + Y = 0, \\ \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + Z = 0, \end{aligned} \tag{18.1}$$

$$\begin{gathered} X_x = \lambda\theta + 2\mu e_{xx}, \quad Y_y = \lambda\theta + 2\mu e_{yy}, \quad Z_z = \lambda\theta + 2\mu e_{zz}, \\ Y_z = 2\mu e_{yz}, \quad Z_x = 2\mu e_{zx}, \quad X_y = 2\mu e_{xy}, \end{gathered} \tag{18.2}$$

$$\begin{gathered} e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{zz} = \frac{\partial w}{\partial z}, \\ e_{yz} = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \quad e_{zx} = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad e_{xy} = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \end{gathered} \tag{18.3}$$

where u, v, w are the components of displacement and

$$\theta = e_{xx} + e_{yy} + e_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \tag{18.4}$$

These equations must still be supplemented by the formulae giving the components of the stress vector acting on a plane with normal n (§ 3):

$$\begin{aligned} X_n &= X_x \cos(n, x) + X_y \cos(n, y) + X_z \cos(n, z), \\ Y_n &= Y_x \cos(n, x) + Y_y \cos(n, y) + Y_z \cos(n, z), \\ Z_n &= Z_x \cos(n, x) + Z_y \cos(n, y) + Z_z \cos(n, z). \end{aligned} \tag{18.5}$$

Next, a general remark will be made with reference to the sets of equations (18.1) and (18.2). These equations are linear and homogeneous in the displacement components u, v, w, the stress components $X_x, \ldots, X_y$ and the body forces X, Y, Z. Hence, if

$$u', v', w', X'_x, \ldots, X'_y \text{ and } u'', v'', w'', X''_x, \ldots, X''_y$$

are two solutions of (18.1) and (18.2) corresponding to body forces X', Y', Z' and X'', Y'', Z'' respectively, then

$$\begin{aligned} &u = u' + u'', \qquad v = v' + v'', \qquad w = w' + w'', \\ &X_x = X'_x + X''_x, \qquad \ldots, \qquad X_y = X'_y + X''_y \end{aligned} \tag{18.6}$$

is a solution of the same system of equations for the body forces

$$X = X' + X'', \quad Y = Y' + Y'', \quad Z = Z' + Z''. \tag{18.7}$$

It will be said that the solution (18.6) has been obtained by *superimposition* of the two given solutions. Formulae (18.5) show that the external stresses, applied to the surface of the body (for this purpose n refers to the outward normal) and corresponding to the last solution, are given by the sums of the surface tractions corresponding to the given solutions. In particular, if $u'', v'', w'', X''_x, \ldots, X''_y$ is any solution when there is no body force ($X'' = Y'' = Z'' = 0$), then (18.6) will satisfy the same equations with the same body forces as the solution u', v', w', $X'_x, \ldots, X'_y$.

§ 19. The simplest cases of elastic equilibrium. The basic elastic constants. Before going further, several very simple cases of elastic equilibrium will be considered for the purpose of studying the physical meaning of the constants characterizing the elastic properties of a body.

First it will be noted that in the absence of body forces, i.e., if

$$X \qquad Z = 0, \tag{19.1}$$

the static equations of the elastic body may be satisfied, in particular, by assuming the strain components $e_{xx}, \ldots, e_{xy}$ to be (arbitrary) constants, i.e., by assuming homogeneous deformation. In fact, by (18.2), the stress components will likewise be constants and hence the equations (18.1) will be identically satisfied (since by supposition $X = Y = Z = 0$). Further, the compatibility conditions of St. Venant (§ 15) will be satisfied, since one may always find displacements u, v, w corresponding to the given strain components. In this simple case the above may be proved directly by finding expressions for the displacements; namely, direct substitution shows that the displacements

$$\begin{aligned} u &= e_{xx}x + e_{xy}y + e_{xz}z + qz - ry + a, \\ v &= e_{yx}x + e_{yy}y + e_{yz}z + rx - pz + b, \\ w &= e_{zx}x + e_{zy}y + e_{zz}z + py - qx + c, \end{aligned} \tag{19.2}$$

satisfy for constant $e_{xx}, \ldots, e_{xy}$ the relations (18.3). Here a, b, c, p, q, r are arbitrary constants; the corresponding terms express therefore rigid body displacement [these formulae could also have been written down immediately using (12.12)]. By § 15 the solution (19.2) is the only possible one for the given $e_{xx}, \ldots, e_{xy}$.

In exactly the same way it is obvious that the static equations may be satisfied by putting the stress components equal to arbitrarily chosen constants. In fact, (18.2) gives then definite constant values for the strain components and one obtains again the preceding case.

Now certain very simple particular cases will be considered. First put

$$X_x = T = \text{const}, \quad Y_y = Z_z = Y_z = Z_x = X_y = 0. \tag{19.3}$$

Then, by (18.2) or by (17.4),

$$\frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} T, \quad e_{yy} = e_{zz} = - \frac{\quad}{2\mu(3\lambda + 2\mu)} T, \tag{19.4}$$

$$e_{yz} = e_{zx} = e_{xy} = 0. \tag{19.5}$$

It will now be assumed that the body under consideration is a prism or a cylinder with generators parallel to Ox and with ends perpendicular to this axis. Then it is obvious from (18.5) that on the side surfaces $X_n = Y_n = Z_n = 0$, i.e., they are free from surface tractions. On the end

facing in the positive direction of the axis Ox : $Y_n = Z_n = 0$, $X_n = T$, and on the other end

$$Y_n = Z_n = 0, \quad X_n = -X_x = -T.$$

Consequently the external forces acting on the cylinder are uniformly distributed over the ends and produce tension, if $T > 0$, and compression, if $T < 0$. The quantity T denotes the tensile or compressive traction, exerted per unit area of the ends. Now the obvious assumption (which may be based upon experimental evidence) will be made that for these conditions and for $T > 0$ the cylinder extends in the longitudinal and contracts in the transverse direction, i.e., for $T > 0$ one must have: $e_{xx} > 0$, $e_{yy} < 0$, $e_{zz} < 0$.

Then, by (19.4),

$$\frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} > 0, \quad \frac{\lambda}{2\mu(3\lambda + 2\mu)} > 0. \tag{19.6}$$

Therefore, in particular, $(\lambda + \mu) \neq 0$; further, it follows from these inequalities (dividing one by the other) that

$$\frac{\lambda}{2(\lambda + \mu)} > 0.$$

Introduce the notation

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \qquad \sigma = \frac{\lambda}{2(\lambda + \mu)} \tag{19.7}$$

On the basis of the above the quantities E and σ are positive for all materials. The quantity E is called *modulus of elasticity* or *Young's modulus* (Th. Young 1773—1829) and σ *Poisson's ratio*. The physical meaning of E is obtained from the first of the formulae (19.4) which gives

$$T = Ee_{xx}. \tag{19.8}$$

Thus E is the ratio of the applied stress to the strain caused by it in the longitudinal direction. The physical meaning of σ follows also from (19.4) which show that

$$\frac{|e_{yy}|}{|e_{xx}|} = \frac{|e_{zz}|}{|e_{xx}|} = \sigma, \tag{19.9}$$

i.e., the ratios of the transverse strains to the longitudinal strain are a constant quantity which does not depend on the shape of the cross-section of the rod nor on the magnitude of applied traction.

Next consider another particular case. Let

$$Y_z = T = \text{const}, \quad X_x = Y_y = Z_z = Z_x = X_y = 0. \tag{19.10}$$

Then, by (18.2),

$$e_{yz} = \frac{1}{2\mu} T, \quad e_{xx} = e_{yy} = e_{zz} = e_{zx} = e_{zy} = 0, \tag{19.11}$$

i.e., the corresponding deformation is pure shear. If the body under consideration is in the undeformed state a right parallelepiped with sides parallel to the coordinate planes, then it is easily seen from (19.10) that the sides perpendicular to the axis Ox are free from surface traction. The tractions applied to the other sides lead to the shearing forces shown for the case $T > 0$ in Fig. 10, where only a cross-section in a plane parallel to Oyz is drawn. The angle between the sides originally parallel to Oxy and Oyz is shown to differ from a right one by $\varepsilon_{yz} = 2e_{yz}$ (cf. § 12). Hence, by (19.11),

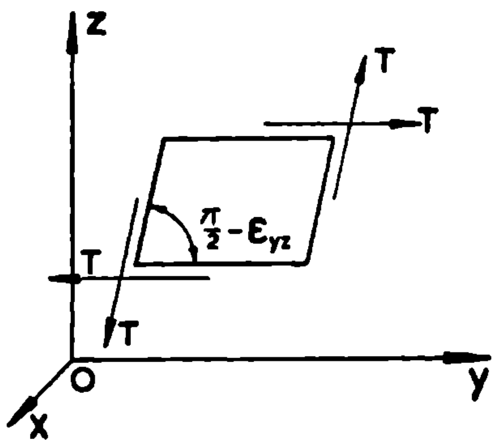


Fig. 10.

$$T = \mu\varepsilon_{yz}. \tag{19.12}$$

Thus μ is the ratio of the shearing stress T and the corresponding angle of shear. For this reason μ is called the *shear modulus*.

Finally consider the case

$$X_x = Y_y = Z_z = -p = \text{const}, \quad Y_z = Z_x = X_y = 0. \tag{19.13}$$

In this case (18.5) shows that the stress acting on any plane with normal n is given by the formulae

$$X_n = -p\cos(n, x), \quad Y_n = -p\cos(n, y), \quad Z_n = -p\cos(n, z),$$

expressing that the stress vector is parallel to the normal and that its magnitude is $|p|$. Hence only a normal stress acts on any plane; if one assumes $p > 0$, the stress will be compressive. The surface of any part of the body under consideration will only be subjected to uniform normal external pressure ("hydrostatic pressure").

Adding the first three formulae of (18.2) one finds

$$p = -(\lambda + \tfrac{2}{3}\mu)\theta.$$

Since θ is the cubical dilatation (and consequently $-\theta$ is the cubical compression), the quantity

$$k = \lambda + \tfrac{2}{3}\mu \tag{19.14}$$

is called the *modulus of compression* or *bulk modulus*. The obvious assumption will be made (which may be based on experimental evidence) that for $p > 0$ a decrease in volume actually takes place, and hence that $k > 0$ for all materials.

In addition to λ and μ the following constants have been introduced in the above work: the modulus of elasticity E, Poisson's ratio σ, the modulus of compression k. The quantities λ and μ may be expressed in terms of any two of these constants. For example, solving the equations (19.7) for λ and μ, one obtains

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)}, \tag{19.15}$$

and substituting these expressions in (19.14)

$$k = \frac{E}{3(1-2\sigma)} \tag{19.16}$$

The last formula shows that one must have for all materials

$$\sigma < \tfrac{1}{2}. \tag{19.17}$$

The formulae (19.15) show that

$$\lambda > 0, \ \mu > 0$$

which is now also obvious on physical grounds (cf. § 17).

Note that by the old theory of Cauchy and Poisson for all bodies $\sigma = \frac{1}{4}$, or, what is the same thing, $\lambda = \mu$. But this is not confirmed by experiment. However, for many materials, σ has approximately the same value of $\frac{1}{3}$ (and not $\frac{1}{4}$).

If one introduces in (17.4) the constants E, σ instead of λ, μ, the formulae take the simpler form

$$\begin{aligned} e_{xx} &= \frac{1}{E}[X_x - \sigma(Y_y + Z_z)], \\ e_{yy} &= \frac{1}{E}[Y_y - \sigma(Z_z + X_x)], \\ e_{zz} &= \frac{1}{E}[Z_z - \sigma(X_x + Y_y)], \end{aligned} \tag{19.18}$$

$$e_{yz} = \frac{1+\sigma}{E} Y_z, \quad e_{zx} = \frac{1+\sigma}{E} Z_x, \quad e_{xy} = \frac{1+\sigma}{E} X_y.$$

NOTE. In the literature one often finds the quantity $m = \frac{1}{\sigma}$ which is called Poisson's coefficient (e.g. R. Grammel [1]). The shear modulus μ is often denoted by G. Recently determined values of the above constants for different materials may likewise be found, for example, in Grammel's book.

§ 20. The fundamental boundary value problems of static elasticity. Uniqueness of solution. Consider now the basic equations of the static elastic body (§ 18) which will be written in the form

$$\begin{aligned}
&\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + X = 0,\\
&\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + Y = 0,\\
&\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + Z = 0,
\end{aligned} \tag{20.1}$$

$$\begin{aligned}
&X_x = \lambda\theta + 2\mu\frac{\partial u}{\partial x}, \quad Y_y = \lambda\theta + 2\mu\frac{\partial v}{\partial y}, \quad Z_z = \lambda\theta + 2\mu\frac{\partial w}{\partial z},\\
&Y_z = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \quad Z_x = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad X_y = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),
\end{aligned} \tag{20.2}$$

where

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

The nine equations (20.1) and (20.2) contain just as many unknown functions $u, v, w, X_x, \ldots, X_y$. The system (20.1) and (20.2) has earlier been called the complete system of static equations of the elastic body. In order to prove this statement, it has to be shown that the system (20.1) and (20.2) completely determines the elastic equilibrium of the body, if the external forces to which it is subject and the "internal" body forces are known.

It has been assumed here that the elastic equilibrium of a body is known, if the stress components or what is the same thing, thanks to the equations (20.2), if the strain components are known at every point of the body. It should not be concluded that body forces are exclusively external, since, for example, gravitational forces between parts of the

elastic body are "internal" body forces. The external forces mentioned above comprise, firstly, external body forces and, secondly, external tractions applied to the boundaries of the body.

In connection with all this there arises the first fundamental boundary value problem:

I. *Find the elastic equilibrium of a body, if the external stresses acting on its boundaries are given.* Here, as in all the following work, it will be assumed that *the body forces are given once and for all.*

In practice, this last point arises in the following manner: body forces acting on a body element depend as a rule on the mass contained in it and on its position with respect to other masses (e.g. gravity forces, centrifugal forces due to rotation, etc.). Under deformation the position of the element as well as its density will change, so that the body forces (X, Y, Z), referred to unit volume, generally speaking will also vary. But in view of the smallness of the deformations and displacements these variations are insignificantly small and may be disregarded.

With (20.1) and (20.2) in mind, this problem leads to the following one: Find functions $u, v, w, X_x, \ldots, X_y$, satisfying (20.1) and (20.2) in the region V originally occupied by the body (cf. § 16), and, in addition, satisfying on the surface (boundary) S of the body the following boundary conditions [cf. (18.5)]:

$$
\begin{aligned}
X_x \cos(n, x) + X_y \cos(n, y) + X_z \cos(n, z) &= f_1,\\
Y_x \cos(n, x) + Y_y \cos(n, y) + Y_z \cos(n, z) &= f_2,\\
Z_x \cos(n, x) + Z_y \cos(n, y) + Z_z \cos(n, z) &= f_3,
\end{aligned}
\tag{20.3}
$$

where n denotes the outward normal to S and f_1, f_2, f_3 are functions, given on the boundary (and representing the components of the known stress vector acting on the surface of the body).

In addition to the first fundamental problem stated above the second fundamental boundary value problem is of considerable interest:

II. *Find the elastic equilibrium of a body, if the displacements of the points of its boundary are given.* Physically this corresponds to the case when, by means of suitable tractions applied to the points of the surface, these points are subjected to known displacements and the boundary is correspondingly deformed. In relation to the equations (20.1) and (20.2) this leads to the determination of solutions which satisfy on the surface of the body the following boundary conditions:

$$
u = g_1, \quad v = g_2, \quad w = g_3, \tag{20.4}
$$

where g_1, g_2, g_3 are functions known on the boundary.

Finally, in many investigations an important part is played by the mixed fundamental boundary value problem which arises whenever displacements are known on one part, and external stresses on the remaining part of the boundary.

In addition to the problems stated already a number of others may be formulated which are no less important in applications; some of these will be considered later, when dealing with the plane case.

In the sequel, unless stated otherwise, it will be assumed that u, v, w are single-valued functions having continuous derivatives up to and including the third order. Under these conditions the strain and stress components will also be single-valued and continuous functions having continuous second order derivatives.

Having in mind the need to prove a "*uniqueness theorem*", i.e., to prove that the system of equations (20.1) and (20.2) has one and only one solution for each of the fundamental problems, one important lemma will first be deduced.

Consider the double integral, extended over the surface of the body,

$$J = \iint_S (X_n u + Y_n v + Z_n w)\, dS, \tag{20.5}$$

where X_n, Y_n, Z_n are determined by (18.5) and by n must be understood the outward normal to S. Substituting from (18.5) into (20.5) one finds

$$J = \iint_S [P \cos(n, x) + Q \cos(n, y) + R \cos(n, z)]\, dS,$$

where for convenience

$$P = X_x u + Y_x v + Z_x w, \quad Q = X_y u + Y_y v + Z_y w,$$
$$R = X_z u + Y_z v + Z_z w.$$

By Green's formula

$$J = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\, dy\, dz.$$

But in the present case

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = u\left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right) + v\left(\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}\right) +$$

$$+ w\left(\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}\right) + X_x\frac{\partial u}{\partial x} + Y_y\frac{\partial v}{\partial y} + Z_z\frac{\partial w}{\partial z} +$$

$$+ Y_z\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) + Z_x\left(\frac{\partial}{\partial z} + \frac{\partial w}{\partial x}\right) + X_y\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

or, by (20.1),

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = -(Xu + Yv + Zw) + 2W,$$

where

$$2W = X_x e_{xx} + Y_y e_{yy} + Z_z e_{zz} + 2Y_z e_{yz} + 2Z_x e_{zx} + 2X_y e_{xy}. \quad (20.6)$$

Thus one finally obtains

$$\iint\limits_S (X_n u + Y_n v + Z_n w)\,dS + \iiint\limits_V (Xu + Yv + Zw)\,dx\,dy\,dz =$$

$$= 2\iiint\limits_V W\,dx\,dy\,dz. \quad (20.7)$$

The expression W in this formula represents, as will be proved in §24, the *potential or strain energy per unit volume*; but at the moment this is of no importance.

Introducing on the right-hand side of (20.6) the expressions (18.2) for the stress components in terms of the strains one finds

$$2W = \lambda(e_{xx} + e_{yy} + e_{zz})^2 + 2\mu(e_{xx}^2 + e_{yy}^2 + e_{zz}^2 + 2e_{yz}^2 + 2e_{zx}^2 + 2e_{xy}^2), \quad (20.6')$$

which proves that W is a *positive* quadratic form involving the components of strain, and, in addition, that it is *definite*, i.e., that it becomes zero, if and only if all the strain components are zero; this follows from the fact that λ and μ have already been shown to be positive quantities.

Similarly, W may be expressed in terms of the stress components; obviously it will again be a positive definite form in these components.

Let it now be assumed that one of the earlier stated problems has t w o solutions. Let $u', v', w', X_x', \ldots, X_y'$ be the components of displacement and stress corresponding to the first solution, and u'', v'', w'', $X_x'', \ldots, X_y''$ be the analogous quantities of the second solution. Form the "difference" of these two solutions, i.e., put

$$u = u'' - u', \ldots, X_y = X_y'' - X_y'.$$

Obviously (cf. § 18) the functions $u, v, w, X_x, \ldots, X_y$ satisfy the same

equations (20.1), (20.2) in which one has only to put

$$X = Y = Z = 0;$$

in other words, the "difference" solution satisfies the basic equilibrium equations in the absence of body forces. Thus, by (20.7) for $X = Y = Z = 0$,

$$\iint_S (X_n u + Y_n v + Z_n w)\, dS = 2 \iiint_V W\, dx\, dy\, dz. \tag{20.7'}$$

Now the following will be noted: In the case of Problem I the quantities X_n, Y_n, Z_n, formed by subtracting the two solutions, will be zero on S, since both solutions, by supposition, satisfy the conditions (20.3) for the same functions f_1, f_2, f_3. Hence

$$X_n = X_x \cos(nx) + X_y \cos(n, y) + X_z \cos(n, z) = 0,$$

$$Y_n = 0, \; Z_n = 0.$$

In the case of Problem II one will in the same way find $u = v = w = 0$ on S. Finally, in the case of the mixed problem, u, v, w will be zero on one part, and X_n, Y_n, Z_n on the remaining part of the boundary. In all three cases the expression $X_n u + Y_n v + Z_n w$ is zero on S. Hence (20.7') becomes

$$\iiint_V W\, dx\, dy\, dz = 0.$$

However, since $W \geqslant 0$, the above relation is only possible, if $W = 0$ at all points of V. It has been seen earlier that this condition implies $e_{xx} = e_{yy} = e_{zz} = e_{yz} = e_{zx} = e_{xy} = 0$ throughout the body. But $e_{xx} = e''_{xx} - e'_{xx}$ etc., where $e''_{xx}, \ldots, e''_{xy}$ and $e'_{xx}, \ldots, e'_{xy}$ are the components of strain, corresponding to the two solutions under consideration. This means that both solutions give identical strain components, and consequently also identical stress components. Hence both solutions are identical in the sense that they give an identical state of stress (and deformation). This proves the uniqueness theorem. [The theorem and the proof given here is due to G. Kirchhoff (1858).]

However, it should be noted that the displacements may not be completely identical. In fact, from the vanishing of $e_{xx}, \ldots, e_{xy}$ does not follow that $u = v = w = 0$, but only

$$u = a + qz - ry, \quad v = b + rx - pz, \quad w = c + py - qx, \tag{20.8}$$

(where a, b, c, p, q, r are constants) expressing rigid body motion. Thus,

when solving the first fundamental problem, one will always obtain the same stresses (and strains), but one may find for the displacements values, differing from each other by terms expressing rigid body motion. This could, of course, have been predicted, because such displacements do not affect the stresses and deformations. Such differences in the solutions are, however, unimportant.

In the cases of the second and the mixed boundary value problems such differences cannot occur, since the displacements are given beforehand for the whole or part of the boundary.

Finally note the following proposition which is a particular case of the uniqueness theorem proved above: If the body forces are zero and if, in addition, either *a*) the external stresses or *b*) the displacements of points of the boundary or *c*) the external stresses on one part and displacements on the remaining part of the boundary vanish, then the stresses throughout the body are zero (and hence there is also no deformation).

The above proof of uniqueness of solution holds true for simply as well as for multiply connected bodies, because at no stage has the assumption of simple connectivity been introduced. However, the hypothesis that the components of displacement are *single-valued* functions of the coordinates is essential for the proof. As has already been stated, in the case of multiply connected bodies one may admit also the existence of displacements which are not single-valued. For such a generalized study of the problem the above uniqueness proof loses its validity and the theorem is no longer true. For a physical interpretation of this case see Part II of this book.

Note that only the following has been proved: if the fundamental boundary value problems of elasticity have a solution, then that solution is unique. But this, of course, it not a proof of the existence of such solutions. The proof of the existence of a solution is much more difficult than the uniqueness proof and it requires application of the most powerful methods of modern analysis. This explains the fact that the proofs of the existence of solutions of the fundamental problems have only been found comparatively recently.

The above-mentioned proofs of existence are given in a great number of original publications. Reference will be made here to only a few of these. *For the secon. fundamental problem*: I. Fredholm [1], G. Lauricella [1, 2], A. Korn [1, 2], L. Lichtenstein [1], D. I. Sherman [21]. *For the first fundamental problem*: A. Korn [3], H. Weyl [1]. Note that as a rule the first problem of this book is called in literature the second boundary value problem and vice versa.

The scope and character of the present book do not allow a general treatment of these problems. Therefore it will only be stated here that the existence of solutions of the first and second fundamental boundary value problems has been proved recently with full mathematical rigour under sufficiently general conditions. The proof for the plane case will be given in Part V of this book.

For the existence of a solution of the first fundamental problem obviously the following condition must be satisfied: the resultant vector and moment of the body forces and (known) external stresses applied to the boundary must be equal to zero. This condition follows from the fundamental principle of statics and may also be deduced from (20.1). In fact, the projection of the resultant vector of these forces, for example on the axis Ox, is

$$\iiint_V X\,dx\,dy\,dz + \iint_S X_n\,dS.$$

But, as has been shown in § 4, this expression is equal to

$$\iiint_V \left(X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right) dx\,dy\,dz;$$

this triple integral, however, is zero by (20.1).

Further, the resultant moment, for example about the axis Ox, is given by

$$\iiint_V (yZ - zY)dx\,dy\,dz + \iint_S (yZ_n - zY_n)\,dS.$$

This expression, as shown in § 4, is equal to

$$\iiint_V (Z_y - Y_z)\,dx\,dy\,dz,$$

where again use has been made of (20.1); but since $Z_y = Y_z$, the last triple integral vanishes.

§ 21. Basic equations in terms of displacement components. The system of equations (20.1) and (20.2) involves simultaneously the components of stress and displacement. However, it is possible to obtain systems containing only one or the other type of components. It is simplest to deduce the system containing the components of displacement. For this purpose it is sufficient to substitute from (20.2) in (20.1) which

gives, after some obvious simplifications,

$$
\begin{aligned}
(\lambda + \mu)\frac{\partial \theta}{\partial x} + \mu \Delta u + X &= 0, \\
(\lambda + \mu)\frac{\partial \theta}{\partial y} + \mu \Delta v + Y &= 0, \\
(\lambda + \mu)\frac{\partial \theta}{\partial z} + \mu \Delta w + Z &= 0,
\end{aligned}
\tag{21.1}
$$

where again

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

and Δ denotes the Laplace operator, i.e.,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Starting from the representation of the elastic body as a system of material points, Navier (1785—1836) obtained in his memoir, presented to the Paris Academy in 1821 and published in 1827, the equations which must be satisfied by the displacements of the points of an elastic body in the dynamic as well as in the static cases. Navier's equations for the latter case agree essentially with the equations (21.1), if one puts in these $\lambda = \mu$. The discovery of these equations may be considered one of the most important stages in the development of the theory of elasticity, and therefore Navier is rightly ranked among the most important of its founders.

The equations (21.1) are very convenient, because of their symmetry and because they contain only three unknowns.

§ 22. Equations in terms of stresses. However, it is often more convenient to deal with equations containing only stresses. It should not be thought that for this purpose one may limit consideration to the equations

$$
\begin{aligned}
\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + X &= 0, \\
\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + Y &= 0, \\
\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + Z &= 0,
\end{aligned}
\tag{22.1}
$$

which have been called "equilibrium equations"

In fact, if $X_x, \ldots, X_y$ satisfy these equations, this does not mean that these quantities express some actually possible state of stress; it is also necessary, in addition, that displacements (u, v, w) can be found which are related to these stresses by (20.2). For this, on the other hand, it is necessary and sufficient (with certain reservations in the case of multiply connected bodies; cf. the end of this section) that the strain components which will now be written [cf. (19.18)]

$$e_{xx} = \frac{1+\sigma}{E} X_x - \frac{\sigma}{E}\Theta, \qquad e_{yy} = \frac{1+\sigma}{E} Y_y - \frac{\sigma}{E}\Theta,$$

$$e_{zz} = \frac{1+\sigma}{E} Z_z - \frac{\sigma}{E}\Theta, \tag{22.2}$$

$$e_{yz} = \frac{1+\sigma}{E} Y_z, \qquad e_{zx} = \frac{1+\sigma}{E} Z_x, \qquad e_{xy} = \frac{1+\sigma}{E} X$$

where

$$\Theta = X_x + Y_y + Z_z,$$

satisfy the compatibility equations of St. Venant [cf. (15.6)].

Substituting from (22.2) in (15.6) one obtains from the formulae in the first row of (15.6)

$$\frac{\partial^2 Y_y}{\partial z^2} + \frac{\partial^2 Z_z}{\partial y^2} - \frac{\sigma}{1+\sigma}\left\{\frac{\partial^2\Theta}{\partial y^2} + \frac{\partial^2\Theta}{\partial z^2}\right\} = 2\frac{\partial^2 Y_z}{\partial y\,\partial z}, \tag{22.3}$$

$$\frac{\partial^2 X_x}{\partial y\,\partial z} - \frac{\sigma}{1+\sigma}\frac{\partial^2\Theta}{\partial y\,\partial z} = \frac{\partial}{\partial x}\left\{-\frac{\partial Y_z}{\partial x} + \frac{\partial Z_x}{\partial y} + \frac{\partial X_y}{\partial z}\right\}; \tag{22.4}$$

by cyclic interchange of symbols one obtains four similar relations corresponding to the remaining compatibility conditions. Equations (22.3) and (22.4) may be somewhat simplified using (22.1). Thus, differentiating the second equation of (22.1) with respect to y and the third with respect to z and adding them, one obtains

$$2\frac{\partial^2 Y_z}{\partial y\,\partial z} + \frac{\partial^2 Y_y}{\partial y^2} + \frac{\partial^2 Z_z}{\partial z^2} + \frac{\partial}{\partial x}\left\{\frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right\} = -\left(\frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right).$$

But, by the first of the equations (22.1),

$$\frac{\partial}{\partial x}\left(\frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right) = -\frac{\partial X}{\partial x} - \frac{\partial^2 X_x}{\partial x^2};$$

substituting this last expression in the preceding formula gives

$$2\frac{\partial^2 Y_z}{\partial y\,\partial z} = \frac{\partial^2 X_x}{\partial x^2} - \frac{\partial^2 Y_y}{\partial y^2} - \frac{\partial^2 Z_z}{\partial z^2} - \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + 2\frac{\partial X}{\partial x}$$

which, when introduced on the right hand-side of (22.3), leads to

$$\frac{\sigma}{1+\sigma}\left(\frac{\partial^2\Theta}{\partial y^2} + \frac{\partial^2\Theta}{\partial z^2}\right) + \frac{\partial^2(Y_y + Z_z)}{\partial z^2} + \frac{\partial^2(Y_y + Z_z)}{\partial y^2} \quad \frac{\partial^2 X_x}{\partial x^2}$$
$$= -\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + 2\frac{\partial X}{\partial x}.$$

Finally, noting that $Y_y + Z_z = \Theta \quad X_x$, one obtains

$$\frac{1}{1+\sigma}\Delta\Theta - \Delta X_x \qquad \frac{}{1+\sigma}\frac{\partial^2\Theta}{\partial x^2} = -\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + 2\frac{\partial X}{\partial x}. \quad (a)$$

Adding this equation to the two analogous equations, obtained from it by cyclic transposition of symbols, one finds a formula which is important in itself

$$\Delta\Theta \quad -\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right)\frac{1+\sigma}{1-\sigma} \tag{22.5}$$

Substitution of (22.5) in (a) finally gives

$$\Delta X_x + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x^2} = -\frac{\sigma}{1-}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial X}{\partial x}. \tag{22.6}$$

This is one of the required formulae, the other two being obtainable by cyclic transposition of symbols.

Now consider equation (22.4). Differentiating the second equation of (22.1) with respect to z and the third with respect to y and adding, one obtains

$$\frac{\partial^2 Y_x}{\partial x\,\partial z} + \frac{\partial^2 Y_y}{\partial y\,\partial z} + \frac{\partial^2 Y_z}{\partial z^2} + \frac{\partial^2 Z_x}{\partial x\,\partial y} + \frac{\partial^2 Z_y}{\partial y^2} + \frac{\partial^2 Z_z}{\partial y\,\partial z} - \quad \left(\frac{\partial Z}{\partial y} + \frac{\partial Y}{\partial z}\right).$$

Adding this equation to (22.4) which may be written

$$\frac{\partial^2 X_x}{\partial y\,\partial z} + \frac{\partial^2 Y_z}{\partial x^2} \quad \frac{\partial^2 Z_x}{\partial x\,\partial y} - \frac{\partial^2 X_y}{\partial x\,\partial z} \quad \frac{\sigma}{1+\sigma}\frac{\partial^2\Theta}{\partial y\,\partial z} = 0,$$

one finds

$$\Delta Y_z + \frac{}{1+\sigma}\frac{\partial^2\Theta}{\partial y\,\partial z} = -\left(\frac{\partial Z}{\partial y} + \frac{\partial Y}{\partial z}\right). \tag{22.7}$$

The other two equations of this type may be obtained by cyclic transposition.

Thus it is seen that the stress components must satisfy nine equations, i.e., (22.1), (22.6) and (22.7) with their analogous equations. The equations (22.6) and (22.7) were obtained by J. H. Michell [1] (pp. 112—113); for the case of zero body forces, these equations were found earlier by E. Beltrami (1892). Therefore the equations (22.6) and (22.7) with their four analogues will be called *conditions of compatibility* of Beltrami-Michell.

It follows from the above that, if the six equations of the type (22.6) and (22.7) are satisfied, the strain components corresponding to the stress components, satisfying the equilibrium equations (22.1), will fulfill the compatibility conditions of St. Venant. Thus, the equations (22.1), (22.6) and (22.7) with their analogues are not only necessary, but also *sufficient*.

Some reservations must be made only in the case of a multiply connected body, when the displacements, corresponding to the stress components satisfying all the above conditions, may be found to be multi-valued. In that case one has to introduce either an additional condition of single-valuedness of the displacements or to admit the existence of multi-valued displacements which, as has been mentioned earlier, may be given a definite physical interpretation.

§ 23. Remarks on the effective solution of the fundamental problems. St. Venant's Principle. Solution of the above-mentioned fundamental boundary value problems for the general case presents in practice great difficulties, if one has in mind *effective* calculations. The so-called general methods give (in the general cases) only theoretical solutions, i.e., in the end they only prove existence of the solution. (These general methods are given e.g. in the papers quoted in § 20).

Solution of one or the other problem is often considerably simplified by application of *St. Venant's Principle* which may be formulated as follows: If one applies to a *small* part of the surface of the body a set of forces which are statically equivalent to zero, then this system of forces will not noticeably affect parts of the body lying away from the above region. Alternatively: If a set of forces, acting on a small part of the surface of a body, is replaced by a system of forces (acting on the same part) which is statically equivalent to the former, then such replacement does not cause a noticeable change in the elastic equilibrium of parts of

the body which do not lie too near to the above-mentioned region. Both formulations of St. Venant's Principle are obviously equivalent.

The Principle was first pronounced in St. Venant's memoir [1] of 1855. It agrees very well with reality. However, its mathematical foundation (which must consist of an estimate of the influence of a system of forces which are statically equivalent to zero) is rather difficult, at least in the general case. By a system of forces, which is statically equivalent to zero, will be understood a system, equivalent to zero from the point of view of the statics of absolutely rigid bodies, i.e., a system, the resultant vector and moment of which are equal to zero. Systems are called statically equivalent, if they have the same resultant vectors and moments.

Thus St. Venant's Principle offers the means of modifying (under the definite conditions stated above) the given stress distribution on the boundary, and thus of simplifying problems. The Principle will be widely used in the later parts of this book.

§ 24. Dynamic equations. The fundamental problems of the dynamics of an elastic body. Although this book deals only with problems of static equilibrium, nevertheless the dynamic equations of an elastic body will be deduced, the simplest fundamental problems for these equations stated and the uniqueness of their solutions proved. In passing, an expression will be obtained for the potential energy of a deformed body.

The deduction of the dynamic equations of an elastic body does not offer any difficulties. These equations may be obtained directly from the static equations by use of D'Alembert's Principle. In fact, it is sufficient for this purpose to write down the static equations and to add the *inertia forces* to the body forces.

In the present case the components of displacement, strain and stress will be functions of x, y, z as well as of the time t. The components of acceleration of a point, occupying a position (x, y, z) in the undeformed state of the body, will be

$$\frac{\partial^2 u(x, y, z, t)}{\partial t^2} \qquad \frac{\partial^2 v(x, y, z, t)}{\partial t^2} \qquad \frac{\partial^2 w(x, y, z, t)}{\partial t^2}$$

The components of the inertia force, applied to a volume element dV containing mass dm, will be

$$-\frac{\partial^2 u}{\partial t^2}\, dm, \qquad -\frac{\partial^2 v}{\partial t^2}\, dm, \qquad -\frac{\partial^2 w}{\partial t^2}\, dm.$$

But since $dm = \rho\, dV$, where ρ is the density, the components of the inertia force per unit volume become

$$\rho \frac{\partial^2 u}{\partial t^2}, \quad -\rho \frac{\partial^2 v}{\partial t^2}, \quad -\rho \frac{\partial^2 w}{\partial t^2}.$$

Adding the inertia force to the body force and introducing these values into (18.1), one finds

$$\begin{aligned}
\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + X &= \rho \frac{\partial^2 u}{\partial t^2}, \\
\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + Y &= \rho \frac{\partial^2 v}{\partial t^2}, \\
\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + Z &= \rho \frac{\partial^2 w}{\partial t^2}.
\end{aligned} \tag{24.1}$$

These equations now take the place of the "equilibrium equations" i.e., of the equations (18.1). The equations relating stresses to strains and expressing the generalized Hooke's Law remain unaltered, since the body forces do not figure in them. In the case of an isotropic body these equations are (18.2) and (18.3). The equations (18.5) remain likewise unchanged.

In the present case it is convenient to use equations in terms of displacements which can be obtained in the same way as it was done in § 21 and which in the case of an isotropic body have the form

$$\begin{aligned}
(\lambda + \mu) \frac{\partial \theta}{\partial x} + \mu \Delta u + X &= \rho \frac{\partial^2 u}{\partial t^2}, \\
(\lambda + \mu) \frac{\partial \theta}{\partial y} + \mu \Delta v + Y &= \rho \frac{\partial^2 v}{\partial t^2}, \\
(\lambda + \mu) \frac{\partial \theta}{\partial z} + \mu \Delta w + Z &= \rho \frac{\partial^2 w}{\partial t^2}.
\end{aligned} \tag{24.2}$$

These equations differ from those obtained by Navier in 1821 (cf. § 21) in that Navier's equations contained only one elastic constant, i.e., one gets his equations from (24.2) by putting $\lambda = \mu$.

Analogous to the fundamental boundary value problems, which were formulated in § 20 for the static case, one may similarly state problems with regard to the dynamic equations. An essential difference is that the boundary conditions have to be augmented by "initial conditions"

i.e., given displacements and velocities of points of the body at some "initial" instant of time t_o. Mathematically these problems may be formulated as follows:

First fundamental problem. *Find functions $u(x, y, z, t)$, $v(x, y, z, t)$, $w(x, y, z, t)$ satisfying* (24.2) *and the following supplementary conditions*:

$$X_n = f_1, \quad Y_n = f_2, \quad Z_n = f_3 \tag{24.3}$$

on the surface S of the body at all times, starting from $t = t_0$, and

$$u = u_0, \quad v = v_0, \quad w = w_0, \quad \frac{\partial u}{\partial t} = \dot{u}_0, \quad \frac{\partial v}{\partial t} = \dot{v}_0, \quad \frac{\partial w}{\partial t} = \dot{w}_0 \tag{24.4}$$

in the region V occupied by the body at time $t = t_0$.

In these formulae f_1, f_2, f_3 are functions given on the surface S of the body and depending, in general, also on the time. Further, $u_0, v_0, w_0, \dot{u}_0, \dot{v}_0, \dot{w}_0$ are known functions of x, y, z. The equations (24.3) are the boundary, and (24.4) the initial conditions.

The second fundamental problem differs from the first only in that the boundary condition (24.3) is replaced by

$$u = g_1, \qquad g_2, \quad w = \tag{24.5}$$

on S; g_1, g_2, g_3 are given functions on S depending, in general, also on the time.

For the mixed problem the condition (24.3) will refer to one part and (24.5) to the remaining part of the boundary S.

Apart from these problems there are a number of other important problems which will, however, not be stated here.

In the above cases it has been assumed that the body forces are known at all points of the body and at all instants of time (beginning with $t = t_0$). No consideration will be given here to the difficult question of the mathematical proof of the existence of solutions of these problems, and it will only be proved that, *if a solution of the given problems exists*, then *it is unique.*

Before giving this proof, a formula will be deduced which is of considerable independent interest and which expresses the law of conservation of energy, as applied to the case under consideration.

Consider any definite motion of a given elastic body and choose as the initial instant t_0 the moment when the body lies in a "natural" state of equilibrium, i.e., when body forces and stresses, and consequently also

deformations, are absent. Let $R(t)$ denote the work done by the external stresses and body forces between the starting time t_0 and the instant t under consideration. This work will now be determined and for this purpose calculate the work dR done by these forces in the time interval $t, t + dt$, assuming dt infinitely small.

A point, occupying before deformation the position (x, y, z), will at time t have the coordinates

$$x + u(x, y, z, t), \quad y + v(x, y, z, t), \quad z + w(x, y, z, t).$$

The displacement of this point in the time interval $t, t + dt$ has obviously the following components:

$$\dot{u}\,dt, \quad \dot{v}\,dt, \quad \dot{w}\,dt,$$

where

$$\dot{u} = \frac{\partial u}{\partial t} \text{ etc.}$$

The work of the external stresses, acting on the surface element dS of the body, in the time interval dt is

$$(X_n\dot{u} + Y_n\dot{v} + Z_n\dot{w})dS\,dt,$$

and the work of the body forces, applied to a body element dV, is

$$(X\dot{u} + Y\dot{v} + Z\dot{w})dV\,dt.$$

Thus the work dR, done by all the above forces during the time interval dt, is given by

$$\frac{dR}{dt} = \iint\limits_S (X_n\dot{u} + Y_n\dot{v} + Z_n\dot{w})dS + \iiint\limits_V (X\dot{u} + Y\dot{v} + Z\dot{w})dV. \quad (a)$$

Replacing under the first of these integrals X_n, Y_n, Z_n by their expressions (18.5), transforming the integral in the same way as the integral J was transformed in § 20, one finds using (24.1)

$$\frac{dR}{dt} = \iiint\limits_V \rho(\ddot{u}\dot{u} + \ddot{v}\dot{v} + \ddot{w}\dot{w})\,dV +$$

$$+ \iiint\limits_V (X_x\dot{e}_{xx} + Y_y\dot{e}_{yy} + Z_z\dot{e}_{zz} + 2Y_z\dot{e}_{yz} + 2Z_x\dot{e}_{zx} + 2X_y\dot{e}_{xy})dV. \quad (b)$$

But

$$\iiint\limits_V \rho(\ddot{u}\dot{u} + \ddot{v}\dot{v} + \ddot{w}\dot{w})dV = \iiint\limits_V \tfrac{1}{2}\rho \frac{\partial}{\partial t}(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,dV = \frac{dT}{dt},$$

where

$$T = \tfrac{1}{2}\iiint\limits_V \rho(\dot{u}^2 + \dot{u}^2 + \dot{w}^2)dV. \tag{24.6}$$

Obviously T is the *kinetic energy* of the elastic body, i.e., the sum of the kinetic energies of its different elements

$$\tfrac{1}{2}dm(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) = \tfrac{1}{2}\rho(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\,dV.$$

Next transform the second term on the right-hand side of (b). Assume now that the body under consideration is isotropic and introduce the function

$$W = \tfrac{1}{2}\lambda(e_{xx} + e_{yy} + e_{zz})^2 + \mu(e_{xx}^2 + e_{yy}^2 + e_{zz}^2 + 2e_{yz}^2 + 2e_{zx}^2 + 2e_{xy}^2); \tag{24.7}$$

it is immediately seen that

$$\begin{gathered} X_x = \frac{\partial W}{\partial e_{xx}}, \quad Y_y = \frac{\partial W}{\partial e_{yy}}, \quad Z_z = \frac{\partial W}{\partial e_{zz}}, \\ 2Y_z = \frac{\partial W}{\partial e_{yz}}, \quad 2Z_x = \frac{\partial W}{\partial e_{zx}}, \quad 2X_y = \frac{\partial W}{\partial e_{xy}}, \end{gathered} \tag{24.8}$$

and hence that the expression under the second integral of (b) is equal to $\dfrac{\partial W}{\partial t}$ and

$$\iiint\limits_V \frac{\partial W}{\partial t}\,dV = \frac{d}{dt}\iiint\limits_V W\,dV.$$

Thus (b) takes the form

$$\frac{dR}{dt} = \frac{dT}{dt} + \frac{d}{dt}\iiint\limits_V W\,dV. \tag{24.9}$$

Integrating both sides of this equation from t_0 to t and taking into consideration that at the initial instant the body is in a natural state of rest (i.e., $T = W = 0$ at $t = t_0$), one finds for the work R done by the

external stresses and body forces in the time interval (t_0, t)

$$R = T + U, \tag{24.10}$$

where

$$U = \iiint_V W\,dV. \tag{24.11}$$

Formula (24.7) shows that W depends solely on the state of deformation at a given moment at a given point; hence U depends on the state of deformation of the considered body at a given instant t. The quantity U is the *potential energy of deformation* of the body, i.e., the work which must be done by the body forces and external stresses, in order to cause a given state of deformation. In fact, if under the influence of these forces the body changed from a "natural" state of rest to a new, deformed state of rest, then, by (24.10), $R = U$, because for a state of rest $T = 0$.

Formula (24.10) indicates that the work of the body forces and external stresses is transformed into kinetic energy and strain energy; it thus expresses the law of conservation of energy.

The quantity W, defined by (24.7), is the strain energy per unit volume. In fact, it follows from (24.11) that the amount of potential energy, belonging to the body element dV, is WdV. The expression W had already been introduced in § 20; it will be remembered that W is a positive definite quadratic form in terms of the strain components. This follows directly from (24.7).

Next consider the question of the uniqueness of the solutions of the fundamental problems. Let any one of them have two solutions for identical boundary and initial conditions and identical body forces. Form the "difference" of these solutions (cf. § 20). The new solution (u, v, w) will satisfy the same equations as the two former solutions, but in the *absence of body forces*; in addition, in the case of the first problem, one will have

$$X_n = Y_n = Z_n = 0 \text{ on } S, \tag{24.3'}$$

and in the case of the second problem,

$$u = v = w = 0 \text{ on } S; \tag{24.5'}$$

in the case of the mixed problem, condition (24.3') will hold on one part of the surface and (24.5') on the remainder. In all cases one has

$$X_n\dot{u} + Y_n\dot{v} + Z_n\dot{w} = 0 \text{ on } S.$$

In fact, in the case (24.3′): $X_n = Y_n = Z_n = 0$; in the case (24.5′): $u = v = w = 0$ (on S) at all instants of time (beginning with $t = t_0$), and hence

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0 \text{ on } S;$$

similarly for the mixed problem.

Further, one obviously has at the initial instant

$$u = v = w = \dot{u} = \dot{v} = \dot{w} = 0,$$

because both known solutions satisfy the same initial conditions.

It follows from the above that the work R for the solution u, v, w is zero, and hence by (24.10) that

$$T = U = 0.$$

But obviously this is only possible, when $T = 0$, $U = 0$, and therefore at all instants of time, starting with $t = t_0$, one will have

$$\dot{u} = \dot{v} = \dot{w} = 0, \qquad = e_{yy} = e_{z} \qquad = e_{zx} = e_{xy} = 0.$$

The first set of these equations shows that the displacements do not depend on the time, i.e., that one is dealing with a static problem. It follows from the second set of conditions that all the strains are zero, i.e., the solution u, v, w can only represent rigid body motion. Finally, it follows from the condition, that at the initial instant all displacements are zero, that there can be no body motion. Thus one has for all points of the body and at all times $u = v = w = 0$. It is seen from this that the two solutions of the problem, mentioned earlier, must be identical and this proves the assertion.

NOTE. It follows from (20.7) that the *potential energy*

$$U = \iiint\limits_V W \, dx \, dy \, dz$$

of the deformed body in equilibrium may be expressed by the formula

$$U = \tfrac{1}{2} \iint\limits_S (X_n u + Y_n v + Z_n w) dS + \tfrac{1}{2} \iiint\limits_V (Xu + Yv + Zw) dV, \quad (24.12)$$

and, in the absence of body forces, by

$$U = \tfrac{1}{2} \iint\limits_{S} (X_n u + Y_n v + Z_n w) dS, \tag{24.13}$$

where the double integral is taken over the entire surface of the body. Note that, by (24.11) and (24.7), $U > 0$ for all states of non-zero-deformation.

The formulae (24.12) and (24.13) are easily remembered; they show that the strain energy of a body is equal to half the work done by the external stresses and body forces of the final equilibrium state, acting through the displacements of the equilibrium state.

PART II

GENERAL FORMULAE OF THE PLANE THEORY OF ELASTICITY

The considerable mathematical difficulties which arise during any attempt to solve the fundamental problems of the theory of elasticity necessitate the search for practical methods of solution in special classes of particular cases. One of the most important of such classes is concerned with the so called "plane theory of elasticity" or "the plane problems of the theory of elasticity" to which are devoted Parts II—VI of this book.

The development of the theory will here be based on the complex representation of the general solution of the equations of the plane theory of elasticity which will be stated below. This complex representation, originally introduced by G. V. Kolosov (cf. his papers [1], [2] and his book [6]), has been found very useful for the effective solution of the fundamental boundary value problems as well as for investigations of a general character, as is shown by a large number of important papers which have been published lately in Russia. Several of these will be studied or referred to in this or later Parts.

From time to time papers have appeared outside Russia in which complex representation of partly incomplete solutions has been used and results have been given which are either contained in the work of Russian authors or which follow directly from the results obtained by the latter. Among these are, for example, the papers by A. C. Stevenson [1] and H. Poritsky [2] about which some remarks will be made in § 32.

It will only be mentioned now that some of the methods, fundamental to complex representation, may be successfully generalized to the case of anisotropic bodies, but this will not be done in this book. Important and interesting results in this direction have been obtained by S. G. Lekhnitzky [1], S. G. Mikhlin [11], D. I. Sherman [9, 19], G. N. Savin [3—6] and others. A systematic study of a number of these results may be found in Lekhnitzky's book which contains some of the results obtained by its author.

Finally, it should be noted that the fundamental nature of the results of the plane theory of elasticity (Parts II—VI), stated below, must of course be seen not in the new deduction of Kolosov's and other formulae, but rather in the application of these formulae to the solution of the fun-

damental boundary value problems by systematic utilization of the properties of Cauchy type integrals and conformal transformation.

In fact, Kolosov's formulae may be deduced in many ways some of which are extraordinarily simple. The method chosen here requires somewhat lengthier calculations than some of the others, because it is completely elementary; but it has been retained here, since it gives, as by-products, a number of formulae useful in the sequel; it also guarantees complete generality of the obtained solutions and does not assume beforehand that these solutions are analytic. Note also that before Kolosov several authors (e.g. L. N. G. Filon) have obtained some complex representations of solutions, but no one (or almost no one) has actually applied them.

CHAPTER 4

BASIC EQUATIONS OF THE PLANE THEORY OF ELASTICITY

The equations of the plane theory of elasticity apply to two cases of equilibrium of elastic bodies which are of considerable interest in practice, namely: to the case of *plane strain* and to the case of the *deformation of a thin plate* under forces applied to its boundary and acting in its plane. These two cases will be discussed in detail in the following two sections.

§ 25. Plane strain. A body will be said to be in the state of *plane strain*, parallel to the plane Oxy, if the displacement component w is zero and if the components u, v depend only on x and y, but not on z. In this case

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

and the formulae (20.2) give

$$X_x = \lambda\theta + 2\mu \frac{\partial u}{\partial x}, \quad Y_y = \lambda\theta + 2\mu \frac{\partial v}{\partial y}, \quad X_y = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

$$Z_z = \lambda\theta, \quad X_z = Y_z = 0.$$

These formulae show that the stress components are likewise independent of z (since u, v and hence θ do not depend on it).

Further, the first two of the equations (20.1) take the form

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + X = 0, \quad \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} + Y = 0,$$

and the third becomes $Z = 0$, indicating that for plane deformation, parallel to the plane Oxy, the component of the body force in the direction perpendicular to the plane of deformation must vanish. The preceding equations also show that the components X, Y of the body force do not depend on z.

Thus, in the end, the static equations of an elastic body in the case

of plane strain, parallel to the plane Oxy, reduce to the following:

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + X = 0, \quad \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} + Y = 0, \tag{25.1}$$

$$X_x = \lambda\theta + 2\mu\frac{\partial u}{\partial x}, \quad Y_y = \lambda\theta + 2\mu\frac{\partial v}{\partial y}, \quad X_y = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \tag{25.2}$$

where all the quantities appearing in these equations are independent of z; the component Z_z (likewise independent of z) is given by $Z = \lambda\theta$ or, noting that by (25.2)

$$X_x + Y_y = 2(\lambda + \mu)\theta, \quad \theta = \frac{1}{2(\lambda + \mu)}(X_x + Y_y),$$

by

$$Z_z = \lambda\theta = \frac{\lambda}{2(\lambda + \mu)}(X_x + Y_y) = \sigma(X_x + Y_y), \tag{25.3}$$

where σ is Poisson's ratio. The formula (25.3), determining Z_z, has been intentionally deduced, since solution of the system (25.1) and (25.2) represents the *fundamental* problem, and Z_z is determined from (25.3) after its solution. There remains now to state those cases when plane deformation takes place.

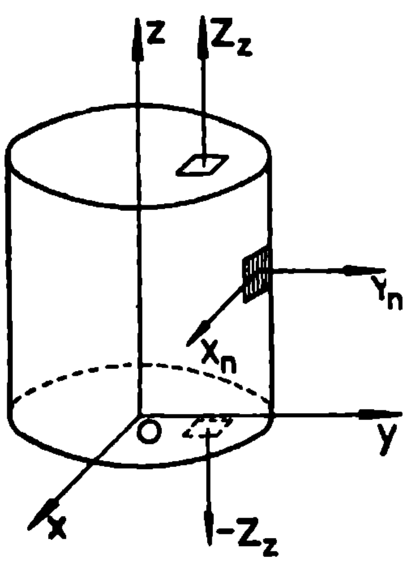


Fig. 11.

It will be assumed that one is dealing with cylindrical (prismatic) bodies, bounded by surfaces parallel to the axis Oz (sides) and by two plane faces normal to the generating surface (ends) (Fig. 11). Further, assume that the external stresses, acting on the sides, are parallel to the plane Oxy and do not depend on z and that the same condition is satisfied by the body forces. The latter as well as the external stresses will be assumed known.

Consider whether under these conditions plane deformation of the cylinder is possible. For this it is necessary and sufficient that the equations (25.1) and (25.2) have solutions u, v, X_x, Y_y, X_y, satisfying on the sides of the cylinder the boundary conditions

$$\begin{aligned} X_x \cos(n, x) + X_y \cos(n, y) &= X_n, \\ Y_x \cos(n, x) + Y_y \cos(n, y) &= Y_n, \end{aligned} \tag{25.4}$$

where X_n, Y_n are the known components of the external stress vector, acting on the side surface, and n is the outward normal; the condition (25.4) is obtained from (3.2) which gave the stress vector acting on the plane with normal n. (The third of these formulae is identically satisfied, since, by hypothesis, $Z_n = 0$, $Z_x = Z_y = 0$ and $\cos(n, z) = 0$ on the ends.) One is thus led to a problem, completely analogous to the first fundamental boundary value problem of the theory of elasticity in the general case (§ 20); but one is dealing here with a simpler case, because the unknown functions u, v, X_x, Y_y, X_y depend only on the two variables x and y and, instead of considering the entire region occupied by the body, one may restrict the investigation to one of its sections in a plane, parallel to Oxy. In other words, one is dealing with the two-dimensional analogue of the problem of § 20.

Under certain general conditions, referring to the shape of the cross-section of the cylinder, it may be shown (cf. Part V) that the two-dimensional problem has always a solution which is unique, provided the resultant of the body forces and the stresses acting on the sides is statically equivalent to zero.

Let u, v, X_x, Y_y, X_y be the solution of the two-dimensional problem. Calculating Z_z from (25.3) and assuming $w = Z_x = Z_y = 0$, one obtains the solution satisfying all the conditions above. It is seen that the ends of the cylinder are not free from stresses, but that they are subject to normal stresses. In fact, the normal stress Z_z acts on the upper and $(-Z_z)$ acts on the lower end, where, for simplicity, the end facing in the positive z direction has been called "upper". Application of these stresses is seen to be necessary for the maintenance of plane deformation. As has been stated, the given body forces and stresses, acting on the sides, determine the functions u, v, X_x, Y_y, X_y, and hence also Z_z. Thus the choice of the longitudinal stress is not arbitrary.

At first sight, this fact seems to reduce the value of the study of plane strain. But in practice this inconvenience is very easily removed in the case of a *long* cylinder (the height of which is large compared with the transverse dimensions). In fact, in order to remove the above stresses on the ends, it is sufficient to superimpose on the obtained solution the solution of the problem of the equilibrium of the cylinder under the condition that the sides are free from external stresses and the ends are subject to tractions equal in magnitude and opposite in sign to those which are to be removed.

Consider these latter tractions, exerted on one of the ends; since

they are parallel to the axis Oz, their resultant is statically equivalent to a force parallel to the same axis, acting, say, at the centroid of the end, and a couple the plane of which is likewise parallel to Oz. The resultant of the stresses, acting on the other end, is statically equivalent to a force and couple, statically balancing the former. But the question of the elastic equilibrium of a (long) cylinder under the influence of tractions, applied to the ends and statically equivalent to a tensile force and a bending couple, belongs to a number of very simple problems of the theory of elasticity and can be solved by elementary methods (cf. Part VII). Therefore one can always remove the tractions on the ends by very simple means.

Thus, from the solution of the problem of plane strain of a cylinder under tractions of the stated type, applied to the side surfaces, one obtains the solution of the problem of equilibrium of a cylinder under the influence of the same forces, but subject to the conditions that the ends are free from stresses; in this latter case, generally speaking, deformation will no longer be plane.

§ 26. Deformation of a thin plate under forces acting in its plane. The equations of the plane theory of elasticity apply also to another case, yet more important in practice, namely to the case of thin plates for definite types of loading.

By a plate will be understood a cylinder of very small height or *thickness* $2h$. The *middle surface* of the plate (i.e., the plane parallel to the ends and half way between them) is taken as the plane Oxy (Fig. 12).

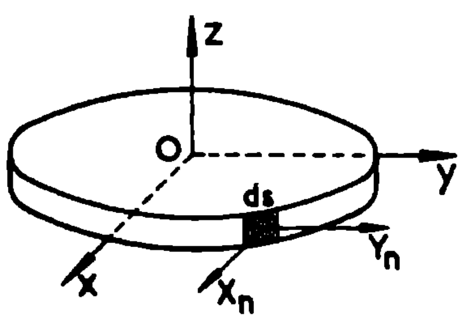


Fig. 12

It will be assumed that the faces are free from external stresses and it will be postulated that the external stresses acting on the edges are parallel to the faces and symmetrically distributed with respect to the middle surface. The same will be assumed to hold true for the body forces. From the practical point of view, it is sufficient, as far as the stresses acting on the edges are concerned, to assume that the resultant of the stresses acting on any element of the edge, included between the two faces, is statically equivalent to a force, applied at the centre of the element and lying in the middle surface; in fact, by St. Venant's Principle (§ 23), every such

resultant may be replaced by a statically equivalent resultant satisfying the earlier conditions.

For reasons of symmetry, it is obvious that the points of the middle surface will remain in it after deformation, that the displacement component w will be very small and that the variations of the components u and v over the thickness of the plate will be insignificant. Therefore it is clear that it is possible to obtain a completely satisfactory representation of the elastic equilibrium of the plate by considering the mean values of the quantities u and v over the thickness of the plate; these mean values which will be denoted by u^* and v^* are defined by

$$u^*(x, y) = \frac{1}{2h}\int_{-h}^{+h} u(x, y, z)dz, \quad v^*(x, y) = \frac{1}{2h}\int_{-h}^{+h} v(x, y, z)dz.$$

By assumption, the functions $X_z(x, y, z)$, $Y_z(x, y, z)$ and $Z_z(x, y, z)$ vanish on the ends, i.e., for $z = \pm h$ (since the ends are free from external stresses). Therefore it follows from

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = 0$$

that

$$\frac{\partial Z_z}{\partial z} = 0$$

for $z = \pm h$. In fact, it follows from $Z_x(x, y, \pm h) = 0$ that

$$\frac{\partial Z_x(x, y, \pm h)}{\partial x} = 0,$$

and similarly that

$$\frac{\partial Z_y(x, y, \pm h)}{\partial y} = 0.$$

Thus the quantity $Z_z(x, y, z)$ is not only zero for $z = \pm h$, but also its derivative with respect to z vanishes for these values. Therefore it is obvious that Z_z will be a very small quantity throughout the thickness of the plate and one may assume, as a good approximation, that $Z_z = 0$ everywhere.

Consider now the equations

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + X = 0, \quad \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + Y \quad 0,$$

and take the mean values of both these equations, i.e., integrate them with respect to z from $-h$ to $+h$ and divide by $2h$. One has

$$\frac{1}{2h}\int_{-h}^{+h}\frac{\partial X_z}{\partial z}\,dz = [X_z]_{-h}^{+h} = 0, \quad \frac{1}{2h}\int_{-h}^{+h}\frac{\partial Y_z}{\partial z}\,dz = [Y_z]_{-h}^{+h} = 0,$$

and hence the preceding equations become

$$\frac{\partial X_x^*}{\partial x} + \frac{\partial X_y^*}{\partial y} + X^* = 0, \quad \frac{\partial Y_x^*}{\partial x} + \frac{\partial Y_y^*}{\partial y} + Y^* = 0. \tag{26.1}$$

Further, it follows from

$$\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z} = Z_z = 0$$

that

$$\frac{\partial w}{\partial z} = -\frac{\lambda}{\lambda + 2\mu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right).$$

Substituting this value of $\frac{\partial w}{\partial z}$ in

$$X_x = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial u}{\partial x}, \quad Y_y = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial v}{\partial y},$$

one obtains

$$X_x = \frac{2\lambda\mu}{\lambda + 2\mu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x}, \quad Y_y = \frac{2\lambda\mu}{\lambda + 2\mu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y}.$$

Taking the mean value of these two equations and of the equation

$$X_y = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$

one finally finds

$$\begin{gathered} X_x^* = \lambda^*\theta^* + 2\mu\frac{\partial u^*}{\partial x}, \quad Y_y^* = \lambda^*\theta^* + 2\mu\frac{\partial v^*}{\partial y} \\ X_y^* = \mu\left(\frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x}\right), \end{gathered} \tag{26.2}$$

where

$$\begin{gathered} \lambda^* = \frac{2\lambda\mu}{\lambda + 2\mu} = \frac{E\sigma}{1 - \sigma^2}, \\ \theta^* = \frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}. \end{gathered} \tag{26.3}$$

Comparison of (26.1) and (26.2) with (25.1) and (25.2) shows that the mean values of the displacement components u, v and the stress components X_x, Y_y, X_y satisfy the same equations which govern the case of plane strain, the only difference being that one has to replace λ by λ^* defined in (26.3).

Following A. E. H. Love [1] (§§ 94 and 146), the stressed state of a plate, for which $Z_z = 0$ everywhere and X_z, Y_z vanish on its faces, will be called "generalized plane stress". Such a state of stress was first considered by L. N. G. Filon [1] (cf. also: Filon [2], E. G. Coker and Filon [1]) who established the above equations for the mean values. These equations are, of course, applicable to plates of finite thickness. It has been seen that for thin plates and under the conditions, stated above, the state of stress may, with good approximation, be assumed to be one of generalized plane stress. For further justification of the assumption that in the case of a thin plate: $Z_z = 0$, reference may be made to J. H. Michell [1] who furnished additional evidence with regard to this point.

Let ds be any line element in the plane Oxy. Consider a rectangular area of height $2h$, perpendicular to Oxy, the trace of which in that plane is ds (Fig. 12). The components of the mean stress, acting on this area, in the directions Ox, Oy are

$$X_n^*\,ds,\quad Y_n^*\,ds,$$

where

$$\begin{aligned} X_n^* &= X_x^* \cos(n, x) + X_y^* \cos(n, y), \\ Y_n^* &= Y_x^* \cos(n, x) + Y_y^* \cos(n, y), \end{aligned} \tag{26.4}$$

and n is the positive normal.

§ 27. Basic equations of the plane theory of elasticity.

It has been seen that in the two cases, considered in § 25 and § 26, one is led to the study of the following system of equations:

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + X = 0,\quad \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + Y = 0, \tag{27.1}$$

$$X_x = \lambda\theta + 2\mu\frac{\partial u}{\partial x},\quad Y_y = \lambda\theta + 2\mu\frac{\partial v}{\partial y},\quad X_y = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{27.2}$$

where

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \tag{27.3}$$

In the case of generalized plane stress (§ 26) the components of displacement and stress have to be replaced by their mean values over the thickness of the plate and λ by $\lambda^* = \frac{2\lambda\mu}{\lambda + 2\mu}$.

Since all quantities depend only on x and y, consideration may be limited to points of the plane Oxy. Therefore, when talking, for example, of a region occupied by a body, one will have in mind a two-dimensional region, i.e., the intersection of the considered body with the plane Oxy; further, instead of talking about tractions acting on areas, one will speak of tractions acting on line elements of that cross-section.

As in Part I it will be assumed that the components of displacement are single-valued continuous functions with continuous derivatives up to and including the third order throughout the region occupied by the body. Then, by (27.2), the stress components will be single-valued functions with continuous second order derivatives.

Just as in § 21, the system (27.1) and (27.2) may be replaced by one involving only displacements. For this purpose one has only to put $w = 0$ in the equations of § 21, or simply to substitute from (27.2) in (27.1). In either way one finds

$$(\lambda + \mu)\frac{\partial\theta}{\partial x} + \mu\Delta u + X = 0, \quad (\lambda + \mu)\frac{\partial\theta}{\partial y} + \mu\Delta v + Y = 0, \tag{27.4}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Having found some solution of this system, the corresponding stresses are obtained from (27.2) by differentiation.

It is likewise not difficult to form the *equations which involve only stresses.* It is now seen that these equations comprise the equations (27.1) and *one* supplementary equation which replaces in the present case the six conditions of compatibility of Beltrami-Michell. This additional equation expresses the condition which must be fulfilled so that one may find, corresponding to functions X_x, Y_y, X_y satisfying (27.1), functions u, v related to X_x, Y_y, X_y by (27.2). This condition may, of course, be obtained as a particular case of the general compatibility conditions, but it will be deduced here independently in two ways.

The first method is based on St. Venant's conditions of compatibility, as was the deduction of the conditions of Beltrami-Michell in the general case. Thus in the case of plain strain, when e_{xx}, e_{yy} and e_{xy} are independent

of z and $e_{yz} = e_{zx} = e_{zz} = 0$, the conditions (15.6) obviously reduce to

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2\,\frac{\partial^2 e_{yx}}{\partial x\,\partial y}.$$

Substituting here the expressions

$$\begin{aligned} e_{xx} &= \frac{1}{2\mu}\left\{X_x - \frac{\lambda}{2(\lambda+\mu)}(X_x + Y_y)\right\}, \\ e_{yy} &= \frac{1}{2\mu}\left\{Y_y - \frac{\lambda}{2(\lambda+\mu)}(X_x + Y_y)\right\}, \\ e_{xy} &= \frac{1}{2\mu}X_y, \end{aligned} \tag{27.5}$$

deduced from (27.2), one easily obtains

$$\frac{\partial^2 X_x}{\partial y^2} + \frac{\partial^2 Y_y}{\partial x^2} - \frac{\lambda}{2(\lambda+\mu)}\Delta(X_x + Y_y) - 2\,\frac{\partial^2 X_y}{\partial x\,\partial y} = 0. \tag{27.6}$$

This is the required condition. It may be considerably simplified by taking into consideration that X_x, Y_y, X_y satisfy (27.1). In fact, differentiating the first equation of (27.1) with respect to x and the second with respect to y and adding, one finds

$$-2\,\frac{\partial^2 X_y}{\partial x\,\partial y} = \frac{\partial^2 X_x}{\partial x^2} + \frac{\partial^2 Y_y}{\partial y^2} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}.$$

Substituting this expression for $-2\,\dfrac{\partial^2 X_y}{\partial x\,\partial y}$ in (27.6) one obtains, after some obvious simplifications,

$$\Delta(X_x + Y_y) = -\frac{2(\lambda+\mu)}{\lambda+2\mu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right). \tag{27.7}$$

The second method of deduction of (27.7) is based directly on the equations (27.1) and (27.2) and it presents at the same time a method for calculating displacements from given stress components (or, what is the same thing, strain components). It is more elementary and more convenient in practice than the one given in § 15 for the general case.

Since it is desired to find conditions which must be satisfied by the stress components X_x, Y_y, X_y, so that there exist functions u, v, related to the former by (27.2), it will now be attempted to actually calculate

u, v from (27.2), assuming that X_x, Y_y, X_y represent a given solution of (27.1).

The first two equations of (27.2) may be written

$$
\begin{aligned}
2\mu \frac{\partial u}{\partial x} &= X_x - \frac{\lambda}{2(\lambda+\mu)}(X_x + Y_y), \\
2\mu \frac{\partial v}{\partial y} &= Y_y - \frac{\lambda}{2(\lambda+\mu)}(X_x + Y_y).
\end{aligned}
\tag{27.5'}
$$

Let (a, b) be an arbitrary point of the body. For the present, consideration will be limited to points lying inside some rectangle with centre (a, b) which lies completely inside the body. Putting $P = X_x + Y_y$, one finds from (27.5')

$$
\begin{aligned}
2\mu\, u(x, y) &= \int_a^x \left\{X_x - \frac{\lambda P}{2(\lambda+\mu)}\right\} dx + f_1(y), \\
2\mu\, v(x, y) &= \int^y \left\{Y_y - \frac{\lambda P}{2(\lambda+\mu)}\right\} dy + f_2(x),
\end{aligned}
\tag{27.8}
$$

where $f_1(y)$, $f_2(x)$ are functions, at present unknown. The expressions (27.8) satisfy (27.5'), i.e., the first two relations of (27.2).

In order to satisfy the third equation of (27.2), substitute in it from (27.8). Differentiating under the integral sign, one obtains

$$
\int^x \left\{\frac{\partial X_x}{\partial y} - \frac{\lambda}{2(\lambda+\mu)}\frac{\partial P}{\partial y}\right\} dx + \int^y \left\{\frac{\partial Y_y}{\partial x} - \frac{2}{2(\lambda+\mu)}\frac{\partial P}{\partial x}\right\} dy - 2X_y =
$$
$$
= - f_1'(y) - f_2'(x). \tag{27.9}
$$

This equation may only be satisfied, if the left-hand side can be conceived as the sum of two functions one of which depends only on x and the other only on y. For this to be so, it is necessary and sufficient that the second derivative $\frac{\partial^2}{\partial x\, \partial y}$ of the left-hand side is identically zero. Applying this criterion to (27.9) one finds equation (27.6), and hence (27.7).

If (27.7) is fulfilled, then the left-hand side of (27.9) has the form

$$F_1(y) + F_2(x)$$

and (27.9) leads to the condition

$$F_2(x) + f_2'(x) = - F_1(y) - f_1'(y)$$

which is only possible if both sides are equal to one and the same constant to be denoted by $2\mu\varepsilon$. Then, by the last equation,

$$\begin{aligned} f_1(y) &= -\int_b^y F_1(y)dy - 2\mu\varepsilon y + 2\mu\alpha, \\ f_2(x) &= -\int_a^x F_2(x)dx + 2\mu\varepsilon x + 2\mu\beta, \end{aligned} \tag{27.10}$$

where α, β are arbitrary constants. Substituting from (27.10) into (27.8) one finds expressions for u and v which are definite apart from terms of the form

$$u' = -\varepsilon y + \alpha, \quad v' = \varepsilon x + \beta, \tag{27.11}$$

where α, β, ε are arbitrary constants. These terms express only rigid body displacement (in the plane Oxy) and they do not influence stresses and strains. The constants α, β, ε attain definite values, if one assumes as given the values of the components of displacement u, v and of rotation

$$r = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{27.12}$$

at some point of the region under consideration, e.g. at (a, b).

So far consideration has been limited to points (x, y) lying inside a rectangle with centre (a, b) which is entirely inside the region occupied by the body. In order to find values of u, v at other points of the region, one has to select some point (a', b') inside that rectangle and near its boundary and to construct a second rectangle with (a', b') as centre. This rectangle must again be chosen in such a way that it does not leave the region occupied by the body, *although it will extend beyond the boundaries of the first rectangle.* In this way one may find the values of u, v at all points of the second rectangle by the method presented above. In order that the values of u, v, obtained in this manner, agree in those parts common to both rectangles, one has to select the arbitrary constants, entering into the formulae for the second rectangle, so that the values of u, v and r at (a', b') coincide with those calculated for this point from the formulae for the first rectangle. Hence it is seen that the formulae for the second rectangle will not involve any new arbitrary constants.

By repeating this procedure sufficiently often one may calculate the displacements for any point of the body. (This method may be compared with the well known process of analytic continuation of functions of a complex variable.)

However, there arises the following question. Let (x_1, y_1) be some point of the body different from the initial point (a, b). In order to calculate the values of u, v at (x_1, y_1), one has, by the above method, to construct a set of rectangles, partly covering each other, the first of which is the rectangle with centre (a, b) and the last a rectangle containing (x_1, y_1). But there is an infinite number of such sets. The question is then whether the particular choice of one of these sets will influence the values of u, v at (x_1, y_1); in other words, whether u, v will be *single-valued* functions of (x_1, y_1).

This question is easily resolved by methods differing from those of the present section, using formulae expressing the displacement components u, v in terms of the stress components X_x, Y_y, X_y by means of curvilinear integrals taken along arbitrary curves linking the points (a, b) and (x_1, y_1). These formulae follow from (15.4) by putting there $w = e_{yz} = e_{zx} = e_{zz} = 0$ and by replacing the components of strain e_{xx}, e_{yy}, e_{xy} by their expressions (27.5) in terms of the stress components X_x, Y_y, X_y. Proceeding in quite an analogous manner as in § 15, it is easily verified that u, v are necessarily single-valued functions, provided the region occupied by the body is *simply connected*.

In the case of multiply connected regions the components u, v may be found to be multi-valued functions, in spite of the fact that (27.7) is satisfied. Therefore, in the case of multiply connected regions, (27.7) must be supplemented by a *condition of single-valuedness of displacements*, where, of course, it has been assumed that the stress components are always single-valued functions. Later on this question will be considered in greater detail.

The necessity of the condition (27.7) may also be inferred in the following manner: Differentiating the first of the equations (27.4) with respect to x and the second with respect to y and adding, one obtains

$$(\lambda + 2\mu)\Delta\theta + \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = 0.$$

Further, noting that by the first two relations of (27.2)

$$\theta = \frac{X_x + Y_y}{2(\lambda + \mu)}$$

and substituting this value of θ in the preceding equation, one obtains again (27.7).

§ 28. Reduction to the case of absence of body forces. The solution of the equations of the plane theory of elasticity is considerably simplified in the case of absence of body forces, i.e., when $X = Y = 0$. On the other hand, the general case may always be reduced to the last: for this purpose it is sufficient to find *any* particular solution of the system of equations (27.1) and (27.2). Let $X_x^{(0)}, Y_y^{(0)}, X_y^{(0)}, u^{(0)}, v^{(0)}$ be such a particular solution. Putting

$$X_x = X_x^{(1)} + X_x^{(0)} \text{ etc.}, \quad u = u^{(1)} + u^{(0)} \text{ etc.}, \tag{28.1}$$

it is seen that the functions $X_x^{(1)}, \ldots, v^{(1)}$ satisfy the same equations as $X_x, \ldots, v$, but for $X = Y = 0$.

The determination of particular solutions $X_x^{(0)}, \ldots, v^{(0)}$ will be limited here to two cases which cover most practical applications: the case of *gravity* and the case of *centrifugal forces* for rotation about an axis parallel to Oz. However, the determination of a particular solution for arbitrarily given body forces does not present any particular difficulties.

In order to find the particular solutions, one may, from a point of view of convenience, either use the equations (27.1) and (27.7) which involve stresses or the equations (27.4) in terms of displacements. The first set of equations will here be used for the problem of gravity forces and the second for the case of inertia forces.

Consider first the case of gravity forces. Assuming that the axis Oy is directed vertically upwards, one has $X = 0$, $Y = -g\rho$, where g is the gravitational acceleration and ρ is the density which will be assumed constant.

Therefore (27.1) and (27.7) take the form

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = 0, \quad \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} = \rho g, \quad \Delta(X_x + Y_y) = 0.$$

Clearly these equations will be satisfied by putting, for example,

$$X_x = X_y = 0, \quad Y_y = \rho g y. \tag{28.2}$$

The displacements corresponding to this particular solution may be calculated in the manner stated earlier. In fact, by (27.8),

$$2\mu u = \int -\frac{\lambda \rho g y}{2(\lambda + \mu)}\, dx = -\frac{\lambda \rho g}{2(\lambda + \mu)}\, xy + f_1(y),$$

$$2\mu v = \int \frac{\lambda + 2\mu}{2(\lambda + \mu)}\, \rho g y \, dy = \frac{\lambda + 2\mu}{4(\lambda + \mu)}\, \rho g y^2 + f_2(x).$$

Substituting these values in

$$\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=X_y=0,$$

one obtains the equation

$$-\frac{\lambda\rho g}{2(\lambda+\mu)}\,x+f_1'(y)+f_2'(x)=0$$

which may be satisfied by putting, for example,

$$f_1(y)=0,\quad f_2(x)=\frac{\lambda\rho g}{4(\lambda+\mu)}\,x^2.$$

Thus, one has for the displacements

$$u=-\frac{\lambda}{4\mu(\lambda+\mu)}\,\rho gxy,\quad v=\frac{\lambda+2\mu}{8\mu(\lambda+\mu)}\,\rho gy^2+\frac{\lambda}{8\mu(\lambda+\mu)}\,\rho gx^2. \tag{28.3}$$

Next use (27.4) to solve the problem of inertia forces. If the body is rotating uniformly about an axis, perpendicular to the plane Oxy and passing through O, the inertia (centrifugal) forces are given by

$$X=\rho\omega^2x,\quad Y=\rho\omega^2y,$$

where ω is the angular velocity. Hence (27.4) takes the form

$$(\lambda+\mu)\frac{\partial\theta}{\partial x}+\mu\Delta u+\rho\omega^2x=0,\quad (\lambda+\mu)\frac{\partial\theta}{\partial y}+\mu\Delta v+\rho\omega^2y=0.$$

It is easily seen that these equations will be satisfied by expressions of the form

$$u=ax^3+bxy^2,\quad v=ay^3+bx^2y.$$

In fact, substituting these values in the preceding equations, it is seen that both will be satisfied, if

$$2(3a+b)\,(\lambda+2\mu)+\rho\omega^2=0,$$

or

$$3a+b=-\frac{\rho\omega^2}{2(\lambda+2\mu)}. \tag{28.4}$$

Thus one of the constants a, b may be chosen arbitrarily. For example, put

$$a=b=-\frac{\rho\omega^2}{8(\lambda+2\mu)}$$

in which case the displacement is purely radial, since then

$$u = -\frac{\rho\omega^2}{8(\lambda + 2\mu)}(x^2 + y^2)x, \quad v = -\frac{\rho\omega^2}{8(\lambda + 2\mu)}(x^2 + y^2)y. \tag{28.5}$$

The corresponding stresses are given by

$$\begin{aligned} X_x &= \frac{2\lambda + \mu}{4(\lambda + 2\mu)}\rho\omega^2(x^2 + y^2) \qquad \frac{}{2(\lambda + 2\mu)}\rho\omega^2 x^2 \\ Y_y &= \frac{2\lambda + \mu}{4(\lambda + 2\mu)}\rho\omega^2(x^2 + y^2) - \frac{\mu}{2(\lambda + 2\mu)}\rho\omega^2 y^2, \\ X_y &= -\frac{\mu\rho\omega^2}{2(\lambda + 2\mu)}xy. \end{aligned} \tag{28.6}$$

CHAPTER 5

STRESS FUNCTION. COMPLEX REPRESENTATION OF THE GENERAL SOLUTION OF THE EQUATIONS OF THE PLANE THEORY OF ELASTICITY

§ 29. Stress function. In the sequel (unless stated otherwise) attention will be concentrated on the equations of the plane theory of elasticity when no body forces are present. In that case the stresses may be expressed by means of one single auxiliary function which is called a *stress function* or *Airy function* and which plays an important part in the plane theory of elasticity.

In fact, under the conditions considered, one has

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = 0, \quad \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} = 0. \tag{29.1}$$

The first of these equations represents the necessary and sufficient condition for the existence of some function $B(x, y)$ such that

$$\frac{\partial B}{\partial x} = X_y, \quad \frac{\partial B}{\partial y} = X_x.$$

The second of the equations (29.1) is the necessary and sufficient condition for the existence of some function $A(x, y)$ such that

$$\frac{\partial A}{\partial x} = Y_y, \quad \frac{\partial A}{\partial y} = -X_y.$$

Comparison of the two expressions for X_y shows that one must have

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x},$$

whence follows the existence of some function $U(x, y)$ such that

$$A = \frac{\partial U}{\partial x}, \quad B = \frac{\partial U}{\partial y}.$$

Substituting these values for A and B in the preceding equations, it is

seen that (in the absence of body forces) there always exists some function $U(x, y)$ by the help of which the stresses may be expressed in the following manner:

$$X_x = \frac{\partial^2 U}{\partial y^2}, \quad X_y = -\frac{\partial^2 U}{\partial x \partial y}, \quad Y_y = \frac{\partial^2 U}{\partial x^2}. \tag{29.2}$$

This fact was first noticed by G. B. Airy (1862). The function U is called a *stress function* or *Airy function.*

Since, by a hypothesis in § 27, the functions X_x, Y_y, X_y are single-valued and continuous together with their second order derivatives, the function U must have continuous derivatives up to and including the fourth order and these derivatives, from the second order onwards, must be single-valued functions throughout the region, occupied by the body.

Conversely, it is obvious that, if U has these properties, the functions X_x, Y_y, X_y, defined by (29.2), will satisfy (29.1). However, it is known that this does not yet mean that these functions correspond to some actual deformations. For this purpose also the condition (27.7) must be satisfied which in the absence of body forces becomes

$$\Delta(X_x + Y_y) = 0; \tag{29.3}$$

or, noting that

$$X_x + Y_y = \Delta U,$$

one obtains the equation

$$\Delta\Delta U = 0 \qquad \text{or} \qquad \frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0. \tag{29.4}$$

Equation (29.4) is called ***biharmonic*** and its solutions ***biharmonic functions.*** J. C. Maxwell was the first person to notice that the stress function must satisfy (29.4).

However, in the sequel, ***biharmonic functions*** will be understood to be only ***functions***, which satisfy the biharmonic equation, ***the derivatives of which are continuous up to and including the fourth order*** and ***the derivatives of which, starting from the second order, are single-valued throughout the region under consideration.***

If the considered region is simply connected, single-valuedness of the second derivatives implies that of the function itself. In multiply connected regions, however, this is not necessarily so, as will be shown later.

Thus it has been proved that the stress functions must be biharmonic. It is known that this condition, which is nothing else but the condition

(27.7), is also a sufficient condition that the corresponding stresses may be produced by some actual deformation, if for the time being no importance is attached to the fact that the corresponding displacements may (in the case of multiply connected regions) turn out to be multi-valued.

NOTE. In § 27 some restrictions have been imposed on the considered displacements and stresses. In fact, it has been agreed to assume that the functions u, v are single-valued and have continuous derivatives up to and including the third order; the continuity and single-valuedness of the stress components and their derivatives up to the second order was a direct consequence of the relations

$$X_x = \lambda\theta + 2\mu\frac{\partial u}{\partial x},\quad Y_y = \lambda\theta + 2\mu\frac{\partial v}{\partial y},\quad X_y = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \tag{29.5}$$

From the point of view of certain deductions of a general character it is convenient to relax these conditions slightly. Thus, everything to be said below will remain true, if from now onwards the following conditions are introduced:

a) *Conditions referring to stresses.* The components X_x, Y_y, X_y are single-valued continuous functions having continuous derivatives up to the second order and satisfying equations (29.1) and (29.3). A consequence of these conditions is that the function U is biharmonic (in the sense stated above).

b) *Conditions referring to displacements.* The components u, v are single-valued, continuous functions having first order derivatives connected with the stress components by (29.5).

It will be seen below that the conditions *a*) ensure the existence of derivatives of any order of the functions X_x, Y_y, X_y; furthermore, it will be seen that these functions are analytic (cf. § 32).

Similarly, the conditions *b*) together with *a*) ensure the existence of derivatives of any order of the functions u, v (cf. § 32) (and even their being analytic). Note that in many cases it is sufficient to adopt the preceding conditions, omitting the condition of single-valuedness of the functions u, v. For example, in the case of simply connected regions this single-valuedness is a necessary consequence of the remaining conditions *a*) and *b*); this follows from the results of the next section.

§ 30. Determination of displacements from the stress function.

If a (biharmonic) stress function U be given, the corresponding stresses

follow from the formulae of § 29, viz.:

$$X_x = \frac{\partial^2 U}{\partial y^2}, \quad Y_y = \frac{\partial^2 U}{\partial x^2}, \quad X_y = -\frac{\partial^2 U}{\partial x \partial y}; \tag{30.1}$$

the displacements, corresponding to these stresses, may be found by the methods of § 27. However, different formulae will be given here which are more convenient than the former and which were first stated by A. E. H. Love [1] who obtained them in a somewhat different manner.

Let *the region S, occupied by the body, for the time being* (up till § 35) *be assumed to be simply connected* (cf. § 15 and Appendix 2 for a definition of connectivity). The present problem is to find functions u, v from the equations

$$\begin{gathered} \lambda\theta + 2\mu \frac{\partial u}{\partial x} = \frac{\partial^2 U}{\partial y^2}, \quad \lambda\theta + 2\mu \frac{\partial v}{\partial y} = \frac{\partial^2 U}{\partial x^2} \\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = -\frac{\partial^2 U}{\partial x \partial y}. \end{gathered} \tag{30.2}$$

The first two of these equations, solved for

$$\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial y},$$

give

$$2\mu \frac{\partial u}{\partial x} = \frac{\partial^2 U}{\partial y^2} - \frac{\lambda}{2(\lambda + \mu)} \Delta U, \quad 2\mu \frac{\partial v}{\partial y} = \frac{\partial^2 U}{\partial x^2} - \frac{\lambda}{2(\lambda + \mu)} \Delta U.$$

Introducing the notation

$$\Delta U = P, \tag{30.3}$$

replacing in the first of the above equations $\frac{\partial^2 U}{\partial y^2}$ by $P - \frac{\partial^2 U}{\partial x^2}$ and in the second $\frac{\partial U}{\partial x^2}$ by $P - \frac{\partial^2 U}{\partial y^2}$, one obtains

$$2\mu \frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x^2} + \frac{\lambda + 2\mu}{2(\lambda + \mu)} P, \quad 2\mu \frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + \frac{\lambda + 2\mu}{2(\lambda + \mu)} P. \tag{30.4}$$

From (30.3) the function P is seen to be harmonic, because

$$\Delta P = \Delta\Delta U = 0.$$

Let Q be the harmonic function, conjugate to P, i.e., the function

satisfying the Cauchy-Riemann conditions

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x};$$

this function is determined for a given P apart from an arbitrary constant term (cf. Appendix 3). Then the expression

$$f(z) = P(x, y) + iQ(x, y) \tag{30.5}$$

will represent a function of the complex variable $z = x + iy$, holomorphic in the region S occupied by the body.

A *function, holomorphic* in a given (simply or multiply connected) region, will always be understood to be *single-valued.* In the sequel, an ***analytic function*** of the complex variable z in a given region S will be a function which may be multi-valued, but each continuously varying branch of which will be holomorphic (and, hence, single-valued) in any finite simply connected part of S. The word „analytic" means that such a function (or rather every branch of it) may be developed in the neighbourhood of any point a of the region S into a series of the form

$$A_0 + A_1(z - a) + A_2(z - a)^2 + \ldots\ldots$$

Sometimes a function, analytic in S, will be understood to be a function, analytic (in the above sense) in a region obtained from S by the exclusion of certain definite points; in such cases the necessary stipulations will always be made.

Furthermore, put

$$\varphi(z) = p + iq = \tfrac{1}{4} \int f(z)\, dz. \tag{30.6}$$

Obviously

$$\varphi'(z) = \frac{\partial p}{\partial x} + i\, \frac{\partial q}{\partial x} = \tfrac{1}{4}(P + iQ),$$

whence, noting that by the Cauchy-Riemann conditions

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}, \quad \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x},$$

one obtains

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} = \tfrac{1}{4}P, \quad \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x} = -\tfrac{1}{4}Q. \tag{30.7}$$

Thus

$$P = 4\frac{\partial p}{\partial x} = 4\, \frac{\partial q}{\partial y},$$

and hence (30.4) may be written

$$2\mu\frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x^2} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\frac{\partial p}{\partial x}, \quad 2\mu\frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\frac{\partial q}{\partial y}$$

Integrating, one obtains

$$2\mu u = -\frac{\partial U}{\partial x} + \frac{2(\lambda+2\mu)}{\lambda+\mu}p + f_1(y),$$

$$2\mu v = -\frac{\partial U}{\partial y} + \frac{2(\lambda+2\mu)}{\lambda+\mu}q + f_2(x).$$

Substituting these expressions in the third of the equations (30.2) and noting that

$$\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = 0,$$

one finds

$$f_1'(y) + f_2'(x) = 0,$$

and hence (cf. § 27) that the functions $f_1(y)$ and $f_2(x)$ have the form

$$f_1 = 2\mu(-\varepsilon y + \alpha), \quad f_2 = 2\mu(\varepsilon x + \beta),$$

where α, β, ε are arbitrary constants (the factor 2μ having been introduced for convenience). Omitting these terms, which only give rigid body displacement, one obtains formulae coinciding essentially with those of A. E. H. Love [1]:

$$2\mu\, u = -\frac{\partial U}{\partial x} + \frac{2(\lambda+2\mu)}{\lambda+\mu}p, \quad 2\mu\, v = -\frac{\partial U}{\partial y} + \frac{2(\lambda+2\mu)}{\lambda+\mu}q. \qquad (30.8)$$

Since the function $\varphi(z)$, defined by (30.6), is obviously holomorphic (cf. Appendix 3) in S (which, as will be remembered, was assumed simply connected), the functions u and v will be found to be single-valued throughout S.

Thus it is seen that every biharmonic function, subject to the conditions of § 29, determines some deformation satisfying all the required conditions.

§ 31. Complex representation of biharmonic functions. It will now be shown that every biharmonic function $U(x, y)$ of the two variables x, y may be represented in a very simple manner by the help of two

functions of the complex variable $z = x + iy$. This fact is of greatest importance for the theory of the biharmonic equation and, in particular, for the plane theory of elasticity, since the properties of functions of a complex variable are generally well known.

The function

$$\varphi(z) = p + iq$$

has already been introduced by (30.6). It is easily verified directly, using (30.7), that the function $U - px - qy$ is harmonic, i.e., that

$$\Delta(U - px - qy) = 0.$$

Hence

$$U = px + qy + p_1,$$

where p_1 is some function harmonic in the region S under consideration. Now let $\chi(z)$ denote the function of the complex variable z, the real part of which is p_1. (In order to find $\chi(z)$, one has to calculate the harmonic function q_1, conjugate to p_1.) If the region S is simply connected, the function $\chi(z)$ will be holomorphic there.

Obviously one may then write

$$U = \Re\{\bar{z}\varphi(z) + \chi(z)\}, \tag{31.1}$$

where $\Re$ denotes "the real part" and

$$\bar{z} = x - iy;$$

in general, if A is some complex number $a + ib$, then $\bar{A}$ will denote its conjugate complex value $a - ib$, so that, for example,

$$\varphi(z) = p - iq.$$

With this notation, (31.1) may be written

$$2U = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}. \tag{31.2}$$

This is the required expression. It was first given by E. Goursat [2] in a somewhat different form, his method of deduction being likewise different. However, in the sequel, no use will be made of this expression for U, but of expressions for its partial derivatives, since these derivatives have direct physical meaning.

The method of deduction used by Goursat is as follows. Let there be given the equation

$$\Delta\Delta U = 0.$$

Introduce instead of x and y the new variables $z = x + iy$ and $\bar{z} = x - iy$; then the preceding equation takes the form

$$\frac{\partial^4 U}{\partial z^2\, \partial \bar{z}^2} = 0,$$

whence it follows directly that

$$U = \varphi_1(z) + \varphi_2(\bar{z}) + \bar{z}\chi_1(z) + z\chi_2(\bar{z}),$$

where φ_1, φ_2, χ_1, χ_2 are "arbitrary" functions. This formal approach may be well justified, if one assumes beforehand that U is analytic. If U is a real function, it is easily seen that one must put

$$\varphi_2(\bar{z}) = \overline{\varphi_1(z)}, \quad \chi_2(\bar{z}) = \overline{\chi_1(z)};$$

hence one obtains (31.2).

The proof, produced in the main text, was first given by the Author [4]. It is somewhat lengthier than that of Goursat, but it does not assume beforehand that every biharmonic function is analytic. On the contrary, this last property follows from (31.2).

It is easily found that

$$\begin{aligned} 2\frac{\partial U}{\partial x} &= \varphi(z) + \bar{z}\overline{\varphi'(z)} + \overline{\varphi(z)} + z\varphi'(z) + \chi'(z) + \overline{\chi'(z)}, \\ 2\frac{\partial U}{\partial y} &= i[-\varphi(z) + \bar{z}\varphi'(z) + \overline{\varphi(z)} - z\overline{\varphi'(z)} + \chi'(z) - \overline{\chi'(z)}]. \end{aligned} \tag{31.3}$$

It is immediately seen from (31.3) that, instead of considering the expressions for

$$\frac{\partial U}{\partial x}, \quad \frac{\partial U}{\partial y},$$

it will be more convenient to deal with the expression for

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}$$

which is by far simpler. In fact, one has

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{31.4}$$

where

$$\psi(z) = \frac{d\chi}{dz} \tag{31.5}$$

Returning to (31.2) it is noticed that, conversely, every expression of the form (31.2) represents a biharmonic function, if $\varphi(z)$, $\chi(z)$ are holomorphic functions of z. In fact, differentiating the first of the equations (31.3) with respect to x and the second with respect to y and adding, one finds

$$\Delta U = 2[\varphi'(z) + \overline{\varphi'(z)}] = 4\Re[\varphi'(z)], \tag{31.6}$$

and hence it follows that ΔU is a harmonic function. Consequently,

$$\Delta\Delta U = 0.$$

The formula (31.6) shows, in addition, that ΔU is completely determined by the real part of the function $\varphi'(z)$.

§ 32. Complex representation of displacements and stresses. Multiplying the second formula of (30.8) by i and adding it to the first, one obtains

$$2\mu(u + iv) = -\left(\frac{\partial U}{\partial x} + i\,\frac{\partial U}{\partial y}\right) + \frac{2(\lambda + 2\mu)}{\lambda + \mu}\,\varphi(z),$$

whence one finds, by (31.4), the very important and convenient formula

$$2\mu(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \tag{32.1}$$

which essentially agrees with a formula, first stated by G. V. Kolosov [1] who obtained it in a different way; in (32.1)

$$\varkappa = \frac{\lambda + 3\mu}{\lambda + \mu} = 3 - 4\sigma. \tag{32.2}$$

In the case of thin plates ("generalized plane stress", § 26) one has to replace $\varkappa$ by $\varkappa^*$, obtained from (32.2) by substituting λ^* for λ. Thus, in this case,

$$\varkappa^* = \frac{\lambda^* + 3\mu}{\lambda^* + \mu} = \frac{3 - \sigma}{1 + \sigma}. \tag{32.2'}$$

Obviously $\varkappa > 1$, $\varkappa^* > 1$.

Next consider the representation of the stress components by means of the same functions φ and ψ. For this purpose an expression will be found for the forces acting on an element of any shape lying in the plane Oxy.

Consider some arc AB in the plane Oxy. Let its positive direction be from A to B and draw the normal n to the right of the arc when looking

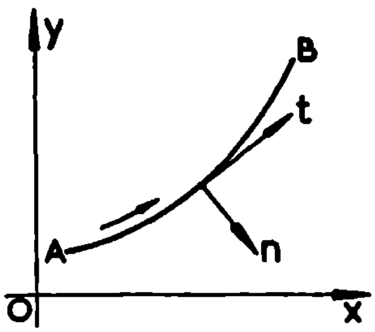


Fig. 13.

along it in the positive direction. In other words, postulate that the positive direction of the normal and the tangent are orientated with respect to each other as the axes Ox and Oy (Fig. 13).

As always, the force $(X_n ds, Y_n ds)$, acting on an element ds of the arc AB, will be understood to be the force exerted on the side of the positive normal. One has

$$X_n = X_x \cos(n, x) + X_y \cos(n, y) = \frac{\partial^2 U}{\partial y^2} \cos(n, x) - \frac{\partial^2 U}{\partial x \partial y} \cos(n, y),$$

$$Y_n = Y_x \cos(n, x) + Y_y \cos(n, y) = -\frac{\partial^2 U}{\partial x \partial y} \cos(n, x) + \frac{\partial^2 U}{\partial x^2} \cos(n, y).$$

But

$$\cos(n, x) = \cos(t, y) = \frac{dy}{ds}, \quad \cos(n, y) = -\cos(t, x) = -\frac{dx}{ds},$$

where t is the positive direction of the tangent. Introducing these values into the preceding formulae, one finds

$$X_n = \frac{d}{ds}\left(\frac{\partial U}{\partial y}\right), \quad Y_n = -\frac{d}{ds}\left(\frac{\partial U}{\partial x}\right), \tag{32.3}$$

or in complex form

$$X_n + iY_n = \frac{d}{ds}\left(\frac{\partial U}{\partial y} - i\frac{\partial U}{\partial x}\right) = -i\frac{d}{ds}\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right), \tag{32.4}$$

or

$$(X_n + iY_n)ds = -i\, d\left(\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right). \tag{32.5}$$

Substituting from (31.4) in (32.5), one obtains

$$(X_n + iY_n)ds = -i\, d\{\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}\}. \tag{32.6}$$

First let the element ds have the direction of the axis Oy. Then

$$ds = dy, \quad dz = i\,dy, \quad d\bar{z} = -i\,dy, \quad X_n = X_x, \quad Y_n = X_y,$$

and (32.6) gives

$$X_x + iX_y = \varphi'(z) + \overline{\varphi'(z)} - z\overline{\varphi''(z)} - \overline{\psi'(z)}. \tag{32.7}$$

Next let ds have the direction of Ox. Then

$$ds = dx, \quad dz = d\bar{z} = dx, \quad X_n = -X_y, \quad Y_n = -Y_y,$$

and (32.6) gives, after multiplication by i,

$$Y_y - iX_y = \varphi'(z) + \overline{\varphi'(z)} + \bar{z}\varphi''(z) + \psi'(z). \tag{32.8}$$

The formulae (32.7) and (32.8) are the required expressions for the stress components. Adding and subtracting (32.7) and (32.8) and replacing in the latter case i by $-i$, one obtains the simpler formulae

$$X_x + Y_y = 2[\varphi'(z) + \overline{\varphi'(z)}] = 4\Re\varphi'(z) = 4\Re\Phi(z) = 2[\Phi(z) + \overline{\Phi(z)}], \tag{32.9}$$

$$Y_y - X_x + 2iX_y = 2[\bar{z}\varphi''(z) + \psi'(z)] = 2(\bar{z}\Phi'(z) + \Psi(z)], \tag{32.10}$$

where

$$\Phi(z) = \varphi'(z), \quad \Psi(z) = \psi'(z). \tag{32.11}$$

The very useful formulae (32.9) and (32.10) are likewise due to G. V. Kolosov [1] who obtained them in a different manner without recourse to the stress function.

The expressions, deduced here for the components of displacement and stress, show that these components, under the earlier stated conditions, are *analytic functions* of the variables x, y inside the considered region, because the functions $\varphi(z)$, $\psi(z)$, $\Phi(z)$, $\Psi(z)$ possess this property.

A function of the real variables x, y is called *analytic* in a given region S, if at each point (x_0, y_0) inside S it may be developed into a (double) series of non-negative powers of $(x - x_0)$, $(y - y_0)$, i.e., into a series of the form

$$\sum_{p,q} a_{p,q}(x - x_0)^p (y - y_0)^q.$$

(This definition may be extended to any number of variables).

As is known, each function of the complex variable $z = x + iy$, holomorphic in a given region, is analytic in the sense that it may be expanded into series of non-negative powers of $(z - z_0)$ near any point $z_0 = x_0 + iy_0$ of that region. On the other hand, it is easily shown (cf. for example E. Goursat [1]) that every analytic function of $z = x + iy$ is an analytic function of x and y.

Finally, a remark will be made with regard to Non-Russian work along the lines of this section. In a recently published paper, A. C. Stevenson [1] deduced formulae which, in essence, agree with those of G. V. Kolosov and also with some of those obtained by the Author of this book, all of which have been published considerably earlier (not only privately, but also in journals well known outside Russia); however, no reference has been made there to this fact.

In a still later publication, H. Poritsky [2] uses formulae which differ only in appearance from those deduced above; in a rather vague reference the author ascribes some of these formulae to the Author, quoting his paper [8] of 1933. However, no mention is made of the Author's earlier work and of that by G. V. Kolosov, although this work (which contained the formulae used and had been published much earlier) is referred to in the quoted paper.

[The following statements were obtained by the translator from the two authors, referred to above.

A. C. Stevenson wrote that at the time when he worked on the paper, quoted by the Author (i.e., 1939—40), he was admittedly ignorant of prior work along these lines. However, in a paper of later origin, published rather earlier and not quoted in this book, he was equally clearly at pains to acknowledge the priority of Kolosov and Muskhelishvili by referring to a total of six papers by G. V. Kolosov, dating as far back as 1909, of four papers by N. I. Muskhelishvili the first of which appeared in 1919 and to the combined paper by both authors, published in 1915.

H. Poritsky indicated that he deduced his formulae in 1931, although his paper was not published until 1945. By that time the Russian work had been given a fair amount of publicity in the U.S.A. and he quoted one of Muskhelishvili's papers merely for the purpose of acknowledging that he had been anticipated.]

§ 33. Expressions for the resultant force and moment. Expressions will be deduced here for the resultant force and moment of the tractions exerted on an arc AB (from the positive side).

Let (X, Y) be the resultant force. It follows from (32.5) and (32.6) that

$$X + iY = \int_{AB} (X_n + iY_n)ds = -i\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right]_A^B =$$
$$= -i[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}]_A^B, \qquad (33.1)$$

where $]_B^A$ will always denote the increase undergone by the expression in the brackets as the point z passes along the arc from A to B.

Next, a formula will be obtained for the resultant moment about the origin of the coordinate system. One has

$$M = \int_{AB} (xY_n - yX_n)ds$$

which, by (32.3), becomes

$$M = -\int_{AB}\left\{x d\frac{\partial U}{\partial x} + y d\frac{\partial U}{\partial y}\right\};$$

integrating by parts, one finds

$$M = -\left[x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}\right]_A^B + \int_{AB}\left\{\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy\right\},$$

and finally

$$M = -\left[x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}\right]_A^B + [U]_A^B. \qquad (33.2)$$

But

$$x \frac{\partial U}{\partial x} + y \frac{\partial U}{\partial y} = \Re\left\{z\left(\frac{\partial U}{\partial x} - i \frac{\partial U}{\partial y}\right)\right\},$$

and, by (31.4),

$$\frac{\partial U}{\partial x} - i \frac{\partial U}{\partial y} = \overline{\varphi(z)} + \bar{z}\varphi'(z) + \psi(z).$$

Further,

$$U = \Re\{\bar{z}\varphi(z) + \chi(z)\},$$

so that (33.2) becomes

$$M = \Re[\chi(z) - z\psi(z) + z\bar{z}\varphi'(z)]_A^B. \tag{33.3}$$

These formulae were first given in the Author's paper [11].

Hitherto it has been assumed that the region S is simply connected, and as a result the functions $\varphi(z)$, $\psi(z)$, $\chi(z)$ will be single-valued in S. Thus, if A and B coincide, i.e., if the considered curve is a contour, the values of these functions will be the same at A and B, and hence one finds

$$X = Y = M = 0, \tag{33.4}$$

as was to be expected. The formulae (33.4) express the fact that the sum of the external forces acting on a part of the body, contained inside any contour, is statically equivalent to zero.

§ 34. Arbitrariness in the definition of the introduced functions.

The important question will now be studied as to how far the functions Φ, Ψ, φ, ψ define the state of stress or the displacements of points of the body.

First consider the problem of the uniqueness of these functions *for a given state of stress*. Expressed in greater detail the problem is as follows. Let X_x, Y_y, X_y be the components of stress for some given state of elastic equilibrium of a body. As has been shown in § 32, there exist functions $\Phi(z)$, $\Psi(z)$ of the complex variable z which are related to X_x, Y_y, X_y by the formulae

$$X_x + Y_y = 4\Re\Phi(z), \tag{34.1}$$

$$Y_y - X_x + 2iX_y = 2[\bar{z}\Phi'(z) + \Psi(z)]; \tag{34.2}$$

the questions are then: how completely are the functions $\Phi(z)$, $\Psi(z)$,

and also the functions

$$\varphi(z) = \int \Phi(z)dz, \quad \psi(z) = \int \Psi(z)dz \tag{34.3}$$

determined by the components X_x, Y_y, X_y and does there remain some arbitrariness in their choice? What is the degree of this arbitrariness?

There is no difficulty in answering these questions. Let Φ_1, Ψ_1, φ_1, ψ_1 be some other system of functions, related to the given components X_x, Y_y, X_y and to each other by the same equations (34.1) to (34.3), as were the functions Φ, Ψ, φ, ψ, i.e.,

$$X_x + Y_y = 4\Re\Phi_1(z), \tag{34.1'}$$

$$Y_y - X_x + 2iX_y = 2[\bar{z}\Phi_1'(z) + \Psi_1(z)], \tag{34.2'}$$

$$\varphi_1(z) = \int \Phi_1(z)dz, \quad \psi_1(z) = \int \Psi_1(z)dz. \tag{34.3'}$$

Consider how the functions Φ_1, Ψ_1, φ_1, ψ_1 may differ from the functions Φ, Ψ, φ, ψ. Comparing (34.1) with (34.1′) it is seen that the functions $\Phi_1(z)$ and $\Phi(z)$ have identical real parts; hence these functions may only differ by an imaginary constant Ci (cf. Appendix 3), so that

$$\Phi_1(z) = \Phi(z) + Ci, \tag{34.4}$$

where C is a real constant.

It follows from (34.4), (34.3) and (34.3′) that

$$\varphi_1 = \varphi(z) + Ciz + \gamma, \tag{34.5}$$

where $\gamma = \alpha + i\beta$ is an arbitrary complex constant. Further, noting that by (34.4): $\Phi_1'(z) = \Phi'(z)$, comparison of (34.2) and (34.2′) obviously gives

$$\Psi_1(z) = \Psi(z), \tag{34.6}$$

and finally, by (34.3) and (34.3′), one finds

$$\psi_1(z) = \psi(z) + \gamma', \tag{34.7}$$

where $\gamma' = \alpha' + i\beta'$ is an arbitrary complex constant. Thus one arrives at the following result:

For a given state of stress the function $\Psi(z)$ is completely defined, the functions $\Phi(z)$, $\varphi(z)$, $\psi(z)$ are defined apart from the terms Ci, $Ciz + \gamma$, γ' respectively, where C is a real and γ, γ' are arbitrary complex constants.

Conversely, it is obvious that a state of stress is not altered, if one replaces

$$\begin{aligned} &\varphi(z) \quad \text{by} \quad \varphi(z) + Ciz + \gamma, \\ &\psi(z) \quad ,, \quad \psi(z) + \gamma', \end{aligned} \tag{A}$$

where C is a real and γ, γ' are arbitrary complex constants. By this substitution $\Phi(z) = \varphi'(z)$ obviously becomes $\Phi(z) + Ci$ and $\Psi(z)$ remains unchanged.

Next investigate the question as to how far the arbitrariness of these functions is removed, *if not only the components of stress but also those of displacement are given.*

The components of displacement completely determine the stress components. Therefore it is clear that, when the former are given, one may not make substitutions different from those of the type (A). Consider how these substitutions affect the components of displacement which were seen in § 32 to be determined by the formula

$$2\mu(u + iv) = \varkappa\varphi(z) - z\varphi'(z) - \psi(z).$$

Direct substitution shows that

$$2\mu(u + iv) \quad \text{becomes} \quad 2\mu(u_1 + iv_1),$$

where

$$2\mu(u_1 + iv_1) = 2\mu(u + iv) + (\varkappa + 1)Ciz + \varkappa\gamma - \bar{\gamma}', \tag{34.8}$$

and hence, putting $\gamma = \alpha + i\beta$, $\gamma' = \alpha' + i\beta'$,

$$u_1 = u + u_0, \qquad v_1 = v + v_0, \tag{34.9}$$

where

$$u_0 = -\frac{(\varkappa + 1)C}{2\mu} y + \frac{\varkappa\alpha - \alpha'}{2\mu}, \quad v_0 = \frac{(\varkappa + 1)C}{2\mu} x + \frac{\varkappa\beta + \beta'}{2\mu}. \tag{34.10}$$

It is thus seen that the additional terms have the form

$$u_0 = -\varepsilon y + \alpha_0, \; v_0 = \varepsilon x + \beta_0, \tag{34.11}$$

where

$$\varepsilon = \frac{(\varkappa + 1)C}{2\mu}, \quad \alpha_0 = \frac{\varkappa\alpha - \alpha'}{2\mu}, \quad \beta_0 = \frac{\varkappa\beta + \beta'}{2\mu}, \tag{34.12}$$

and that they express pure rigid body motion. This result had, of course, to be expected, since the displacements, corresponding to a given state

of stress, are uniquely determined apart from a term describing rigid body displacement.

Formula (34.8) shows that a substitution of the form (A) will affect the displacements, unless

$$C = 0, \quad \varkappa\gamma - \bar{\gamma}' = 0. \tag{34.13}$$

Thus, *for given displacements*, it is impossible to select the constants C, γ, γ' arbitrarily; if, for example, *one* of the constants γ, γ' has been chosen, the other is determined by (34.13).

The arbitrary constants, entering into the above functions, may be given one or the other value which may be convenient. Assuming, for simplicity, that the origin lies within the region S, occupied by the body, these constants will be chosen in the following manner (unless stated otherwise).

When *the stresses are given*, the three constants C, γ, γ' will be chosen in such a way that

$$\varphi(0) = 0, \quad \mathfrak{J}\varphi'(0) = 0, \quad \psi(0) = 0, \tag{34.14}$$

where $\mathfrak{J}$ denotes the imaginary part.

The first of these conditions leads to a suitable choice of γ, the second to that of C and the third to that of γ'. These conditions obviously remove all arbitrariness as far as the functions φ and ψ are concerned.

When *the displacements are given*, a suitable choice of γ will be assured by the condition

$$\varphi(0) = 0 \tag{34.15}$$

which will completely determine the functions φ and ψ.

Note still the following fact. It is obvious that the expression

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} \quad \varphi(z) + z\varphi'(z) + \psi(z) \tag{34.16}$$

completely determines the state of stress of the body, since it determines the quantities $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}$, and hence the second derivatives of U which specify the components of stress. Consider now the question as to what conditions must be fulfilled by the constants C, γ, γ', so that the transformation (A) does not only leave the state of stress, but also the expression (34.16) unchanged.

It is easily verified that, applying (A), this expression becomes

$$\frac{\partial U}{\partial x} \quad i\frac{\partial U}{\partial y} + \gamma + \bar{\gamma}'.$$

Hence, if $\dfrac{\partial U}{\partial x} + i\dfrac{\partial U}{\partial y}$ be given, one must have $\gamma + \bar{\gamma}' = 0$. Thus the constant C and one of the constants γ and γ' may be chosen arbitrarily. One may, for example, put

$$\varphi(0) = 0, \ \Im\{\varphi'(0)\} = 0 \tag{34.17}$$

and in this way completely determine the functions φ and ψ.

§ 35. General formulae for finite multiply connected regions.

Consider now the case when the region S, occupied by the body, is *multiply* connected. For simplicity assume that the region is bounded by several simple closed contours $L_1, L_2, \ldots, L_m, L_{m+1}$ (i.e., by contours which do not intersect themselves; for more detail see § 37); the last of these contours is to contain all the others, as is shown in Fig. 14 (e.g. a plate with holes). Further, assume that these contours have no points in common.

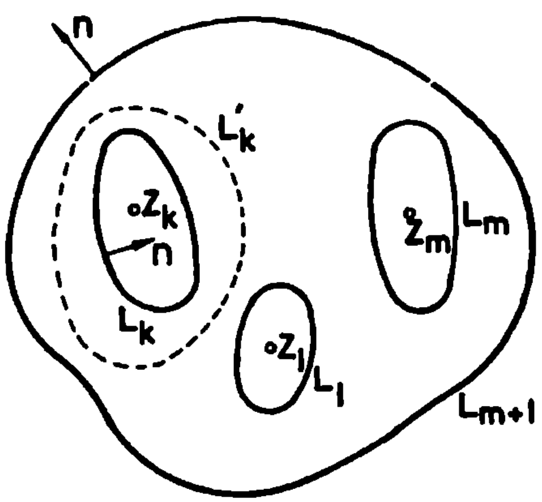


Fig. 14.

It will be remembered that, by supposition, the components of stress and displacement are to be single-valued functions. In spite of this fact the functions φ and ψ may, in this case, be found to be multi-valued. However, it will be noted, on the basis of the statements of the preceding sections, that these functions will be holomorphic and hence single-valued in any simply connected part of the region S occupied by the body. Thus the functions φ and ψ are analytic in S (cf. § 30).

The above circumstances will now be explained in detail. Let S' be some simply connected part of S. One may define the functions φ, ψ, corresponding to a given state of elastic equilibrium of S', by (arbitrarily) fixing the undetermined constants introduced in § 34. These functions have been shown there to be holomorphic in S'. But if one continues these functions analytically beyond S' (remaining, of course,

in S), then, by describing a closed path and returning into S', one may not return to the former values of φ and ψ. However, it is easily seen that the new values of these functions can differ from the old ones only by terms of the form stated in § 34, because both values correspond to one and the same state of elastic equilibrium. This fact also follows from (35.10) and (35.11) below.

The type of multi-valuedness of the relevant functions will now be studied. First of all, the formula

$$X_x + Y_y = 4\Re\Phi(z)$$

shows that the real part of $\Phi(z)$ is single-valued (since, by supposition, the left-hand side of the equation is so). But this does not yet mean that also its imaginary part is single-valued. In fact, for one circuit (e.g. anti-clockwise) around some contour L_k', surrounding one of the contours L_k, this imaginary part may undergo an increase $B_k i$, where B_k is a real constant (cf. Appendix 3). Introduce, instead of the constants B_k, other real constants A_k, defined by

$$B_k = 2\pi A_k.$$

Next consider the function

$$\Phi^*(z) = \Phi(z) - \sum_{k=1} A_k \log (z - z_k), \tag{35.1}$$

where $z_1, z_2, \ldots, z_m$ denote fixed points, arbitrarily chosen inside the contours $L_1, L_2, \ldots, L_m$ (i.e., outside S). Since $\log (z - z_k)$ undergoes an increase $2\pi i$ when z passes once around L_k (anti-clockwise), the expression $A_k \log (z - z_k)$ increases by $2\pi i A_k$; the remaining terms under the summation sign in (35.1) will revert to their former values. Hence $\Phi^*(z)$ returns to its original value for a circuit around any closed contour in S.

Thus one has

$$\Phi(z) = \sum_{k=1} A_k \log (z - z_k) + \Phi^*(z), \tag{35.2}$$

where $\Phi^*(z)$ is holomorphic and hence single-valued in S. Further, one obtains for $\varphi(z)$

$$\varphi(z) = \int_{z_0}^{z} \Phi(z)dz + \text{const.} =$$

$$= \sum_{k=1}^{m} A_k\{(z - z_k) \log (z - z_k) - (z - z_k)\} + \int_{z_0}^{z} \Phi^*(z)dz + \text{const.}$$

where z_0 is an arbitrarily fixed point of S. But the integral $\int_{z_0}^{z} \Phi^*(z)dz$ represents itself a function of the complex variable z which for a circuit of one of the contours L_k may undergo an increase of the form

$$2\pi i c_k,$$

where c_k is a constant which, in general, will be complex (and the factor $2\pi i$ has been introduced for convenience). Hence, proceeding as before, one can write

$$\int_{z_0}^{z} \Phi^*(z)dz = \sum_{k=1}^{m} c_k \log (z - z_k) + \text{a single-valued function.}$$

Introducing this expression into the preceding formula and combining terms of the form $A_k z_k \log (z - z_k)$ and $c_k \log (z - z_k)$, one obtains

$$\varphi(z) = z \sum_{k=1}^{m} A_k \log (z - z_k) + \sum_{k=1}^{m} \gamma_k \log (z - z_k) + \varphi^*(z), \tag{35.3}$$

where $\varphi^*(z)$ is a function, holomorphic in S, and γ_k are constants (which, in general, are complex).

Finally, it is seen from

$$Y_y - X_x + 2iX_y = 2[\bar{z}\Phi'(z) + \Psi(z)]$$

that $\Psi(z)$ is a holomorphic function. Whence it follows that the function

$$\psi(z) = \int \Psi(z)dz,$$

as before, may be written

$$\psi(z) = \sum_{k=1}^{m} \gamma_k' \log (z - z_k) + \psi^*(z), \tag{35.4}$$

where γ_k' are certain (generally complex) constants and $\psi^*(z)$ is a holomorphic function.

In an analogous manner one has for the function

$$\chi(z) = \int \psi(z)dz$$

the expression

$$\chi(z) = z \sum_{k=1}^{\cdots} \gamma_k' \log (z - z_k) + \sum_{k=1}^{\cdots} \gamma_k'' \log (z - z_k) + \chi^*(z), \tag{35.5}$$

where γ_k'' are (generally complex) constants and $\chi^*(z)$ is a holomorphic function.

Hitherto no consideration has been given to the *condition of single-valuedness of displacements*. By (32.1) one has

$$2\mu(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}.$$

Substituting in this formula the expressions found above for $\varphi(z)$ and $\psi(z)$, it is immediately seen that

$$2\mu[u + iv]_{L'_k} = 2\pi i\{(\varkappa + 1)A_k z + \varkappa\gamma_k + \bar{\gamma}'_k\}, \tag{35.6}$$

where $[\quad]_{L'_k}$ denotes the increase undergone by the expression in brackets for one anti-clockwise circuit of the contour L'_k. Hence it is necessary and sufficient for the single-valuedness of displacements that in the formulae (35.1) — (35.5)

$$A_k = 0, \quad \varkappa\gamma_k + \bar{\gamma}'_k = 0, \quad k = 1, 2, \ldots, m. \tag{35.7}$$

It will now be shown that the quantities γ_k, γ'_k may be very simply expressed in terms of X_k, Y_k, where (X_k, Y_k) denotes the resultant vector of the external forces, exerted on the contour $L_k (k = 1, 2, \ldots, m)$. By (33.1), applying it to the contour L_k, one has

$$X_k + iY_k \qquad \cdot i[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}]_{L_k}.$$

In this formula it has been assumed that the contour is traversed in the direction for which the normal n points to the right. But in the present case the normal n must be directed outwards with respect to the region S (Fig. 14), because one requires the resultant vector of the *external* forces. Consequently, in the preceding formula, the contour L_k must be traversed in the clockwise direction (assuming, of course, that the axes Ox, Oy are right-handed; see Fig. 14). Taking this fact into consideration, one easily obtains, using (35.3) and (35.4),

$$X_k + iY_k = -2\pi(\gamma_k - \bar{\gamma}'_k). \tag{35.8}$$

Formulae (35.7) and (35.8) give then

$$\gamma_k = -\frac{X_k + iY_k}{2\mu(1 + \varkappa)}, \quad \gamma'_k \quad \frac{\varkappa(X_k - iY_k)}{2\pi(1 + \varkappa)} \tag{35.9}$$

Using (35.9) (and also the fact that $A_k = 0$), formulae (35.3) and

(35.4) may finally be written

$$\varphi(z) = -\frac{1}{2\pi(1+\varkappa)} \sum_{k=1}^{m} (X_k + iY_k) \log (z - z_k) + \varphi^*(z), \quad (35.10)$$

$$\psi(z) = \frac{\varkappa}{2\pi(1+\varkappa)} \sum_{k=1}^{m} (X_k - iY_k) \log (z - z_k) + \psi^*(z). \quad (35.11)$$

NOTE. It is clear from the deduction above (and also from physical considerations) that (X_k, Y_k) represents the resultant vector of the forces exerted (from the relevant side) on any simple contour L'_k, surrounding L_k.

§ 36. Case of infinite regions. From the point of view of application the consideration of infinite regions is likewise of major interest. For the present the investigation will be limited to the case when the region S consists of the entire plane Oxy from which finite parts, bounded by simple contours, have been removed (infinite plate with holes). The boundary of such a region consists of several simple contours $L_1, L_2, \ldots, L_m$, which is the limiting case of the region considered in the preceding section, when the contour L_{m+1} has entirely moved to infinity.

The formulae of § 35 hold, of course, for any finite part of S. There only remains to study the behaviour of the functions φ and ψ in the neighbourhood of the point at infinity in the plane Oxy.

Draw about the origin as centre a circle L_R with radius R sufficiently large so that all the contours L_k lie inside L_R. For every point outside L_R one obviously has

$$z_k$$

and hence

$$\log (z - z_k) = \log z + \log\left(1 - \frac{z_k}{z}\right) = \log z - \frac{z_k}{z} - \frac{1}{2}\left(\frac{z_k}{z}\right)^2 - \ldots =$$

$$= \log z + \text{a function, holomorphic outside } L_R.$$

Therefore, by (35.10) and (35.11),

$$\varphi(z) = -\frac{X+iY}{2\pi(1+\varkappa)} \log z + \varphi^{**}(z), \quad \psi(z) = \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)} \log z + \psi^{**}(z), \quad (36.1)$$

where

$$X = \sum_{k=1}^{m} X_k, \quad Y = \sum_{k=1}^{m} Y_k \quad (36.2)$$

and $\varphi^{**}(z)$, $\psi^{**}(z)$ are functions, holomorphic outside L_R with the possible exclusion of the point at infinity. Obviously X and Y are the components of the resultant vector of all external forces acting on the boundary of S, i.e., on the union of the contours $L_1, L_2, \ldots, L_m$.

A function will be called holomorphic at the point $z = \infty$, if in the neighbourhood of that point (i.e., for sufficiently large $|z|$) it may be represented by a series of the form

$$a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots .$$

By the theorem of Laurent, the functions $\varphi^{**}(z)$ and $\psi^{**}(z)$ may be represented outside L_R by the series

$$\varphi^{**}(z) = \overset{-}{\Sigma}\, a_n z^n, \quad \psi^{**}(z) = \overset{-}{\Sigma}\, a_n' z^n \tag{36.3}$$

which will converge uniformly for every finite region outside L_R. The above theorem is known to hold for a function, holomorphic inside a ring bounded by two concentric circles L_1 and L_2, where L_1 may be shrunk into a single point and L_2 may become infinitely large.

This is all that may be said with regard to the functions φ and ψ, unless additional conditions are introduced with respect to the distribution of stresses in the neighbourhood of the point at infinity of the plane.

Introduce now the following condition: *the components of stress are bounded throughout the region S*. Consider what must be the functions φ and ψ, in order that this condition is satisfied.

By (32.9) and (32.10)

$$X_x + Y_y = 2[\varphi'(z) + \overline{\varphi'(z)}], \tag{a}$$

$$Y_y - X_x + 2iX_y = 2[\bar{z}\varphi''(z) + \psi'(z)]. \tag{b}$$

Introduce into the first of these formulae the expression (36.1) for $\varphi(z)$, replacing $\varphi^{**}(z)$ by (36.3):

$$X_x + Y_y = 2\left\{\frac{X+iY}{2\pi(1+\varkappa)}\frac{1}{z} - \frac{X-iY}{2\pi(1+\varkappa)}\frac{1}{\bar{z}} + \sum_{-\infty}^{\infty} n(a_n z^{n-1} + \bar{a}_n \bar{z}^{n-1})\right\}.$$

The only terms which may grow beyond all bounds with $|z|$ arise from the series

$$\sum_{n=2}^{\infty} n(a_n z^{n-1} + \bar{a}_n \bar{z}^{n-1}) = \sum_{n=2}^{\infty} n r^{n-1}\,[a_n e^{(n-1)i\vartheta} + \bar{a}_n e^{-(n-1)i\vartheta}],$$

where $z = re^{i\vartheta}$. Whence it follows that for $X_x + Y_y$ to remain finite, as $r \to \infty$, one must have

$$a_n = \bar{a}_n = 0 \quad (n \geqslant 2).$$

Assuming this condition to be satisfied, it is easily seen in an analogous manner from (b) that it is neccesary and sufficient for the boundedness of

$$Y_y - X_x + 2iX_y$$

that

$$\sum_2^\infty nr^{n-1} a_n' e^{(n-1)i\vartheta}$$

remains bounded, whence it follows that

$$a_n' = 0 \quad (n \geqslant 2).$$

Conversely, it is obvious that X_x, Y_y, X_y will be bounded, if these conditions are satisfied. Hence one has finally

$$\varphi(z) = -\frac{X + iY}{2\pi(1 + \varkappa)} \log z + \Gamma z + \varphi_0(z), \tag{36.4}$$

$$\psi(z) = \frac{\varkappa(X - iY)}{2\pi(1 + \varkappa)} \log z + \Gamma' z + \psi_0(z), \tag{36.5}$$

where

$$\Gamma = a_1 = B + iC, \quad \Gamma' = a_1' = B' + iC' \tag{36.6}$$

are constants, generally complex, and $\varphi_0(z)$, $\psi_0(z)$ are functions, holomorphic outside L_R, *including the point at infinity*, i.e., for sufficiently large $|z|$ they may be expanded into series of the form

$$\varphi_0(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots, \quad \psi_0(z) = a_0' + \frac{a_1'}{z} + \frac{a_2'}{z^2} + \ldots \tag{36.7}$$

(where, for convenience, a_1, a_2 etc. have been written instead of a_{-1}, a_{-2}, etc.).

On the basis of § 34 the state of stress will not be altered by assuming

$$a_0 = a_0' = 0,$$

i.e.,

$$\varphi_0(\infty) = \psi_0(\infty) = 0,$$

and, in addition,

$$C = 0.$$

The real constants B, B', C', introduced into (36.4) and (36.5) by means

of Γ and Γ', have a very simple physical interpretation. In fact, it follows directly from (*a*) and (*b*) that for $z \to \infty$

$$\lim (X_x + Y_y) = 4B, \ \lim (Y_y - X_x + 2iX_y) = 2\Gamma' = 2(B' + iC'), \tag{36.8}$$

whence

$$X_x^{(\infty)} = 2B - B', \quad Y_y^{(\infty)} = 2B + B', \quad X_y^{(\infty)} = C'. \tag{36.9}$$

This means that in the neighbourhood of the point at infinity the stresses are uniformly distributed (or rather that their distribution differs from a uniform one by infinitely small quantities). Let N_1, N_2 be the values of the principal stresses at infinity and α the angle made by the direction of N_1 with the axis Ox. Comparing (36.8) with (8.12) one finds

$$\begin{gathered} \Re\Gamma = B = \tfrac{1}{4}(N_1 + N_2), \\ \Gamma' = B' + iC' = -\tfrac{1}{2}(N_1 - N_2)e^{-2i\alpha}. \end{gathered} \tag{36.10}$$

The constant C, which does not affect the stresses, may be related to the *rotation* of an infinitely remote part of the plane. The expression for the rotation is

$$= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

(cf. § 14, where it is denoted by r), whence, by (30.8),

$$\varepsilon = \frac{\lambda + 2\mu}{2\mu(\lambda + \mu)}\left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) = \frac{1 + \varkappa}{4\mu}\left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) =$$

$$\frac{1 + \varkappa}{2\mu} \cdot \frac{\varphi'(z) - \overline{\varphi'(z)}}{2i} \tag{36.11'}$$

since, remembering that

$$\frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x} \text{ and } \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x} = \varphi'(z),$$

one has

$$\frac{\partial p}{\partial x} - i\frac{\partial q}{\partial x} = \overline{\varphi'(z)}, \quad \frac{\partial q}{\partial x} \quad \frac{1}{2i}[\varphi'(z) - \varphi'(z)].$$

It follows immediately from (36.11'), (36.4) and (36.5) that

$$\frac{1 + \varkappa}{2\mu} \cdot C,$$

and hence

$$C = \frac{2\mu\varepsilon_\infty}{1 + \varkappa}. \tag{36.12}$$

It will be noted at the same time that the state of stress characterized by the linear functions

$$\varphi(z) = (B + iC)z + \text{const}, \qquad \psi(z) = (B' + iC')z + \text{const}$$

is ***homogeneous***: the stresses are uniformly distributed, i.e., the stress components (and hence the strain components) are constant quantities. In fact, the components of stress are expressed by (36.9), if the superscript ∞ is omitted.

Next consider the behaviour of the displacements at infinity under the assumed conditions in the general case. For this purpose use will be made of (32.1) which by (36.4) and (36.5) becomes

$$2\mu(u + iv) = -\frac{\varkappa(X + iY)}{2\pi(1 + \varkappa)} \log (z\bar{z}) + (\varkappa\Gamma - \bar{\Gamma})z - \bar{\Gamma}'\bar{z} + \ldots, \tag{36.13}$$

where terms remaining bounded for large values of $|z|$ have been omitted. It is easily seen from (36.13) that, *generally speaking, the displacements will not be bounded at infinity under the conditions introduced so far.* In order that they may be bounded, one obviously has to impose the conditions

$$X = Y = 0, \quad \Gamma = \Gamma' = 0. \tag{36.14}$$

The first group of these conditions postulate that the resultant vector of all external forces acting on the boundary of the region is zero, while the second group demand that the stresses at infinity vanish and, besides, that infinitely remote parts of the plane do not undergo any rotation.

Note also that even in the case of rotation the stresses at infinity are zero and that in the absence of rotation ($C = 0$) the displacements increase like $\log(z\bar{z}) = 2 \log r$, if the resultant vector (X, Y) is not zero.

NOTE. In (36.4) and (36.5) the functions $\varphi_0(z)$, $\psi_0(z)$ are holomorphic outside any circle, enclosing all contours $L_1, L_2, \ldots, L_m$. If there is only *one* such contour L_1 (plane with one hole), it is easily seen that $\varphi_0(z)$, $\psi_0(z)$ will be holomorphic *throughout* the region S, provided only the origin of coordinates is taken outside S (i.e., inside the hole). In fact, in this case (35.10) and (35.11) coincide with (36.1), if one puts in the former

$$z_1 = 0$$

and replaces $\varphi^*(z)$, $\psi^*(z)$ by $\varphi^{**}(z)$, $\psi^{**}(z)$ respectively. But the functions $\varphi^*(z)$, $\psi^*(z)$ are known to be holomorphic *throughout* S with the possible exclusion of the point at infinity, which proves the assertion.

§ 37. Some properties following from the analytic character of the solution. On analytic continuation across a given contour.

A number of properties to be considered in the present section follow from the analytic character of the general solution of the equations of the plane theory of elasticity.

1. First, certain terms will be defined which will be used in the sequel and which have to some degree already been used above. In addition, some simple results of the theory of functions of a complex variable will be recalled.

A *simple curve* will be a line which may be represented parametrically by

$$x = f_1(\sigma), \quad y = f_2(\sigma),$$

where $f_1(\sigma)$, $f_2(\sigma)$ are continuous functions of the real parameter σ, varying over some finite interval, say, $0 \leq \sigma \leq l$, and different values of σ correspond to different points (x, y) with the possible exception of the points $\sigma = 0$, $\sigma = l$ (i.e., the line does not intersect itself). If the values $\sigma = 0$ and $\sigma = l$ correspond to different points, the line will be called an arc; if these two values of σ refer to one and the same point, the arc is closed and it will be called *a simple contour*. Such lines, i.e., simple arcs and contours are also called Jordan curves.

A line (and, in particular, a simple arc) will be called *smooth*, if it possesses at each of its points a tangent which changes continuously with the point of contact; more strictly, a line is smooth, if one can find for it a parametric representation: $x = f_1(\sigma)$, $y = f_2(\sigma)$, where $f_1(\sigma)$ and $f_2(\sigma)$ have continuous first order derivatives with respect to σ which are not simultaneously equal to zero.

A *region* will always (unless stated otherwise) be a finite or infinite part of a plane, bounded by one or several simple contours which do not have points in common, just as in the preceding sections; unless it is stated to the contrary, a region will be assumed to be *connected*, i.e., any two points of it can be joined by a continuous line which does not leave the region.

A given region will be denoted by S and its boundary (i.e., the union of the contours, forming the boundary of S) by L. *The boundary L will not be included with the region S*. Thus, if one is dealing with some property which refers to S as well as to its boundary L, such a property will be said to hold for $S + L$. In general, if L' is some part of L and if some property holds in S and on L', it will be said to hold in $S + L'$. A *part* of a boundary

will always be understood to consist of one or several continuous arcs of a contour.

Let $F(z)$ be a function given in S (but not on L) and continuous there. F will, in general, be some complex function of $z = x + iy$, i.e., a function of the form $U(x, y) + iV(x, y)$, where $U(x, y)$, $V(x, y)$ are real functions of x, y, and it will not be assumed to be analytic. The function $F(z)$ will be said to be *continuous* from S on L', if it is possible to prescribe for $F(z)$ values on L', so that the function obtained in this manner is continuous in $S + L'$. In this case it will often be simply said that the given function $F(z)$ is continuous in $S + L'$ or that it is *continuous in S up to L'*, which will mean that the function $F(z)$ has been given the stated values on L'.

Let t be some point of L and let $F(z)$ tend to a definite limit as $z \to t$ from inside S in an otherwise arbitrary manner (i.e., $z \to t$ *along any path* remaining in S, where it has not necessarily been assumed that the "path" is a continuous line; it may for example consist of a set of distinct points). Under these circumstances $F(z)$ will be said to have a definite boundary value at the point t or to be continuous from S at the point t; the boundary value at t will be understood to be the above-mentioned limit.

It is easily shown that, if $F(z)$ is continuous from S at all points t of some part L' of the boundary (where L' may be the entire boundary) and if these boundary values of $F(z)$ at t are denoted by $F(t)$, then $F(t)$ will be a continuous function of t on L'.

It follows from the same definition of continuity that, if $F(z)$ is continuous from S at all points of L', $F(z)$ will be continuous in $S + L'$, i.e., continuous in S up to L', if by $F(z)$, for z on L', one understands the corresponding boundary values.

In the sequel the statement, that $F(t)$ is the *boundary value* of $F(z)$ or that $F(z)$ takes the boundary value $F(t)$, will *always* mean that $F(t)$ is the limit of $F(z)$ as $z \to t$ in an arbitrary manner, the only restriction being that z has to remain in S. In other words, the use of the terms *boundary value at a given point* or *on a given part of the boundary* will *always* imply that the function under consideration is continuous from S at the given point or given part of the boundary.

2. Let the boundaries of the two regions S_1 and S_2, which do not cover each other, have a common part L which is a simple arc or contour and let $F_1(z)$ and $F_2(z)$ be functions of a complex variable, holomorphic in S_1 and S_2 respectively and continuous up to L (Fig. 15). Further, let

$F_1(z) = F_2(z)$ on L. Then the function $F(z)$, defined by

$$F(z) = \begin{cases} F_1(z) \text{ in } S_1 \text{ and on } L, \\ F_2(z) \text{ in } S_2 \text{ and on } L, \end{cases}$$

will be holomorphic in S, obtained by joining S_1 and S_2 including L. (If L is an arc, then its ends are not included with L.) The proof of this theorem can be found in any treatise on complex function theory.

The following conclusion follows directly from this theorem. Let $F(z)$ be holomorphic in some region S and let the boundary values of $F(z)$ be zero on some part L' of the boundary of this region. Then $F(z) = 0$ in the entire region. In fact, add to S some part S' of the plane adjoining the other side of L' (Fig. 16) and put $F(z) = 0$ in S'. Then, by the above theorem, the function $F(z)$ will be holomorphic in $S + S'$ and, since it is equal to zero in S', it will be zero everywhere, because an analytic function vanishing in a part of a region (i.e., in a subregion) is zero in the entire region.

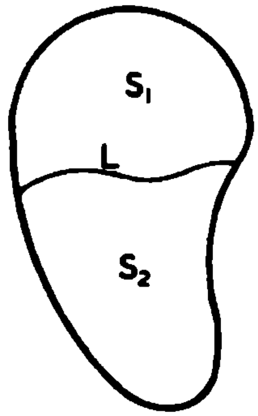


Fig. 15.

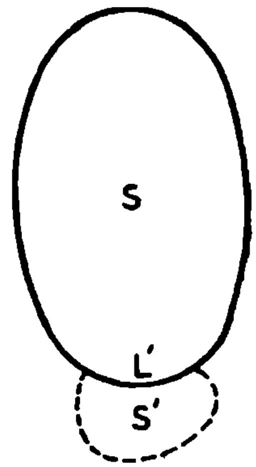


Fig. 16.

3. Next consider the general solution of the equations of the plane theory of elasticity and note first of all that the functions $\varphi(z)$, $\psi(z)$, $\Phi(z)$, $\Psi(z)$, occurring in this general solution, are also analytic functions of z in the entire region, occupied by the body, even in the case when that region is multiply connected. This follows from the expressions for these functions, deduced in the preceding sections. The only difference from the case of a simply connected region is that the functions $\varphi(z)$ and $\psi(z)$ may be found to be multi-valued, as a consequence of the presence of logarithmic terms. (If multi-valued displacements are admitted, the function $\Phi(z)$ may also turn out to be multi-valued). Since an analytic function of $z = x + iy$ leads at the same time to analytic functions of the real variables x, y (cf. end of § 32), the components of stress X_x, Y_y, X_y and displacement u, v are, as in the case of simply connected regions, analytic functions of x, y throughout the region, occupied by the body.

From this property of the solution there follows immediately a proposition which at first sight may appear somewhat unexpected.

If some part of the body (i.e., a subregion which may even be arbitrarily small) is in a "natural" state, i.e., if no stresses occur there, then the whole body is in a natural state or, in other words, no stresses occur anywhere.

In fact, if $X_x = Y_y = X_y = 0$ in some part of S, this will be so in the whole of S, because an analytic function cannot vanish in a part of a region without being zero in the whole region.

The proof will now be given of a simple and important proposition concerning the analytic continuation of the solution across a given contour. Let there be two regions S^+ and S^- which do not cover each other, but the boundaries of which have a part in common consisting of a smooth line L (i.e., an arc or contour). Assume that the components of displacement and stress satisfy in each of the regions S^+ and S^- the conditions of § 27. In that case they will be analytic functions in each of the separate regions S^+ and S^-.

Consider the necessary and sufficient conditions for the components of stress and displacement to be analytic in the region S, obtained by joining S^+ and S^- (including L). If u, v, X_x, Y_y, X_y are analytic in the whole of S, it is obvious that they will be continuous on L from S^+ as well as from S^- and that their boundary values on L from either side will be equal. Denoting the boundary values, obtained by going to the limit from S^+ and S^-, by superscripts (+) and (—) respectively, one finds the *necessary conditions*

$$u^+ = u^-, \quad v^+ = v^-, \; X_n^+ = X_n^-, \quad Y_n^+ = Y_n^- \text{ on } L, \tag{37.1}$$

where (X_n^+, Y_n^+), (X_n^-, Y_n^-) are the stress vectors, applied to an element of L at the point t, when that element is assumed to belong to S^+ and S^- respectively, i.e.,

$$\begin{aligned} X_n^+ &= X_x^+ \cos(n, x) + X_y^+ \cos(n, y), \\ Y_n^+ &= X_y^+ \cos(n, x) + Y_y^+ \cos(n, y), \end{aligned} \tag{37.2}$$

and similarly for X_n^-, Y_n^-, where n is the normal to L at the relevant point, directed to a definite side (which may be chosen arbitrarily).

It will be shown now that the conditions (37.1) are *sufficient* (assuming the existence of the boundary values of the components of displacement and stress from either side). These conditions simply express the fact that the displacements remain continuous for a passage across L and that the stresses acting on an element of this line from either side satisfy the law

of action and reaction. It follows from the first two conditions of (37.1) that

$$2\mu(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \tag{37.3}$$

is continuous on L from both sides and that the boundary values from either side are equal. Further, it follows from the two latter conditions of (37.1) that the same properties may be ascribed to

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}\,, \tag{37.4}$$

provided a proper choice has been made for the arbitrary constants which must be added to one of the functions φ, ψ in the regions S^+ and S^-. This is obvious since, by (33.1), one has for (37.4), in both S^+ and S^-, the formula

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = i\int (X_n + iY_n)ds, \tag{37.5}$$

where the integral is taken along an arbitrary line l, remaining all the time (with the exception of the point a) in S^+ or S^- and joining some fixed point a of L to a point z in S^+ or S^-; by bringing the point z into the neighbourhood of some point t of L, from S^+ or S^-, the line l may be chosen so close to L (i.e., close in the sense of distance as well as of tangential direction) that the integral on the right-hand side of (37.5) is approximated as closely as one pleases by the integral

$$i\int^{t} (X_n^+ + iY_n^+)ds = i\int^{t} (X_n^- + iY_n^-)ds,$$

taken over an arc of L, connecting a and t.

Adding (37.3) and (37.4), it is immediately verified that the function $\varphi(z)$ is also continuous on L from S^+ and S^- and that its boundary values from either side are equal. Hence, by what has been said above under 2, the function $\varphi(z)$ will be analytic in S; but then $\varphi'(z)$ will likewise be analytic. It is then obvious that $\psi(z)$ is continuous on L from both sides and that its boundary values are equal. Thus $\psi(z)$, like $\varphi(z)$, will be analytic in S. This proves the proposition.

The following result is easily deduced from the preceding ones: *If on some part (however small) of the boundary of a body*

$$X_n = Y_n = u = v = 0, \tag{37.6}$$

then the stresses are zero throughout the body. This result is due to E. Almansi [3] who proved it in a different way for the general three-dimensional case.

Let S be the region occupied by the body and L' that part of the boundary where (37.6) is fulfilled. Select some region S', adjacent to L' and outside S. By the above and by (37.6), the functions X_x, Y_y, X_y, u, v may be continued analytically from S into S' by simply putting these functions equal to zero in S'. But then, by what has been said earlier, one finds that $X_x = Y_y = X_y = u = v = 0$, because these functions, being analytic in the region obtained by adding S and S', vanish in the part S'.

NOTE. The results on the analytic continuation through a given contour, proved above, may be somewhat generalized. In fact, retaining the condition that the components of displacement must be continuous on L from S^+ as well as from S^-, one can replace the corresponding condition for the stress components by a weaker requirement which is more natural from the physical point of view, namely that the expression (37.4) must be continuous up to L. This condition is easily seen to lead to the following. Select some (smooth) arc l^+ (or l^-), entirely in S^+ (or S^-) and close to L, and suppose that this arc tends to some arc l of the line L; further, let (X, Y) be the resultant vector of the forces, applied to l^+(or l^-) from the sides, facing S^+(or S^-). Then, as l^+ (or l^-) approaches l, this resultant vector tends to (X^+, Y^+) [or (X^-, Y^-)] which, by supposition, is the resultant vector of the forces, exerted from the sides S^+(or S^-) on the arc l of the boundary of the body.

Provided the stated conditions are fulfilled, it is easily seen that (37.1) can be replaced by

$$u^+ = u^-, \; v^+ = v^-, \; X^+ + X^- = 0, \; Y^+ + Y^- = 0, \qquad (37.1')$$

where (X^+, Y^+) and (X^-, Y^-) are the resultant vectors of the forces applied from the sides S^+ and S^- respectively to an arbitrary arc l of the line L.

In the same manner (37.6) may be replaced by

$$X = Y = u = v = 0, \qquad (37.6')$$

where (X, Y) is the resultant vector of the forces, exerted on an arbitrary arc of the boundary.

§ 38. Transformation of rectilinear coordinates. Consider now how the various functions, *corresponding to a given state of stress of a body*,

change under transformation from one system of rectangular rectilinear coordinates to another.

First investigate the effect of the translation of the origin to a new point (x_0, y_0). Let (x, y) and (x_1, y_1) be the coordinates of the same point in the old and new systems and let

$$z = x + iy, \quad z_1 = x_1 + iy_1.$$

Obviously

$$z = z_1 + z_0, \tag{38.1}$$

where

$$z_0 = x_0 + iy_0.$$

Beginning with the formulae

$$X_x + Y_y = 4\Re\Phi(z), \quad Y_y - X_x + 2iX_y = 2[\bar{z}\Phi'(z) + \Psi(z)], \tag{38.2}$$

denote by $\Phi_1(z_1)$ and $\Psi_1(z_1)$ the functions playing in the new system the same parts as $\Phi(z)$ and $\Psi(z)$ in the old one. Since the stress components are not altered by a translation, one has by the first equation of (38.2)

$$\Re\Phi(z) = \Re\Phi_1(z_1) = \Re\Phi_1(z - z_0),$$

whence

$$\Phi(z) = \Phi_1(z - z_0). \tag{38.3}$$

One might have added on the right-hand side any purely imaginary constant which would have no influence on the distribution of stress.

The second formula of (38.2) gives

$$\begin{aligned}\bar{z}\Phi'(z) + \Psi(z) = \bar{z}_1\Phi_1'(z_1) + \Psi_1(z_1) = (\bar{z} - \bar{z}_0)\Phi_1'(z - z_0) + \Psi_1(z - z_0) = \\ = \bar{z}\Phi_1'(z - z_0) + \Psi_1(z - z_0) - \bar{z}_0\Phi_1'(z - z_0),\end{aligned}$$

whence, by (38.3),

$$\Psi(z) = \Psi_1(z - z_0) - \bar{z}_0\Phi_1'(z - z_0). \tag{38.4}$$

Integrating (38.3) and (38.4) with respect to z, one obtains

$$\varphi(z) = \varphi_1(z - z_0), \quad \psi(z) = \psi_1(z - z_0) - \bar{z}_0\varphi_1'(z - z_0), \tag{38.5}$$

where certain arbitrary constants which do not affect the stress distribution have been omitted.

It is seen that the function $\psi(z)$ is *not invariant* for a translation of the origin, i.e., the values for the old coordinates are not obtained by simply replacing in $\psi_1(z_1)$ the variable z_1 by $z - z_0$. In contrast, the function $\varphi(z)$ is invariant.

Next consider the effect of rotating the axes, leaving the origin fixed. If the axis Ox_1 is turned with respect to Ox by an angle α, then

$$x = x_1 \cos\alpha - y_1 \sin\alpha,$$

$$y = x_1 \sin\alpha + y_1 \cos\alpha,$$

whence

$$x + iy = (x_1 + iy_1)e^{i\alpha},$$

i.e.,

$$z = z_1 e^{i\alpha}, \quad z_1 = ze^{-i\alpha}. \tag{38.6}$$

In view of the invariance of $X_x + Y_y$, one has, on the basis of the first equality of (38.2),

$$\Re\Phi(z) = \Re\Phi_1(z_1) = \Re\Phi(ze^{-i\alpha}),$$

whence, omitting a purely imaginary constant term,

$$\Phi(z) = \Phi_1(ze^{-i\alpha}). \tag{38.7}$$

Further, the expression corresponding to

$$Y_y - X_x + 2iX_y,$$

but referring to the new system, is by (8.8) equal to

$$(Y_y - X_x + 2iX_y)e^{2i\alpha}.$$

Thus, by a formula analogous to the second formula of (38.2),

$$\bar{z}_1\Phi_1'(z_1) + \Psi_1(z_1) = [\bar{z}\Phi'(z) + \Psi(z)]e^{2i\alpha},$$

whence

$$\bar{z}\Phi'(z) + \Psi(z) = [\bar{z}e^{i\alpha}\Phi_1'(ze^{-i\alpha}) + \Psi_1(ze^{-i\alpha})]e^{-2i\alpha}.$$

Further, noting that by (38.7) $\Phi'(z) = e^{-i\alpha}\Phi_1'(ze^{-i\alpha})$, one finds

$$\Psi(z) = \Psi_1(ze^{-i\alpha})e^{-2i\alpha}. \tag{38.8}$$

Integrating (38.7) and (38.8) with respect to z and omitting arbitrary constants which do not influence the stress distribution, one obtains

$$\varphi(z) = \varphi_1(ze^{-i\alpha})e^{i\alpha}, \quad \psi(z) = \psi_1(ze^{-i\alpha})e^{-i\alpha}. \tag{38.9}$$

Finally, integration of the second of the preceding formulae gives

$$\chi(z) = \chi_1(ze^{-i\alpha}), \tag{38.10}$$

where again an arbitrary constant has been omitted.

NOTE. If the arbitrary constants had not been omitted, one would have found, for example, instead of (38.9) and (38.10)

$$\varphi(z) = \varphi_1(ze^{-i\alpha})e^{i\alpha} + Ciz + a + ib, \quad \psi(z) = \psi_1(ze^{-i\alpha})e^{-i\alpha} + a' + ib',$$

where C, a, b, a', b' are arbitrary real constants which do not affect the stress distribution. In the earlier formulae

$$C = a = b \qquad b' = 0.$$

Thanks to that choice of constants, not only the stresses, corresponding to the new and the old functions, but also the displacements will be the same. (Otherwise the latter would have differed by a rigid body displacement.) Further, by omitting an arbitrary constant in (38.10), it has been ensured that the stress functions U, formed by means of the new and the old functions, will be identical. This would not have been the case for a different choice of constants, since stress functions, corresponding to one and the same state of stress, can differ from each other by an arbitrary term of the form: $Ax + By + C$.

§ 39. Polar coordinates. In many cases it is convenient to express stresses and displacements in polar coordinates. Let the origin O of the system Oxy be the pole, and Ox the polar axis. Then, if r and ϑ are the polar coordinates of some point $M(x, y)$, one has, with an obvious choice of the angle,

$$z = x + iy = re^{i\vartheta}. \tag{39.1}$$

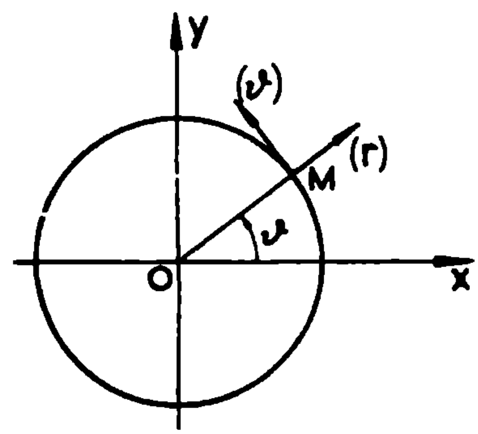


Fig. 17.

Draw through the point M two axes; (r), being a prolongation of the radius vector [on the side of increasing r], (ϑ), perpendicular to the first [to the side of increasing ϑ; Fig. 17].

Let v_r, v_ϑ denote the projections of the displacement at M on to the axes (r) and (ϑ). These quantities are called components of displacement in polar coordinates. By known formulae of analytic geometry

$$u = v_r \cos\vartheta - v_\vartheta \sin\vartheta, \quad v = v_r \sin\vartheta + v_\vartheta \cos\vartheta,$$

where (u, v) are the components of the same displacement in the cartesian

coordinate system Oxy; thus

$$u + iv = (v_r + iv_\vartheta)e^{i\vartheta}, \quad v_r + iv_\vartheta = (u + iv)e^{-i\vartheta}, \tag{39.2}$$

whence by (32.1)

$$2\mu(v_r + iv_\vartheta) = e^{-i\vartheta}[\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}]. \tag{39.3}$$

This formula gives expressions for v_r and v_ϑ in polar coordinates, if one replaces on the right hand-side z by $re^{i\vartheta}$ and separates real and imaginary parts.

The components of stress in polar coordinates are defined in quite the same way as in cartesian coordinates, with the distinction that the part of the axes Ox and Oy is now played by (r) and (ϑ) at the point M where the stresses are to be studied. Denoting temporarily the axis (r) by Mx' and (ϑ) by My', the above-mentioned components are

$$X'_{x'}, \quad Y'_{y'}, \quad X'_{y'}.$$

In the sequel, the following notation which is widely employed in literature will be used for these components:

$$\widehat{rr} = X'_{x'}, \quad \widehat{\vartheta\vartheta} = Y'_{y'}, \quad \widehat{r\vartheta} = X'_{y'}.$$

Thus $\widehat{rr}$ denotes the projection on (r) of the stress acting on the plane normal to (r); $\widehat{\vartheta\vartheta}$ is the projection on (ϑ) of the stress acting on the plane normal to (ϑ). Finally, $\widehat{r\vartheta}$ is the projection on (ϑ) of the stress acting on the plane normal to (r), or the projection on (r) of the stress acting on the plane normal to (ϑ).

By (8.8)

$$\begin{aligned} \widehat{rr} + \widehat{\vartheta\vartheta} &= 4\Re\Phi(z) = 2[\Phi(z) + \overline{\Phi(z)}], \\ \widehat{\vartheta\vartheta} - \widehat{rr} + 2i\widehat{r\vartheta} &= 2[\bar{z}\Phi'(z) + \Psi(z)]e^{2i\vartheta}. \end{aligned} \tag{39.4}$$

These formulae enable one to calculate the components of stress in polar coordinates.

By subtraction one obtains from (39.4) the useful formula

$$\widehat{rr} - i\widehat{r\vartheta} = \Phi(z) + \overline{\Phi(z)} - e^{2i\vartheta}[\bar{z}\Phi'(z) + \Psi(z)], \tag{39.5}$$

giving the stresses acting on an arc of the circle $r = \text{const.}$ from the side opposite the centre.

These formulae are analogous to those given by G. V. Kolosov in a somewhat different form.

§ 40. The fundamental boundary value problems. Uniqueness of solution. The fundamental boundary value problems will be defined in complete analogy with those formulated in § 20 for the three-dimensional case. As before, absence of body forces will be assumed.

First fundamental problem (Problem I): *To find the state of elastic equilibrium for given external stresses applied to the boundary L of the region S.*

Second fundamental problem (Problem II): *To find the state of elastic equilibrium for given displacements of points of the boundary L.*

By S will be understood a region of the form discussed in § 35 and § 36, by L the union of the contours $L_1, L_2, \ldots, L_m, L_{m+1}$ (if the region is finite, cf. § 35), or of the contours $L_1, L_2, \ldots, L_m$ (if the region is infinite, cf. § 36). In the sequel, unless stated otherwise, it will be assumed that all considered contours are *smooth lines* (i.e., that they have continuously varying tangents). If S is infinite, it will be assumed that the stresses in infinitely remote parts of the plane satisfy the conditions of § 36, i.e., that they remain bounded.

In addition, in the case of Problem I for infinite regions, it will be assumed that the values of the stresses at infinity are known; by § 36 they will enter into the constants

$$\Re\Gamma = B, \quad \Gamma' = B' + iC'. \tag{40.1}$$

Further, since the constant C (remembering that $\Gamma = B + iC$) does not influence the stress distribution, let $C = 0$.

In the case of Problem II for infinite regions, it will be assumed that the quantities

$$\Gamma = B + iC, \quad \Gamma' = B' + iC', \quad X, \quad Y \tag{40.2}$$

are given, i.e., that not only the values of the stresses at infinity, but also that of the rotation (§ 36) and, besides, the resultant vector (X, Y) of *all* external forces, applied to the boundary, are given. At first sight, the last condition seems to be unneccessary, but it can be shown that without it the problem remains indefinite, i.e., that it has an infinite number of solutions.

Apart from the stated problems, the fundamental mixed problem plays an important part, i.e., the problem for which displacements are given for one part and stresses for the remaining part of the boundary. In the case of the mixed problem for infinite regions, it will be assumed, as in Problem II, that, in addition, the values of X,

Y, Γ, Γ' are given. In Part VI several problems of a different type will be considered.

It will now be proved that, if the above problems have solutions, these will be unique. For finite regions, the proof is completely analogous to that presented earlier for the general case of three dimensions, while for infinite regions (such regions not having been considered for the three-dimensional case) certain additional considerations are required.

First consider the case of finite regions (simply or multiply connected). Study the integral (cf. § 20)

$$J = \int_L (X_n u + Y_n v) ds,$$

where

$$\begin{aligned} X_n &= X_x \cos(n, x) + X_y \cos(n, y), \\ Y_n &= Y_x \cos(n, x) + Y_y \cos(n, y) \end{aligned} \tag{40.3}$$

denote the stress components, applied to the boundary L, and n is the outward normal to L.

By Green's theorem

$$\begin{aligned} J = \int_L [(X_x u + Y_x v) \cos(n, x) + (X_y u + Y_y v) \cos(n, y)] ds = \\ = \iint \Big\{ u \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} \right) + v \left(\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} \right) + \\ + X_x \frac{\partial u}{\partial x} + X_y \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + Y_y \frac{\partial v}{\partial y} \Big\} dx\, dy. \end{aligned}$$

But, by (29.1),

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = 0, \quad \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} = 0,$$

also

$$\frac{\partial u}{\partial x} = e_x \qquad \frac{\partial v}{\partial y} = e_{yy}, \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 2e_x$$

and

$$X_x = \lambda\theta + 2\mu e_{xx}, \quad Y_y = \lambda\theta + 2\mu e_{yy}, \quad X_y = 2\mu e_{xy}.$$

Hence, the above expression becomes

$$\int (X_n u + Y_n v) ds = \iint \{\lambda\theta^2 + 2\mu(e_{xx}^2 + e_{yy}^2 + 2e_{xy}^2)\} dx\, dy. \tag{40.4}$$

If $u, v, X_n, Y_n, e_{xx}, e_{yy}, e_{xy}$ represent the "difference" of two solutions of the first, the second or the mixed problem, the expression $X_n u + Y_n v$ will be zero on the boundary L (cf. § 20). Hence, the double integral on the right-hand side will be zero. But since the function under the integral is a positive definite quadratic form, this can only be so, if

$$e_{xx} = e_{yy} = e_{xy} = 0.$$

Hence also X_x, Y_y, X_y, arising from the difference of the two solutions, must be zero, i.e., both solutions are identical in the sense that they lead to the same stresses and strains. However, the displacements may differ from each other by terms of the form

$$u_0 = -\varepsilon y + \alpha, \quad v_0 = \varepsilon x + \beta,$$

corresponding to rigid body displacement in the plane Oxy. In the cases of the second and the mixed problems, this difference does not occur, since the displacements of both solutions must be the same on the whole or part of the boundary.

Next consider the case of infinite regions. As before, let it be assumed that any of the three fundamental problems possesses two solutions and that u, v, X_n, Y_n denote the "difference" of these solutions. For Problem I : $X_n = Y_n = 0$ on the boundary, i.e., the resultant vector of all forces, exerted on the boundary, is zero. However, for the second and the mixed problems, this vector had been assumed given for both solutions, and hence it will also vanish for the difference of the two solutions. Thus in all the cases considered: $X = Y = 0$. In addition, the quantities Γ, Γ', corresponding to the difference, will be zero, since they were to be the same for both solutions.

Now apply (40.4) to the finite region, bounded by the contours $L_1, \ldots, L_m$ and the circle L_R with radius R and centre at O, which contains all the contours $L_1, \ldots, L_m$. It will be proved that

$$(X_n u + Y_n v) ds \tag{40.5}$$

tends to zero as $R \to \infty$. In fact, by (36.4), (36.5) and (36.7), where one has to put

$$X = Y = \Gamma = \Gamma' = 0,$$

one has for $|z| > R$

$$\varphi(z) = a_0 + \frac{a_1}{z} + \ldots, \quad \psi(z) = a_0' + \frac{a_1'}{z} + \ldots,$$

$$\Phi(z) = \varphi'(z) = -\frac{a_1}{z^2} + \ldots, \quad \Psi(z) = \psi'(z) \qquad \frac{\alpha_1}{z^2} +$$

The formula

$$2\mu(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}$$

shows that under these conditions u, v remain bounded. Further, the relations

$$X_x + Y_y = 2[\Phi(z) + \overline{\Phi(z)}], \quad Y_y - X_x + 2iX_y = 2[\bar{z}\Phi'(z) + \Psi(z)]$$

indicate that X_x, Y_y, X_y tend to zero with order $|1/z^2|$ (at the least) as $|z| \to \infty$. Hence, the expression $X_n u + Y_n v$ will be of order $\frac{1}{R^2}$ on the circle L_R. On the other hand, the path of integration in (40.5) is of length $2\pi R$ and hence the integral (40.5) is of order $\frac{1}{R}$ and tends to zero as $R \to \infty$.

Applying (40.4) first to the region, contained between L and L_R, and then increasing R beyond all bounds, it is seen that the integral on the left-hand side will tend to the integral taken over the boundary L; hence, the integral on the right-hand side will likewise tend to a limit which, by conventional definition, will represent the integral taken over the infinite region S. Thus (40.4) applies to infinite regions S and, hence, the conclusions regarding uniqueness of solutions remain true also for these cases.

The above proof of uniqueness of solution is analogous to that given by Kirchhoff for three dimensions. It implies that the components of stress and displacement are continuous along the boundary of the region inside which they satisfy the conditions of § 27. A proof may be given for somewhat more general conditions, but this will not be done here (cf. end of § 42).

Regarding the question of the existence of solutions, the following remarks will be made for the present. From a mathematical point of view, the first fundamental problem is completely equivalent, at least for regions bounded by one contour (for the case of several contours cf. Note 1 at end of the next section), to the *problem of equilibrium of a thin elastic plate, clamped at the edges*, under the influence of loads normal to its plane. This latter problem can be reduced to the determination of a biharmonic function U for given values of its partial

derivatives

$$\frac{\partial U}{\partial x}, \quad \frac{\partial U}{\partial y}$$

on the boundary of the region.

This problem is discussed in any of the treatises, mentioned in the list given at the beginning of Part I. As a rule, the problem leads to the determination of U for given boundary values of U and of the normal derivative $\frac{dU}{dn}$. But obviously one may in this case immediately determine the boundary values of

$$\frac{\partial U}{\partial x}, \quad \frac{\partial U}{\partial y},$$

since

$$\frac{\partial U}{\partial x} = \frac{dU}{ds} \cos (t, x) + \frac{dU}{dn} \cos (n, x), \quad \frac{\partial U}{\partial y} = \frac{dU}{ds} \cos (t, y) + \frac{dU}{dn} \cos (n, y),$$

where s denotes the coordinate along the boundary and t the direction of the tangent. Thus one arrives at the problem, stated in the text.

The first fundamental problem will be reduced to just such a mathematical problem (cf. § 41). The problem of finding a biharmonic function for given values of the derivatives $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$ on a contour will be called the *fundamental biharmonic problem*. This problem (or its equivalent problem of the equilibrium of a plate, clamped along the edges) has been the subject of many investigations, especially since 1907 when the Paris Academy of Science declared it the object of a prize. This prize was obtained by J. Hadamard [1], G. Lauricella [3], A. Korn [4] and T. Boggio. The above authors completely solved the problem for the case of *finite* regions, bounded by a simple contour and satisfying several conditions of a general character. (In 1936 S. L. Sobolev [1], using variational methods, gave a proof of the existence of solution of a boundary problem which represented a considerable generalization of the fundamental biharmonic problem).

Use of functions of a complex variable provided recently the means of obtaining the solution of the first as well as of the second fundamental problem for regions, bounded by an arbitrary number of contours. It also solved the fundamental mixed problem and a number of other important general problems. Certain of the above-mentioned general results will be studied below in Part V, while short statements will be given of others.

It will just be noted here that in the case of finite regions the first

fundamental problem has, of course, a solution only when the resultant vector and moment of the given external forces, applied to the boundary L of the region, are zero.

But in the case of infinite regions a solution exists only when this condition is not satisfied, even if it is required that the stresses at infinity vanish. This is explained by the fact that, if one considers part of the body enclosed between a given contour L and a circle, containing this contour, then, although the external stresses acting on the circle tend to zero as the radius increases beyond all bounds, taken over the whole boundary they may give a finite resultant vector and moment, because they are distributed over a circle the length of the circumference of which increases beyond all bounds. The resultant vector and moment of the external forces, applied to the union of the given contour L and the circle, is always zero.

Regarding the above-mentioned general solutions of the fundamental problems it may be noted that, just because of their generality, these solutions are often unsatisfactory from the point of view of application. Therefore one is obliged to study special methods of solution offering the possibility of practical analysis of more or less wide classes of regions, important in applications. Parts III — VI of this book are largely devoted to such methods.

§ 41. Reduction of the fundamental problems to problems of complex function theory. Since the state of stress and the displacements can be expressed by means of the two complex functions $\varphi(z)$ and $\psi(z)$, the problems formulated in the last section lead to the determination of these functions under certain conditions which they must satisfy on the boundary.

For greater clarity it will be assumed for the time being that the *region S is bounded by a single contour L*. This region S may, however, be finite or infinite (infinite plane with a hole).

In the case of *Problem II*, the boundary condition can be expressed in the following manner:

$$\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} = 2\mu(g_1 + ig_2) \text{ on } L, \tag{41.1}$$

where $g_1 = g_1(s)$ and $g_2 = g_2(s)$ are the given displacements of the points of the contour L; they are given functions of the arc coordinate s of the contour which may be measured from an arbitrary point of L.

In the case of *Problem I*, the boundary condition can be expressed by

two different methods which should be used according to their convenience. Only one of these will be stated now, while the other will be given at the end of this section. Let $X_n(s)$, $Y_n(s)$ be the given values of the vector components of the external stresses, applied along L, and let the positive direction of the arc coordinate s be such that the region S lies on the left. By (32.5)

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = i\int_0^s (X_n + iY_n)ds + \text{const. on } L, \tag{41.2}$$

where the additive constant will, in general, be complex. Let

$$i\int_0^s (X_n + iY_n)ds = f_1(s) + if_2(s), \tag{41.3}$$

where f_1, f_2 must be considered as *given* (real) *functions of the arc coordinate* s. Thus (41.2) becomes

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = f_1 + if_2 + \text{const. on L.} \tag{41.2'}$$

Disregarding the arbitrary constant on the right-hand side, the problem of the determination of U is equivalent to the "fundamental biharmonic problem" (cf. § 40). Further, remembering (31.4), viz.,

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{41.4}$$

the condition (41.2') can be rewritten

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = f_1 + if_2 + \text{const. on } L. \tag{41.5}$$

Now the following will be noted. It has been seen in the preceding section that knowledge of $X_n(s)$, $Y_n(s)$ completely determines the state of stress of the body. But the functions $\varphi(z)$, $\psi(z)$ will then not be completely determined; in fact, it has been found in § 34 that the substitutions

$$\begin{aligned} &\varphi(z) + Ciz + \gamma \quad \text{for } \varphi(z), \\ &\psi(z) + \gamma' \qquad\quad \text{for } \psi(z), \end{aligned} \tag{A}$$

where C is a real and $\gamma = \alpha + i\beta$, $\gamma' = \alpha' + i\beta'$ are complex constants, do not alter the state of stress and, conversely, that any transformation,

which does not affect the stresses, must be of the form (A). In addition (§ 34),

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} \text{ is then replaced by } \frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} + \gamma + \bar{\gamma}'. \qquad (B)$$

It thus follows that by a suitable choice of γ and γ' the constant in (41.5) can be given any arbitrary value.

Next, the cases of finite and infinite regions will be considered separately and a beginning will be made with the former. In the case of Problem II, the boundary values completely determine the displacements at all points of the body (§ 40). Therefore, by the results of § 34 (assuming the origin to lie within the region S), only one of the quantities $\varphi(0)$ or $\psi(0)$ may be fixed arbitrarily beforehand. Unless stated otherwise, it will always be supposed that

$$\varphi(0) = 0. \qquad (41.6)$$

In the case of Problem I, when the boundary conditions completely determine the state of stress of the body (the displacements being determined apart from rigid body translation), both quantities $\varphi(0)$, $\psi(0)$ may be fixed arbitrarily, in addition to the imaginary part of $\varphi'(0)$. But, if the constant on the right-hand side of (41.5) is fixed in a definite manner, only one of the two quantities $\varphi(0)$, $\psi(0)$ can be decided upon arbitrarily. Therefore, in the case of Problem I, one may assume

$$\varphi(0) = 0, \quad \mathfrak{J}\varphi'(0) = 0. \qquad (41.7)$$

Regarding the last point, the following remark may be made. If φ and ψ are any functions solving Problem I, application of the transformation (A) gives functions solving the same problem. In order that (41.5) may be fulfilled for a definite value of the constant on the right-hand side, the quantity $\gamma + \bar{\gamma}'$ will be fixed, as can be seen from (B). For example, if γ be given, the constant γ' will be completely determined.

In the case of *infinite regions*, assume that the origin lies outside the region (i.e., inside the hole). Then, by § 36,

$$\begin{aligned} \varphi(z) &= -\frac{X + iY}{2\pi(1 + \varkappa)} \log z + \Gamma z + \varphi_0(z), \\ \psi(z) &= \frac{\varkappa(X - iY)}{2\pi(1 + \varkappa)} \log z + \Gamma' z + \psi_0(z), \end{aligned} \qquad (41.8)$$

where φ_0, ψ_0 are holomorphic in S, *including the point at infinity*.

For Problem II one may assume

$$\varphi_0(\infty) = 0, \tag{41.6'}$$

because, as is known from § 34, an arbitrary constant $\gamma = \alpha + i\beta$ can be added to $\varphi(z)$ without any effect on the displacements. For Problem I, when the constant on the right-hand side of (41.5) has been fixed beforehand, one can assume (cf. the case of finite regions)

$$\varphi_0(\infty) = 0, \; C = 0. \tag{41.7'}$$

The supplementary conditions (41.6), (41.7) or (41.6'), (41.7'), obviously, completely determine the unknown functions φ, ψ, if (in the case of Problem I) the constant on the right-hand side is fixed.

Now turn to the general case, when the boundary consists of several contours $L_1, L_2, \ldots, L_m, L_{m+1}$ (finite regions) or $L_1, \ldots, L_m$ (infinite regions). The functions g_1, g_2 (for Problem II) or X_n, Y_n (for Problem I) will be given on each of the contours L_k. The form of $f_1(s) + if_2(s)$ will be defined on each contour separately and the arc coordinate s will be measured on every contour from an arbitrarily chosen point.

The conditions (41.1) or (41.5) must be fulfilled on each of the contours L_k. For Problem I the constant on the right-hand side of (41.5) may (and generally will) have different values on the different contours L_k. It may be fixed arbitrarily for only one of the L_k. Its values for the remaining contours necessarily remain undetermined; these values must be found for the solution of the problem, as a consequence of its uniqueness (cf. case of a single contour).

Finally, the boundary condition for Problem I will be stated in a different form. Let there be given the normal and the tangential components N and T of the external stresses acting on the boundary L. The components N and T will be the projections of the stresses on the *outward* normal n and on the tangent, pointing to the *left* of n. Then

$$2(N - iT) = X_x + Y_y - (Y_y - X_x + 2iX_y)e^{2i\alpha} \text{ on } L,$$

where α is the angle between the normal n and the axis Ox, measured from the latter. In order to obtain this formula, one only has to think of the normal n as axis $O'x'$ and of the tangent as axis $O'y'$. Then

$$N = X'_{x'}, \qquad T = X'_{y'},$$

and the above formula agrees with (8.8'). Introducing into this formula the expressions (32.9) and (32.10), one finds

$$\Phi(z) + \overline{\Phi(z)} - e^{2i\alpha}\{\bar{z}\Phi'(z) + \Psi(z)\} = N - iT \text{ on } L. \tag{41.9}$$

The boundary condition, written in this form which is mainly used by G. V. Kolosov [1, 2], is often more convenient than the condition (41.5). But in certain cases (41.5) is preferable. One of its advantages is that it is applied in just this form to the problem of a plate with clamped edges (fundamental biharmonic problem). Besides, condition (41.5) is very similar to (41.1) and, thanks to this fact, the methods of solution of the Problems I and II are almost identical.

NOTE 1. In the case of a multiply connected region S, the following difference exists between the fundamental biharmonic problem and the first fundamental problem of the plane theory of elasticity: for the fundamental biharmonic problem, the expression

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = f_1 + if_2$$

is given *completely* on each of the contours L_k, for the first fundamental problem it is given apart from constants C_k on L_k (these constants being unknown beforehand) and only one of the C_k may be fixed arbitrarily.

Further, there is a difference in the conditions imposed on the unknown function $U(x, y)$: for the fundamental biharmonic problem, it is usually required that the derivatives

$$\frac{\partial U}{\partial x}, \quad \frac{\partial U}{\partial y}$$

be single-valued in S, or even that $U(x, y)$ be single-valued there (*e. g.* when dealing with the equilibrium of plates clamped at the edges); for the first fundamental problem, it is required from $U(x, y)$ that the components of stress and displacement, corresponding to it, be single-valued. In the case of simply connected regions, both conditions lead to single-valuedness of $U(x, y)$.

NOTE 2. In spite of the fact that solution of the boundary value problems in the general case offers very great practical difficulties, it is very easy in certain particular cases to guess the solution from the form of the boundary condition (41.5). For example, assume that the boundary of the body is subject to uniformly distributed normal tension P (for $P < 0$ one would have compression). Let n be the outward normal. Then

$$X_n + iY_n = P[\cos(n, x) + i\cos(n, y)] = -Pi\left[\frac{dx}{ds} + i\frac{dy}{ds}\right] = -Pi\frac{dz}{ds},$$

whence

$$f_1 + if_2 = i\int (X_n + iY_n)ds = Pz + \text{const.},$$

and the boundary condition (41.5) becomes

$$\varphi(z) + z\varphi'(z) + \psi(z) = Pz + \text{const.};$$

thus it is seen than one may satisfy this condition by putting

$$\varphi(z) = \tfrac{1}{2}Pz, \qquad \psi(z) = 0. \tag{41.10}$$

By the uniqueness theorem, any other solutions can only differ from (41.10) by rigid body displacements.

The stress components, corresponding to (41.10), are, using the formulae of § 32,

$$X_x = Y_y = P, \qquad X_y = 0. \tag{41.11}$$

Note again a curious case, where the boundary problems are solved directly, almost without any calculations. Consider first Problem I for the case of a single contour and assume that the function

$$f_1 - if_2 = -i\int (X_n - iY_n)ds + \text{const.}, \tag{41.12}$$

given (apart from an arbitrary constant) on L, represents the boundary value of some function $F(z)$, holomorphic in S and continuous along L. Then (41.5) may be written (taking the conjugate expression)

$$\overline{\varphi(z)} + \bar{z}\varphi'(z) + \psi(z) = F(z) \text{ on } L, \tag{41.13}$$

and obviously the solution of the problem is obtained by putting

$$\varphi(z) = 0, \quad \psi(z) = F(z). \tag{41.14}$$

It follows from the uniqueness theorem that the problem has no other solutions, except for those differing from (41.14) by rigid body displacements. Quite an analogous reasoning may be applied to Problem II, and the generalization to multiply connected regions does not present any difficulty.

Consider, as the simplest example, an arbitrary (simply or multiply connected) body and suppose that $F(z) = Qz$, where Q is a real constant.

This corresponds to the case when

$$X_n - iY_n = Qi\,\frac{dz}{ds} = Qi\left[\frac{dx}{ds} + i\,\frac{dy}{ds}\right],$$

i.e.,

$$X_n = -Q\cos(n, x), \quad Y_n = Q\cos(n, y).$$

Thus, uniformly distributed stresses, equal to Q, are applied to the boundary of the body; however, these stresses are not directed along the outward normal, but in the direction of the reflection of the normal in the axis Oy. In the present case, by (41.14),

$$\varphi(z) = 0, \quad \psi(z) = Qz.$$

For the components of stress one finds from the formulae of § 32

$$X_x = -Q, \quad Y_y = Q, \quad X_y = 0.$$

If, for example, the considered region is a rectangle with sides parallel to the coordinate axes, the above is the solution of the problem for the case, when uniformly distributed tensile forces act on the sides, parallel to the axis Ox, and similar compressive forces act on the sides parallel to Oy.

§ 42. Concept of regular solutions. Supplementary remarks. It is known that for simply connected regions the functions $\varphi(z)$ and $\psi(z)$ are holomorphic in S; for multiply connected regions they have the form

$$\begin{aligned}\varphi(z) &= -\frac{1}{2\pi(1+\varkappa)}\sum_{k=1}^{m}(X_k + iY_k)\log(z - z_k) + \varphi_0(z),\\ \psi(z) &= \frac{\varkappa}{2\pi(1+\varkappa)}\sum_{k=1}^{m}(X_k - iY_k)\log(z - z_k) + \psi_0(z),\end{aligned} \tag{42.1}$$

where $\varphi_0(z)$, $\psi_0(z)$ are holomorphic in S.

In the sequel (unless stated otherwise) it will be assumed that $\varphi_0(z)$, $\varphi_0'(z)$, $\psi_0(z)$ are *continuous in S up to the boundary of the region* (the naturalness of this condition will become clear below). Under these circumstances it will be said that the *solution is regular.*

In the case of finite simply connected regions, φ and ψ will take the place of φ_0, ψ_0, and, as shown by the formula

$$\frac{\partial U}{\partial x} + i\,\frac{\partial U}{\partial y} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}, \tag{42.2}$$

the expression

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}$$

will be single-valued and continuous on L and, consequently, will revert to its original value for a circuit of the contour. Hence the same property may be ascribed to the expression

$$f_1(s) + if_2(s) = i\int (X_n + iY_n)ds,$$

differing from the preceding one only by a constant [cf. (41.2)]. But this proves that

$$\int (X_n + iY_n)ds,$$

taken along the contour L, is zero, i.e., that the resultant vector of the external forces, applied to the boundary, must be zero. Thus, ***a necessary consequence of the regularity of the solution is that the resultant vector of the external forces is zero.*** It is easily deduced from (33.3) that ***the resultant moment is also necessarily equal to zero.***

In the case of multiply connected regions, for a circuit of one of the contours L_k $(k = 1, 2, \ldots, m)$ in the direction which leaves S on the left, the expression (42.2) is easily seen, using (42.1), to increase by $i(X_k+iY_k)$. For a circuit of the outer contour L_{m+1}, however, it will increase by

$$-i(\sum_{k=1}^{m} X_k + i\sum_{k=1}^{m} Y_k).$$

This again proves that the resultant vector of all external forces is necessarily zero. It follows in an analogous manner that the resultant moment of these forces must be zero.

The following must be noted with regard to the expression $f_1 + if_2$, defined on each contour L_k by

$$f_1 + if_2 = i\int (X_n + iY_n)ds. \tag{42.3}$$

It is known from § 41 that on each contour L_k

$$f_1 + if_2 = \frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} + \text{const.} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \text{const.} \tag{42.4}$$

For a circuit of L_k, the right-hand side will increase by $i(X_k + iY_k)$; hence $f_1 + if_2$ is not, in the general case, a single-valued and continuous function on the contour L_k. With equal right one of the following conditions may be adopted:

Assume either that $f_1 + if_2$ is a single-valued function of the arc coordinate s; then this function must be assumed to be discontinuous, i.e., for each complete circuit of a contour and passage through the starting point of the circuit, this expression will change discontinuously by $-i(X_k + iY_k)$, in order to revert it to its original value.

Or assume $f_1 + if_2$ to be a multi-valued, but continuous function of the arc coordinate s which increases by $i(X_k + iY_k)$ for every complete circuit of L_k. (All circuits being executed in a direction which leaves S on the left). This condition is in many respects more convenient and will always be adopted here. Under these circumstances, the constant on the right-hand side of (42.4) will maintain one and the same value for motion along L_k, if definite branches of the logarithmic terms, appearing in the expressions for φ and ψ, are chosen and if it is stipulated that these are to vary continuously for continuous changes of s along the contour.

Quite analogous remarks refer to infinite regions, the only difference being that the resultant vector and moment of the external forces, acting on the boundary, must not necessarily be zero.

If the region S is bounded by one simple contour L, the condition that the resultant vector and moment of the external forces, applied to the boundary, must vanish leads, as follows from the above, to the condition of continuity and single-valuedness of the expression $f_1 + if_2$ on L. Consider how the condition that the resultant moment of the external forces should be equal to zero may be expressed in terms of f_1 and f_2. Integrating this condition by parts one finds

$$0 = \int_L (xY_n - yX_n)ds = -\int_L (x\,df_1 + y\,df_2) =$$

$$= -[xf_1 + yf_2]_L + \int_L (f_1 dx + f_2 dy),$$

where $[\ ]_L$ denotes again the increase undergone by the expression in brackets for one circuit of L. If the resultant vector of the forces acting on L is zero, the functions f_1 and f_2 are single-valued and continuous, and hence $[xf_1 + yf_2]_L = 0$. Thus, if the resultant vector is zero, the above

condition can be written in the form

$$\int (f_1\,dx + f_2\,dy) = 0. \tag{42.5}$$

This formula is easily generalized to the case of regions bounded by several contours.

In § 40, the uniqueness theorem for the solutions of the fundamental problems was proved under the supposition that the components of stress and displacement are continuous up to the boundary. It is not difficult to prove the uniqueness of the solutions of the first and second fundamental problems, assuming them to be regular. The proof is given in the Author's paper [11]. The Uniqueness theorem for the fundamental biharmonic problem was proved for somewhat more general conditions by S. G. Mikhlin [6].

§ 43. Concentrated forces, applied to a contour. It has been seen that the requirement of continuity of the expression $f_1 + if_2$ on a given contour L_k may be fulfilled, because of the stipulation of regularity of the solution, i.e., owing to the condition of continuity of $\varphi_0, \psi_0, \varphi_0'$ up to the boundary. Consider what happens, if this condition is violated. Let the arc AB be part of the boundary of the region and let $\varphi_0, \psi_0, \varphi_0'$ cease to be continuous at some point C of this arc. Assume that $\dfrac{\partial U}{\partial x} + i\dfrac{\partial U}{\partial y}$ remains continuous in S, in the neighbourhood of this point, and undergoes only a finite jump for a passage through the point C, when z moves along AB. Further, assume that U is continuous near and on C.

Let

$$\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right]_C = \left[\frac{\partial U}{\partial x}\right]_C + i\left[\frac{\partial U}{\partial y}\right]_C$$

denote the jump in $\dfrac{\partial U}{\partial x} + i\dfrac{\partial U}{\partial y}$ for a passage through C, if z describes AB in the positive direction (i.e., leaving S on the left).

Consider an infinitesimal part $C'DC''$ of the body and the resultant vector of the external forces, acting on the arc $C'DC''$ (Fig. 18) of the boundary of this part. By (33.1), this resultant vector (X, Y) may be written

$$X + iY = -i\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right]_{C'}^{C''}.$$

Letting C' and C'' approach C, one obtains in the limit

$$X + iY = -i\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right]_C,$$
$$\text{i.e., } X = \left[\frac{\partial U}{\partial y}\right]_C, \quad Y = -\left[\frac{\partial U}{\partial x}\right]_C. \tag{43.1}$$

The resultant moment (with respect to the origin) of the same forces is easily calculated by (33.2), its limit being

$$M = -x\left[\frac{\partial U}{\partial x}\right]_C - y\left[\frac{\partial U}{\partial y}\right]_C = xY - yX,$$

where x, y are the coordinates of C. Hence, the forces applied to the infinitesimal arc $C'DC''$ are equivalent to one single force (X, Y) applied at C, its components being given by (43.1).

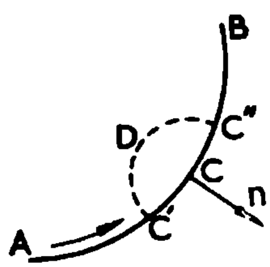


Fig. 18.

Thus, the point of discontinuity C of the expression $\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}$ on the contour (possessing the properties stated above) should be considered as the point of application of the *concentrated force* (X, Y), defined by (43.1).

§ 44. Dependence of the state of stress on the elastic constants.

An important property of the solution of the first fundamental problem will now be discussed. First consider the *case of finite simply connected regions*. The unknown functions φ, ψ are then holomorphic in the region S. Further, since the boundary condition (41.5) does not depend on the elastic constants λ and μ, the functions φ and ψ, giving the solution of the first fundamental problem, will also solve this problem (for the same given external stresses) for a body of the same shape, but made of some other (homogeneous and isotropic) material.

Thus, *for given external stresses, the state of stress of a simply connected (finite) body depends only on its shape, but not on its material.* The displacements and strains will, of course, depend on the material, since the constants λ and μ enter into the formulae, giving the displacements in terms of the functions φ and ψ.

In the case of the exact problem of plane strain this proposition, of course, only holds with respect to the components X_x, Y_y, X_y, because Z_z depends on λ and μ (or more correctly, on relations involving these quantities). But in the case of thin plates, i.e., for "generalized plane stress" (§ 26), the proposition holds fully, because then $Z_z = 0$.

The theorem on the independence of the state of stress on the elastic constants (always with reference to the components X_x, Y_y, X_y) is with little justification called the theorem of M. Levy, for example by G. V. Kolosov [3, 4]. The truth is that M. Levy [1] emphasizes the fact that the equations, to be satisfied by X_x, Y_y, X_y, do not involve the elastic constants. But it does not follow from this fact in the general case that the stressed state does not depend on the elastic constants (cf. later).

Next consider the *case of multiply connected bodies*. Also in this case the constants λ and μ do not figure in the boundary conditions. But they do appear (through $\varkappa$) in (35.10) and (35.11), viz.,

$$\begin{aligned} \varphi(z) &= -\frac{1}{2\pi(1+\varkappa)} \sum_{k=1}^{m} (X_k + iY_k) \log (z - z_k) + \varphi^*(z), \\ \psi(z) &= \frac{\varkappa}{2\pi(1+\varkappa)} \sum_{k=1}^{m} (X_k - iY_k) \log (z - z_k) + \psi^*(z). \end{aligned} \tag{44.1}$$

Assume that the first fundamental problem has been solved for a given material, i.e., that the corresponding functions φ, ψ have been found. Consider whether the same functions may give the solution of the same problem for the same boundary stresses and for a body of the same shape, but of different material with the constants λ', μ' instead of λ and μ. Denote by $\varkappa'$ the corresponding value of $\varkappa$. The functions φ, ψ will, of course, satisfy the given boundary conditions also for the second body, because the elastic constants do not figure in these conditions. However, the displacements, corresponding to these functions, may turn out to be multi-valued. In fact, for single-valuedness of the displacements, one has by (35.7), in which one has now to replace $\varkappa$ by $\varkappa'$,

$$\varkappa'\gamma_k + \bar{\gamma}'_k = 0,$$

where, by (35.9),

$$\gamma_k = -\frac{X_k + iY_k}{2\pi(1+\varkappa)}, \quad \bar{\gamma}'_k = \frac{\varkappa(X_k + iY_k)}{2\pi(1+\varkappa)},$$

and hence

$$(\varkappa - \varkappa')\frac{X_k + iY_k}{2\pi(1+\varkappa)} = 0.$$

But this will only be possible for $\varkappa' \neq \varkappa$, if $X_k = Y_k = 0$. Thus, the same functions φ and ψ will give the solution for bodies of different materials (with different constants $\varkappa$) if, and only if, the resultant vectors of the external forces applied *to each of the contours L_k separately* are zero; then, and only then, the state of stress does not depend on the elastic constants. Otherwise it depends on the value of $\varkappa$, i.e., on the value of λ/μ.

This result is due to J. H. Michell [1]. It is of considerable interest for experiments involving models made of various materials which are convenient for the purpose. It is seen that under the given conditions the material does not affect the results. G. V. Kolosov [3, 4] gave formulae elucidating the influence of the elastic constants also in the case, when body forces are present the components of which are analytic functions of the coordinates. However, the results of Kolosov require additional study in the case of multiply connected regions

A more detailed statement of a practical nature with regard to the influence of the choice of material constants of multiply connected bodies can be found in the paper by L. N. G. Filon [3] and also in the book by E. G. Coker and L. N. G. Filon [1]. It should be noted that the deduction of all the results, obtained by Filon, can be considerably simplified, if one starts from the above formulae.

CHAPTER 6

MULTI-VALUED DISPLACEMENTS. THERMAL STRESSES

§ 45. Multi-valued Displacements. Dislocations. The condition of single-valuedness of the displacements, which hitherto has always been assumed to be fulfilled, seems at first sight to be quite inevitable from a physical point of view. However, it will be seen that a very simple physical interpretation can be given to multi-valued displacements.

As before, it will be assumed that the components of stress, and hence the components of strain, are single-valued functions in the region, occupied by the body; more exactly, it will be assumed that all the conditions stated at the end of § 29 are fulfilled *with the exception of the condition of single-valuedness of the displacements.*

It will be remembered that in the case of *simply connected* regions single-valuedness of the displacement components remains the necessary consequence of the other conditions (cf. §§ 29, 30). Therefore only multiply connected regions need be considered. As in § 35, suppose that the region S, occupied by the body, is bounded by several simple contours L_1, $L_2, \ldots, L_m, L_{m+1}$ the last of which contains the others.

It will also be remembered that the deduction of the formulae (35.1)—(35.6) was not based on the condition of single-valuedness of the displacements; this condition was only introduced starting with (35.7). Therefore, in particular, (35.3) and (35.4) remain valid under the conditions to be considered now.

In order to study the character of multi-valuedness of the components of displacement, convert the region S into a simply connected one by means of m cuts $a_1b_1, a_2b_2, \ldots, a_mb_m$, connecting $L_1, L_2, \ldots, L_m$ with the outer contour L_{m+1} and not intersecting each other (Fig. 19). (These cuts may be produced in any manner whatsoever, e.g. by joining some point of L_1 with some point of L_2, some point of L_2 with some point of L_3 etc. and by reaching in this manner some point of L_{m+1}; but for the sake of simplicity the above stated system of cuts has been adopted.)

In the region cut in the above manner the functions φ, ψ, and hence also the displacements, will be single-valued. At each cut a distinction

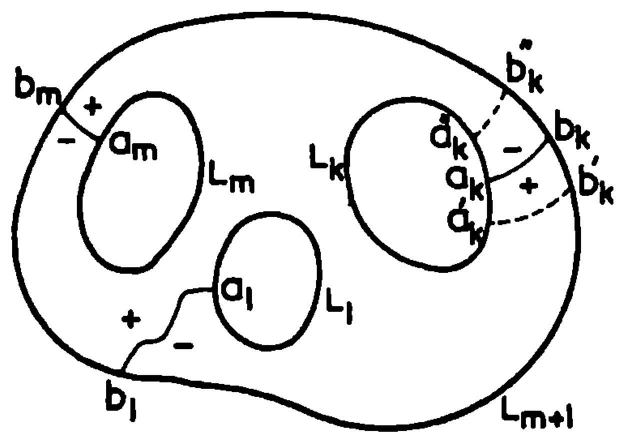


Fig. 19

must be made between the two sides which will be denoted by + and —; these signs will be allotted in such a way that, in order to go from some point (x, y) on the side (—) of the cut $a_k b_k$ (and remaining in the cut region) to the corresponding point of the side (+) of the same cut (i.e., to the point with the same coordinates), one has to encircle the contour L_k in an anti-clockwise direction. By (35.6), one has for such a circuit

$$u^+ - u^- + i(v^+ - v^-) = \frac{\pi i}{\mu} \{(\varkappa + 1)A_k(x + iy) + \varkappa\gamma_k - \bar{\gamma}_k'\}, \quad (45.1)$$

where A_k is the real, $\gamma_k = \alpha_k + i\beta_k$ and $\gamma_k' = \alpha_k' + i\beta_k'$ are the complex constants, appearing in (35.3) and (35.4); here u^+, v^+ and u^-, v^- are the values of the components of displacement, corresponding to the points on the sides (+) and (—) which coincide in the geometrical point (x, y). The formula (45.1) may be rewritten

$$u^+ - u^- = -\varepsilon_k y + \overset{\circ}{\alpha}_k, \quad v^+ - v^- = \varepsilon_k x + \overset{\circ}{\beta}_k, \quad (45.2)$$

where

$$\varepsilon_k = \frac{\pi(\varkappa + 1)A_k}{\mu}, \quad \overset{\circ}{\alpha}_k = \frac{\pi(-\varkappa\beta_k + \beta_k')}{\mu}, \quad \overset{\circ}{\beta}_k = \frac{\pi(\varkappa\alpha_k + \alpha_k')}{\mu}. \quad (45.3)$$

There is no difficulty in giving a physical interpretation of these multi-valued displacements. (It will be remembered that only very small deformations of the body are being considered; hence also the quantities ε_k, $\overset{\circ}{\alpha}_k$, $\overset{\circ}{\beta}_k$ will be very small.) In fact, in order to explain those displacements, it is sufficient to suppose that along each cut $a_k b_k$ the two sides of the body have been connected by removing from it, before deformation, a (very narrow) strip the sides $a_k'' b_k''$ and $a_k' b_k'$ (Fig. 19) of which are congruent and placed in such a way that $a_k'' b_k''$ results from $a_k' b_k'$ by a rigid displacement, consisting of a rotation by an angle ε_k about the origin and a translation with components $\overset{\circ}{\alpha}_k$, $\overset{\circ}{\beta}_k$. It has been implied here that for a reunion the same points are to be combined which would correspond

to each other, but for the above rigid displacement. The notation has been chosen in such a way that the lines $a_k''b_k''$ and $a_k'b_k'$ will, after deformation, become the sides (—) and (+) of the cut a_kb_k.

The relations (45.2) have been obtained, in order to elucidate how the above operation of reunion could be accomplished; for example, let $a_k''b_k''$ remain fixed and let the side $a_k'b_k'$ move as a rigid unit until it meets $a_k''b_k''$. Then $u^- = v^- = 0$, $u^+ = -\varepsilon_k y + \overset{\circ}{\alpha}_k$, $v^+ = \varepsilon_k x + \overset{\circ}{\beta}_k$ and hence (45.2) is fulfilled. If after this process the body is left to its own devices and becomes, in addition, subject to some ordinary deformation, the relations (45.2) will not be disturbed, because adjoining points of the contacting sides will move like one point and no additional differences between (u^+, v^+) and (u^-, v^-) will arise. Clearly the shape of the line a_kb_k in the final state will, in general, differ from that of $a_k'b_k'$ and $a_k''b_k''$.

For simplicity, the above discussion only refers to the r e m o v a l of strips with sides $a_k''b_k''$ and $a_k'b_k'$. But for some values of ε_k, $\overset{\circ}{\alpha}_k$, $\overset{\circ}{\beta}_k$ it may happen that (before deformation) the sides $a_k'b_k'$ and $a_k''b_k''$ will overlap, so that virtually strips have to be added rather than removed. Similarly, it may also happen that $a_k'b_k'$ and $a_k''b_k''$ only partly overlap, in which case material may have to be added in one place and removed in another. However, in the sequel, for the sake of brevity, only "removal" of strips will be mentioned. Likewise it is clear that, when joining the sides $a_k'b_k'$ and $a_k''b_k''$, their end points may not completely coincide with each other so that after reunion they may form (small) steps on the boundaries of the region; but these will not be considered here.

The above interpretation of multi-valued displacements was first stated by A. Timpe [1] for the particular case of a circular ring. (This case will be treated as an example in § 60.) Somewhat later, V. Volterra obtained more general results referring to multiply connected bodies of arbitrary shape.

Cf. V. Volterra [1] which contains a summary of his results, and also his books [2] and [3]. The case of plane deformation has also been considered in a paper by L. N. G. Filon [3] which presents interesting results referring to the problem of the study of a state of stress by means of experiments with models of different materials; cf. also. E. G. Coker and N. G. Filon [1].

Volterra uses for deformations of the body of the type described above the term "distorsion". A. E. H. Love [1] proposed instead the term "dislocation" which will be used here for lack of a better one.

Note the following important property of dislocations, stated by V. Volterra. If the cuts a_kb_k are to shift and change their shape, but in such a way that the points a_k and b_k remain on the contours L_k and L_{m+1} respectively and that the cuts never intersect each other, the quantities

ε_k, $\overset{\circ}{\alpha}_k$, $\overset{\circ}{\beta}_k$, determined by (45.3), obviously remain unchanged. In other words, these quantities do not alter from one system of cuts to another, as long as the latter are topologically equivalent.

It has been seen that, under the requirement of single-valuedness of the displacements, the stresses inside a body are completely determined by the external loads. This requirement is equivalent to the conditions

$$\varepsilon_k = \overset{\circ}{\alpha}_k = \overset{\circ}{\beta}_k = 0 \quad (k = 1, \ldots, m).$$

It is easily shown that the stresses will likewise be fully determined by given external loads and arbitrarily prescribed (small) quantities ε_k, $\overset{\circ}{\alpha}_k$, $\overset{\circ}{\beta}_k$; in fact, the "difference" between two solutions (if there exist two of them) obviously gives a solution for which there are no external stresses and for which

$$\varepsilon_k = \overset{\circ}{\alpha}_k = \overset{\circ}{\beta}_k = 0,$$

i.e., for which the displacements are single-valued. But under these conditions the stresses are known to be zero everywhere. The quantities ε_k, $\overset{\circ}{\alpha}_k$, $\overset{\circ}{\beta}_k$, of which there are $3\,m$, will be called *characteristics of dislocations* (they are the "caractéristiques de la distorsion" of V. Volterra).

NOTE 1. There arises the question: Why is there no possibility of dislocations in a *simply connected body*? For example, a sector may be cut from a circular disc in order to bring into contact and join free edges; thus, of course, stresses will arise in the disc and apparently the same case will occur as for multiply connected bodies. But the difference here is that in this case the stresses will not satisfy the conditions of continuity, stated in § 29, because it has been seen that for a simply connected body the displacements cannot be multi-valued, provided the conditions of continuity are fulfilled.

2. A similar answer must be given to the question as to why one had to restrict consideration to dislocations, caused by removal (or addition) of strips with *congruent* sides and joined in a definite manner.

§ 46. Thermal Stresses. There is a remarkable relation between the dislocations considered above and the stresses caused in a body by a non-uniform temperature distribution; this will now be explained. But first it is necessary to become acquainted with the law expressing the effect of non-uniform temperatures in an elastic body. The equations of the theory of elasticity, hitherto used, refer to the case when the temperature is the same throughout the body. On the basis of a law, enunciated

by J. M. C. Duhamel and F. Neumann (cf. A. E. H. Love [1], Chap. III), the following relations hold in the case of non-uniform heating between the components of strain and stress:

$$X_x = -\nu T + \lambda\theta + 2\mu\frac{\partial u}{\partial x}, \quad Y_y = -\nu T + \lambda\theta + 2\mu\frac{\partial v}{\partial y},$$

$$Z_z = -\nu T + \lambda\theta + 2\mu\frac{\partial w}{\partial z}, \tag{46.1}$$

$$Y_z = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \quad Z_x = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad X_y = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right);$$

here T denotes the temperature at a given point, taking as "zero" of the temperature scale the temperature of the body in its "natural" state; ν is some positive constant depending on a property of the material of the body. (This law is only strictly applicable for not too large temperature variations, because the coefficients λ, μ, ν change with the temperature and these changes cannot otherwise be disregarded.) The equations (46.1) replace, for the present, the generalized Hooke's law and they only differ from those, expressing the latter, by the terms $-\nu T$ on the right-hand sides of the first three formulae.

The components of stress must, of course, satisfy the same equations (18.1), since in their deduction no assumption regarding the temperature distribution had to be made.

Consider now the case of plane strain of a cylindrical body, studied in § 25 ($w = 0$, u, v independent of z), and assume that T does not depend on the coordinate z. Likewise let there be no body forces present. Then

$$Y_z = X_z = 0,$$

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = 0, \quad \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} = 0, \tag{46.2}$$

$$X_x = -\nu T + \lambda\theta + 2\mu\frac{\partial u}{\partial x}, \quad Y_y = -\nu T + \lambda\theta + 2\mu\frac{\partial v}{\partial y},$$

$$X_y = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \quad \theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \tag{46.3}$$

and

$$Z_z = \lambda\theta - \nu T, \tag{46.4}$$

or, noting that by (46.3)

$$X_x + Y_y = -2\nu T + 2(\lambda+\mu)\theta, \quad \theta = \frac{X_x + Y_y}{2(\lambda+\mu)} + \frac{\nu T}{\lambda+\mu},$$

$$Z_z = -\frac{\nu\mu}{\lambda+\mu} T + \frac{\lambda}{2(\lambda+\mu)}(X_x + Y_y). \tag{46.4'}$$

Let it now be assumed that one is dealing with *a steady state of heat flow*, so that the temperature T depends only on x, y and not on the time. Then it is known that

$$\Delta T = 0, \tag{46.5}$$

i.e., T is a harmonic function of x and y. Denote by $F(z)$ the function of the complex variable $z = x + iy$ (there being no danger of a confusion with the coordinate z), having as a real part $T(x, y)$, and put

$$u^*(x, y) + iv^*(x, y) = \int F(z)\, dz. \tag{46.6}$$

Obviously,

$$\frac{\partial u^*}{\partial x} = \frac{\partial v^*}{\partial y} = T, \quad \frac{\partial u^*}{\partial y} = -\frac{\partial v^*}{\partial x} \tag{46.7}$$

Further, let

$$u = u' + \frac{\nu u^*}{2(\lambda+\mu)}, \quad v = v' + \frac{\nu v^*}{2(\lambda+\mu)}, \tag{46.8}$$

where u', v' are two new functions. Substituting from (46.8) into (46.3) and using (46.7), it is easily verified that

$$X_x = \lambda\theta' + 2\mu\frac{\partial u'}{\partial x}, \quad Y_y = \lambda\theta' + 2\mu\frac{\partial v'}{\partial y}, \quad X_y = \mu\left(\frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y}\right), \tag{46.9}$$

where

$$\theta' = \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}.$$

Thus it is seen that *the functions X_x, Y_y, X_y, u', v' satisfy the well known equations of the plane theory of elasticity, as if the body were uniformly heated* (in fact, as if $T = 0$), where u', v' play now the parts of the displacements. (This property was stated in the Author's paper [1] and in a somewhat changed form in his paper [2]; a short study of the results has likewise been given in a section of his paper [3]. A long time afterwards, H. Poritsky [1] published similar results).

Thus, *the problem of the study of stresses* in a given cylindrical body, caused by a steady flow of heat, in the case of plane strain *is reduced to the ordinary problem* (i.e., for $T = 0$) *for a body of the same shape with the same external stresses acting on its sides.* This latter problem (concerned with X_x, Y_y, X_y, u', v') will be called the *auxiliary* problem. The fact that *the stresses X_x, Y_y, X_y are the same in the original and in the auxiliary problems* is very remarkable.

First, consider the case of *a simply connected body.* Suppose that no external forces are acting on its side surfaces. Then the auxiliary problem is known to have only the following solution (omitting rigid body displacements):

$$X_x = Y_y = X_y = 0, \quad u' = v' = 0.$$

Thus, *in a simply connected cylinder, steady heat flow* (which depends only on x and y) *does not cause stresses X_x, Y_y, X_y.* The displacements will be given by the formulae, obtained from (46.8),

$$u = -\frac{\nu u^*}{2(\lambda + \mu)}, \quad v = -\frac{\nu v^*}{2(\lambda + \mu)}, \tag{46.10}$$

where u^*, v^* are determined by (46.6), using the temperature $T(x, y)$. It must not be imagined that there are no stresses whatsoever present. In fact, the component Z_z will, in general, be different from zero and be given by (46.4′) (where one has now to put $X_x = Y_y = 0$):

$$Z_z = -\frac{\nu\mu}{\lambda + \mu} T(x, y). \tag{46.11}$$

It is seen that this direct stress Z_z must be applied to the faces of the cylinder as a necessary condition for the maintenance of plane strain.

If it is desired to find the solution when the faces are free from stress, one may, in the case of a long cylinder, resort to the following method (cf. § 25). The stresses applied, say, to the "upper" face, which are given by (46.11), are statically equivalent to a force, directed parallel to the generators of the cylinder, and a couple, the moment of which is parallel to the face; in an application, one may, for example, place the force at the centroid of the face. Similarly, the stresses acting on the "lower" end may be replaced by a force and couple, opposite to the former. Next, superimpose on the solution obtained above that of the problem of a cylinder, subject to tension and bending by forces and couples opposite to those above. (It will be shown in Part VII that the solution of this

problem for any (long) cylinder may be obtained by quite elementary means.) This will give an (approximate) solution of the problem. Actually, the stresses applied to the faces will now be statically equivalent to zero. Thus, by Saint-Venant's Principle (§ 23), they may be assumed to be, in general, non-existent (provided the dimensions of the faces are small compared with the length of the cylinder). Only near the ends, the solution will differ appreciably from the exact one. It may still be added that, as will be seen in Part VII, the components X_x, Y_y, X_y will be zero for the above-mentioned problem of tension and flexure of a cylinder. Thus, one will have in the final solution, as before, $X_x = Y_y = X_y = 0$ and only $Z_z \neq 0$.

When the dimensions of the ends are not small compared with the height of the cylinder, one has to look for a more exact solution which does not only take account of the resultant forces and moments, applied to the ends, but also of the actual stress distribution there.

Now the case of *multiply connected regions* of the type studied in the preceding section will be considered. In this case the function $F(z)$ the real part of which is the (single-valued) function $T(x, y)$ may be multi-valued. In fact, reasoning in the same way as for the function $\Phi(z)$ in § 35, it is seen that

$$F(z) = \sum_{k=1}^{m} B_k \log (z - z_k) + \text{a holomorphic function}, \qquad (46.12)$$

where B_k $(k = 1, \ldots, m)$ are real constants and z_k are arbitrary fixed points inside the contours L_k. Further [cf. the deduction of (35.3)]

$$u^* + iv^* = \int F(z)\,dz = z \sum_{k=1} B_k \log (z - z_k) + \sum_{k=1} (\alpha_k^* + i\beta_k^*) \log (z - z_k) + \\ + \text{a holomorphic function}, \qquad (46.13)$$

where α_k^*, β_k^* are certain real constants. (The constants B_k, α_k^*, β_k^* will be known, if the temperature $T(x, y)$ is given at each point.) For an (anti-clockwise) circuit of a contour, surrounding L_k, this expression undergoes an increase (cf. the notation of § 45)

$$u^{*+} - u^{*-} + i(v^{*+} - v^{*-}) = 2\pi i(zB_k + \alpha_k^* + i\beta_k^*). \qquad (46.14)$$

Let it be assumed that the body under consideration is not subject to dislocations, i.e., that *the displacements of the original problem* (u, v) *are single-valued*. Then, by (46.8),

$$0 = (u'^+ - u'^-) + i(v'^+ - v'^-) + \{(u^{*+} - u^{*-}) + i(v^{*+} - v^{*-})\}\frac{\nu}{2(\lambda + \mu)},$$

and hence, using (46.14),

$$(u'^{+} - u'^{-}) + i(v'^{+} - v'^{-}) = \frac{\pi i \nu}{\lambda + \mu} (B_k z + \alpha_k^* + i\beta_k^*). \quad (46.15)$$

This formula proves that *the displacements u', v' of the "auxiliary problem" are the same as if the body (which is at uniform temperature) were subject to dislocations with the characteristics* [cf. (45.2)]

$$\begin{gathered} \varepsilon_k = - \frac{\pi\nu}{\lambda + \mu} B_k, \\ \frac{\pi\nu}{\lambda + \mu} \beta_k^*, \quad \beta_k^0 = - \frac{\pi\nu}{\lambda + \mu} \end{gathered} \quad (46.16)$$

Thus, the auxiliary problem is reduced to the determination of the elastic equilibrium for a uniform temperature ($T = 0$) and for given characteristics of dislocations.

If there are no external stresses acting on the side surface, the stresses X_x, Y_y, X_y (in the auxiliary as well as in the original problem) *are the same, as if the body were subject to given dislocations in the absence of external loading and for uniform temperature.*

If the sides of the cylinder are loaded in an arbitrary manner, the solution of the ordinary problem of the plane theory of elasticity for given external stresses applied to the boundary must be superimposed. As regards stresses applied to the ends, all that has been said with regard to the case of simply connected regions remains in force with the only exception that the stress Z_z will not be given by (46.11), but by the general formula (46.4′), because now $X_x + Y_y$ will, generally speaking, be different from zero.

CHAPTER 7

TRANSFORMATION OF THE BASIC FORMULAE FOR CONFORMAL MAPPING

§ 47. Conformal transformation. In this section the simplest properties of conformal transformations will be recalled, without proofs being given. A very detailed study of the relevant theoretical problems may, for example, be found in I. I. Privalov's book [1] or in a very recent book by M. A. Lavrentjev [1].

Let z and ζ be two complex variables such that

$$z = \omega(\zeta), \tag{47.1}$$

where $\omega(\zeta)$ is a single-valued analytic function in some region Σ in the ζ plane. The equation (47.1) relates every point ζ of Σ to some definite point z in the z plane. These latter points will cover in the z plane some region S. Conversely, let it be assumed that each point z of S, by (47.1), corresponds to some definite point of Σ. It will then be said that (47.1) determines an invertible single-valued *conformal transformation* or *conformal mapping* of the region S into the region Σ (or conversely). (In the sequel, when speaking of conformal transformations, they will always be assumed to be reversible and single-valued.)

The transformation is called conformal, because of the following property which is characteristic for relations of the form (47.1), where $\omega(\zeta)$ is a holomorphic function: If in Σ two linear elements be taken which extend from some point ζ and form between them an angle α, the corresponding elements in S will form the same angle α and the sense of the angle will be maintained.

Unless stated otherwise, regions, considered in the sequel, will always be assumed to be bounded by one or several simple contours, as was stated in § 37. The regions Σ and S may be finite or infinite (and, in particular, one of them may be finite, while the other is infinite). If, for example, the region Σ is finite and S is infinite, the function $\omega(\zeta)$ must become infinite at some point of Σ (as otherwise there would not be some point of Σ corresponding to the point at infinity in S). It is easily

proved that $\omega(\zeta)$ must have a simple pole at that point, i.e., assuming for simplicity that $z = \infty$ corresponds to $\zeta = 0$, then

$$\omega(\zeta) = \frac{c}{\zeta} + \text{a holomorphic function,} \tag{47.2}$$

where c is a constant and no other singularities can occur in Σ; otherwise the transformation would not be reversible and single-valued. If Σ and S are both infinite and the points at infinity correspond to each other, the function $\omega(\zeta)$ must for the same reason have the form

$$\omega(\zeta) = R\zeta + \text{a holomorphic function,} \tag{47.2'}$$

where R is a constant. It will be remembered that a function, holomorphic in an infinite region, is understood to be one which is holomorphic in any finite part of this region and which for sufficiently large $|\zeta|$ may be represented by a series of the form

$$a_0 + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \cdots.$$

Further, it may be shown that the derivative $\omega'(\zeta)$ cannot become zero in Σ; otherwise the transformation would not be reversible and single-valued.

Next there arises the following question: If two arbitrary regions Σ and S be given, is it always possible to find a function $\omega(\zeta)$ such that (47.1) gives a transformation of S into Σ (and vice versa)? This problem has been solved in recent times with extremely wide generality. Here only some general remarks will be made. First of all, it is obviously impossible to obtain a (reversible and single-valued) transformation of a simply connected region into a multiply connected one.

Consider now the case when the two regions are simply connected and bounded by simple contours. Then a relation of the form (47.1), mapping the one region on to the other, can always be found and the function will be continuous up to the contours. In addition, the function $\omega(\zeta)$ may always be chosen so that an arbitrarily given point ζ_0 of Σ corresponds to an arbitrarily given point z_0 of S and that the directions of arbitrarily chosen linear elements, passing through ζ_0 and z_0, correspond. These supplementary conditions will fully determine the function $\omega(\zeta)$.

For simplicity, suppose that Σ is the unit circle with its centre at the origin. Denote the circumference of the circle by γ, so that one has on γ

$|\zeta| = 1$. Since the transformation is to be continuous up to the contours, the function $\omega(\zeta)$ will be continuous on γ from the left (taking the anti-clockwise direction as positive); let its boundary values be denoted by $\omega(\sigma)$, where $\sigma = e^{i\theta}$ is a point of γ.

In the sequel, the behaviour of the derivative $\omega'(\zeta)$ near and on γ will be of interest; in particular, the question has to be considered whether $\omega'(\zeta)$ vanishes at any point of the contour. This problem is resolved by the following proposition. For the sake of simplicity it has here been formulated for less general conditions than in the paper by V. I. Smirnov [2]; the same remark applies to the subsequent proposition regarding the second derivative.

If the coordinates of the points of the contour of S have continuous derivatives up to the second order along the arc (i.e., if the curvature of the contour changes continuously), the function $\omega'(\zeta)$ is continuous up to γ and, denoting its boundary values by $\omega'(\sigma)$,

$$\omega'(\sigma) = \frac{d\omega(\sigma)}{d\sigma}; \tag{47.3}$$

in addition,

$$\omega'(\sigma) \neq 0 \text{ everywhere on } \gamma \tag{47.4}$$

(it being already known that $\omega'(\zeta) \neq 0$ inside γ). Further, if the coordinates of the points of the contour of S have also continuous derivatives up to the third order, the second derivative $\omega''(\zeta)$ will be continuous on γ from the left and its boundary value $\omega''(\sigma)$ is given by

$$\omega''(\sigma) = \frac{d\omega'(\sigma)}{d\sigma}. \tag{47.3'}$$

In the sequel, unless stated otherwise, it will be assumed that one is dealing with contours satisfying these conditions.

Note also that, once the region S has been mapped on to the unit circle, it can always be transformed into the infinite plane with a circular hole. For this purpose it is sufficient to make the substitution

$$\zeta = \frac{1}{\zeta_1};$$

in fact, when ζ covers the region $|\zeta| < 1$, ζ_1 covers the infinite region with a circular hole $|\zeta_1| > 1$, and hence, considering z as a function of ζ_1, one obtains the required transformation. In the sequel, finite simply

connected regions will almost always be mapped on to the circle $|\zeta| < 1$, and infinite simply connected regions on to the region $|\zeta| > 1$. In both cases one could limit oneself to transformations into the circle $|\zeta| < 1$, but the stated convention is somewhat more convenient in practical applications.

Next, a few remarks will be made with respect to multiply connected regions. Obviously only regions of equal connectivity may be mapped on to each other. For example, a doubly connected region S (i.e., a region, bounded by two contours, because regions of more general shape will not be considered here) may always be mapped on to a circular ring. But, in contrast to the case of simply connected regions, this ring may not be chosen quite arbitrarily. The ratio of the radii of the inner and outer circles will depend on the shape of S.

Two simple theorems will now be stated which are very useful in practice:

I. *Let Σ be a finite or infinite (connected) region in the ζ plane, bounded by a simple contour* γ (no other assumptions being required with respect to the contour), *and let $\omega(\zeta)$ be a function, holomorphic in Σ* (including the point at infinity, if the region is infinite) *and continuous up to the contour. Further, let the points, defined by $z = \omega(\zeta)$, describe in the z plane (moving always in one and the same direction) some simple contour L, when ζ describes* γ (where it is assumed that different points of γ correspond to different points of L). *Then $z = \omega(\zeta)$ gives the conformal transformation of the region S, contained inside L, on the region Σ (and vice versa)* (cf. W. F. Osgood [1] p. 377 where a completely elementary proof is given, assuming the contours γ and L to consist of a finite number of smooth arcs; see also M. A. Lavrentjev [1], § 61).

This theorem may be generalized to the case of multiply connected regions in the following manner: (The proof of the generalized theorem differs little from that given for the preceding one by Osgood).

II. *Let Σ be a finite or infinite (connected) region, bounded by several simple contours $\gamma_1, \gamma_2, \ldots, \gamma_k$ (having no points in common). Let $\omega(\zeta)$ be a function, holomorphic in Σ and continuous up to the boundary, and let the point z, defined by $z = \omega(\zeta)$, describe in the z plane the simple contours $L_1, L_2, \ldots, L_k$ (not having common points), bounding some (connected) region S, when ζ describes the contours $\gamma_1, \ldots, \gamma_k$. For this purpose it has been assumed that, when ζ describes the boundary of Σ in the positive direction (i.e., leaving Σ all the time on the left), the corresponding point z describes the*

boundary of S likewise in the positive direction. Under these conditions $z = \omega(\zeta)$ represents the conformal transformation of S on to Σ (and vice versa).

These theorems are easily generalized in different directions (e.g., for the case, when the boundaries contain arcs), but this will not be done here.

NOTE. It is easily seen that, if Σ and S are conformally transformed into one another by a relation of the form (47.1), the point z will move in the positive direction along the boundary of S, when ζ describes the boundary of Σ in the positive direction. This condition has not been introduced into the formulation of Theorem I, since it is not required in the proof; the conditions, included there, are already sufficient to prove the theorem. Thus, the direction in which S is described will without fail be as stated above. But for the formulation of Theorem II this condition is necessary; otherwise the theorem may be found to be untrue.

The statement, referring to the directions in which the contours are described, may be proved in the following manner. Let ν be the normal to the boundary of Σ, directed inward, and τ the tangent in the positive direction of the boundary; then ν will be pointing to the left of τ. The same relation will exist between the corresponding directions of n and t at points of the boundary of S, in view of the fact that conformal transformation does not only preserve the magnitudes of angles, but also their sense. Here it has been assumed that the transformation is conformal up to the boundary, but the above stated property is also easily proved for the general case.

§ 48. Simple examples of conformal mapping.

1°. Bilinear function. Consider the case, when z is a *bilinear function of* ζ

$$z = \frac{a\zeta + b}{c\zeta + d}, \tag{48.1}$$

where a, b, c, d are constants (in general, complex) and $ad - bc \neq 0$ (the latter condition having been introduced to exclude the case, when the right-hand side of (48.1) does not depend on ζ). Solving (48.1) for ζ, one obtains the, likewise bilinear, inverse transformation

$$\zeta = \frac{-dz + b}{cz - a}. \qquad (48.1)$$

Thus every point of the ζ plane corresponds to a definite point of the z plane *and vice versa*. The point at infinity has not been excluded. In

fact, the point $\zeta = -\frac{u}{c}$ corresponds to the point $z = \infty$, and $z = \frac{u}{c}$ to $\zeta = \infty$. Hence (48.1) gives an invertible, single-valued relation between the unbounded planes of z and ζ. (It may be shown that the function (48.1) is the only one having the stated property.)

The bilinear transformation has the remarkable property that it *preserves circles*, i.e., that it relates any circle in the ζ plane to a circle in the z plane and vice versa. For this purpose straight lines are to be considered particular cases of circles. This is most simply proved in the following way. The equation of any circle in the z plane is known to be of the form

$$A(x^2 + y^2) + Bx + Cy + D = 0, \tag{a}$$

where A, B, C, D are real constants (the case $A = 0$ corresponding to straight lines). Since $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$, $x^2 + y^2 = z\bar{z}$, this equation may be written

$$Az\bar{z} + Mz + \bar{M}\bar{z} + D = 0, \tag{b}$$

where A and D are real and $M, \bar{M}$ are conjugate complex constants. It is easily verified that, conversely, an equation of the preceding type may always be reduced to the form (a). In order to obtain now the equations of the lines corresponding in the ζ plane to the circles in the z plane, it is sufficient to substitute in (b) from (48.1). After some simplifications, one finds

$$A_0\zeta\bar{\zeta} + M_0\zeta + \bar{M}_0\bar{\zeta} + D_0 = 0,$$

where A_0, D_0 are real, M_0, $\bar{M}_0$ are conjugate complex constants. Hence one has again obtained the equation of circles, as was to be proved.

One of the simplest particular cases of (48.1) is

$$z = \frac{R^2}{\zeta}, \quad \zeta = \frac{R^2}{z}, \tag{48.2}$$

where R is a real constant; let it be assumed that $R > 0$. In order to give a clear description of this transformation, the concept of the *reflection of a point in a circle* will be recalled. Let Γ be the circle with radius R and with the origin as centre. Let z be some point in its plane. Construct another point z', related to z in the following manner:

$$z\bar{z}' = R^2. \tag{48.3}$$

If $z = re^{i\vartheta}$, then obviously $z' = r'e^{i\vartheta}$, where $r = |z|$ and $r' = |z'|$ are

the distances of z and z' from O which are connected by the relation

$$rr' = R^2. \tag{48.3'}$$

Thus, the points z and z' are located on the same ray through O and their distances from that point are related by (48.3'). The point z', related to the point z in the above manner, is called the *reflection of z in* Γ. Clearly z is in the same sense the reflection of z'. The transformation (48.3), relating z and z', is also called an *inversion*. The points z and z' are also called *conjugate points* with respect to the circle Γ. When one of the points is given, the other is easily constructed by the use of a compass and ruler: If, for example, z be given outside Γ, it is sufficient for the construction of z' to draw the tangent from z to Γ and from there the perpendicular to the ray Oz (Fig. 20).

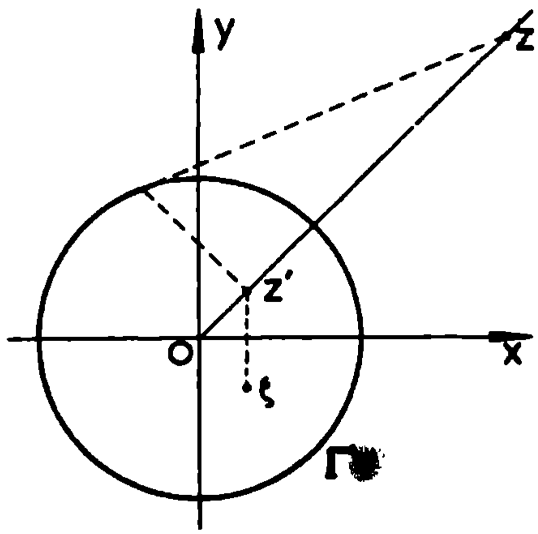


Fig. 20.

Obviously, for an inversion, the points of Γ correspond to themselves and the point $z = \infty$ corresponds to $z' = 0$; points outside Γ go over into points inside, and vice versa.

Now consider the transformation (48.2). Imagine that the ζ plane is placed on top of the z plane in such a way that the origins and axes of their coordinate systems coincide. The point ζ, corresponding to the point $z = re^{i\vartheta}$, will then be given by

$$\zeta = r'e^{-i\vartheta} = \bar{z}'.$$

Hence the point ζ may be found in the following manner: Reflect the point z in the circle Γ and reflect its image, thus obtained, in the real axis; the latter image will be the point ζ (Fig. 20).

Next, another bilinear transformation of the form

$$\frac{\zeta}{1 - a\zeta}, \quad \zeta = \frac{\quad}{1 + az} \tag{48.4}$$

will be studied, where a is a real positive constant. The points $\zeta = 0$ and $\zeta = 1/a$ correspond to the points $z = 0$ and $z = \infty$; the point $\zeta = \infty$ corresponds to the point $z = -1/a$. Thus straight lines, passing through

$\zeta = 0$ in the ζ plane, will correspond to circles, passing through the points

$$O(z = 0) \text{ and } O'(z = -1/a)$$

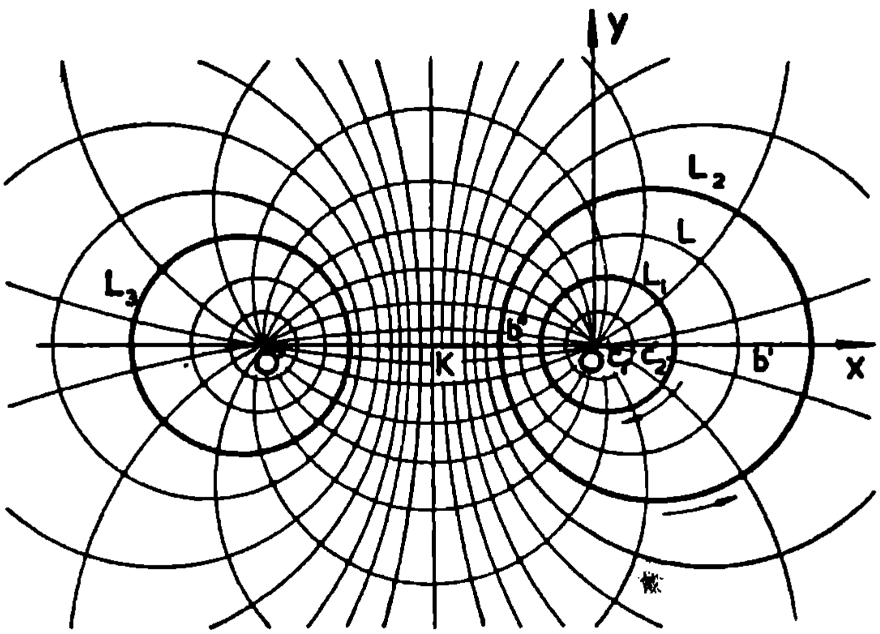


Fig. 21*a*.

in the z plane (Fig.s 21*a*, 21*b*). Further, concentric circles with the centre at $\zeta = 0$ will correspond to circles in the z plane which are orthogonal to the circles, passing through the points $z = 0$ and $z = -1/a$ (as a consequence of the fact that the transformation is conformal); the centres of these circles obviously lie on the axis Ox.

Draw about the origin of the ζ plane the circle γ with radius ρ. The points

$$\zeta = +\rho \text{ and } \zeta = -\rho$$

will correspond in the z plane to the points

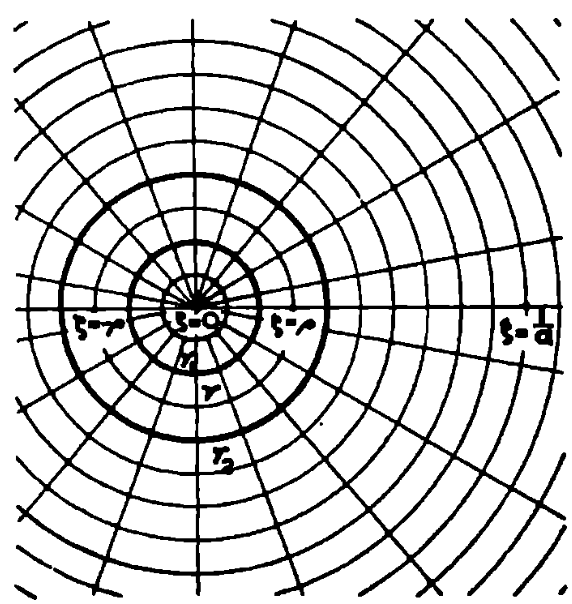


Fig. 21*b*.

$$b' = \frac{\rho}{1 - a\rho}, \quad b'' = -\frac{\rho}{1 + a\rho} < 0 \qquad (48.5)$$

on the Ox axis. Thus the abscissa c of the centre of the circle L, corresponding in the z plane to the circle γ, and its radius are given by

$$c = \tfrac{1}{2}(b' + b'') = \frac{a\rho^2}{1 - a^2\rho^2}, \qquad r = \tfrac{1}{2}(b' - b'') = \frac{\rho}{1 - a^2\rho^2}; \tag{48.6}$$

let it be assumed that $r < 0$, if the point b' lies to the left of the point b''. If $\rho < 1/a$, then $b' > 0$ and $r > 0$. When $\rho \to 1/a$, r and c increase beyond all bounds and L becomes in the limit the straight line perpendicular to the axis Ox and passing through the point K with abscissa $- 1/2a$. If $\rho > 1/a$, the corresponding circle in the z plane lies on the other side of this straight line.

Consider now two circles L_1 and L_2 in the z plane, corresponding to two circles γ_1 and γ_2 with radii ρ_1 and ρ_2 in the ζ plane, and let $\rho_1 < \rho_2 < < 1/a$. Then, obviously, the transformation (48.4) gives the conformal mapping of the region, contained between the two eccentric circles L_1 and L_2, on the ring, bounded by γ_1 and γ_2. Provided the elements, determining the first region, be given, i.e., the radii

$$r_1, \quad r_2 \quad (r_2 > r_1)$$

of the circles L_1, L_2 and the distance l between their centres ($l < r_2 - r_1$), then it is easy to determine the quantity a, appearing in (48.4), and the radii ρ_1, ρ_2 of the circles γ_1, γ_2. In fact, these quantities are given by the formulae

$$r_1 = \frac{\rho_1}{1 - a^2\rho_1^2}, \quad r_2 = \frac{\rho_2}{1 - a^2\rho_2^2}, \quad \frac{a\rho_2^2}{1 - a^2\rho_2^2} - \frac{a\rho_1^2}{1 - a^2\rho_1^2} = l, \tag{48.7}$$

from which one obtains

$$\begin{gathered} \frac{l}{\sqrt{(r_1^2 - r_2^2)^2 - 2l^2(r_1^2 + r_2^2) + l^4}}, \\ \rho_1 = \frac{\sqrt{1 + 4r_1^2a^2} - 1}{2r_1a^2}, \quad \rho_2 = \frac{\sqrt{1 + 4r_2^2a^2} - 1.}{2r_2a^2} \end{gathered} \tag{48.8}$$

The quantities a, ρ_1, ρ_2 are easily constructed by the use of a compass and ruler. It is obvious that the points $z = 0$ and $z = - 1/a$ are simultaneously *conjugate* with respect to the two circles L_1 and L_2, and this property allows the immediate construction of the above points.

By the same method, the infinite region, consisting of the points outside two given circles L_1 and L_3 (Fig. 21a), may be mapped on the ring

bounded by the two concentric circles γ_1 and γ_3 with radii ρ_1 and ρ_3. In this case $\rho_3 > 1/a$.

2°. Pascal's limaçon

Let

$$z = \omega(\zeta) = R(\zeta + m\zeta^2), \; R > 0, \quad 0 \leq m \leq \tfrac{1}{2}. \qquad (48.9)$$

Putting

$$z = x + iy, \quad \zeta = \rho e^{i\vartheta},$$

one finds

$$x + iy = R(\rho e^{i\vartheta} + m\rho^2 e^{2i\vartheta}),$$

whence

$$x = R(\rho \cos \vartheta + m\rho^2 \cos 2\vartheta), \quad y = R(\rho \sin \vartheta + m\rho^2 \sin 2\vartheta). \quad (48.10)$$

When the point ζ describes the unit circle γ, the point (x, y) describes in the z plane the curve L the parametric representation of which is

$$x = R(\cos \vartheta + m \cos 2\vartheta), \quad y = R(\sin \vartheta + m \sin 2\vartheta). \quad (48.11)$$

This curve is called *Pascal's limaçon* and it is a particular case of the epitrochoids studied later on. If, as has been assumed,

$$0 \quad m$$

this curve does not intersect itself and, while ϑ varies from 0 to 2π, the point z traces it out in one and the same direction. Thus, by what has been stated at the end of the preceding section, (48.9) gives the conformal transformation of the region inside Pascal's limaçon on to the unit circle.

For $m = 0$, the limaçon of Pascal becomes a circle and, for $m = \frac{1}{2}$, a cardioid. In the latter case, the curve has a cusp at the point, corresponding to $\zeta = -1$, since $\omega'(\zeta) = 0$ there. (The fact that $\omega'(\zeta)$ becomes zero on γ does not contradict the statements of § 47, since in the case of the cardioid the boundary has a cusp.)

Circles with radii $\rho < 1$ in the ζ plane also correspond to limaçons of Pascal the parametric representation of which is obtained by putting in (48.10)

$$\rho = \text{const.}$$

The radii of the circles γ in the ζ plane are transformed into curves in the z plane, their parametric representation being found by putting

$$\vartheta = \text{const.}$$

in (48.10) (ρ will now be the parameter, $0 \leq \rho \leq 1$); these curves are easily verified to be parabolas. In fig. 22*a* are shown

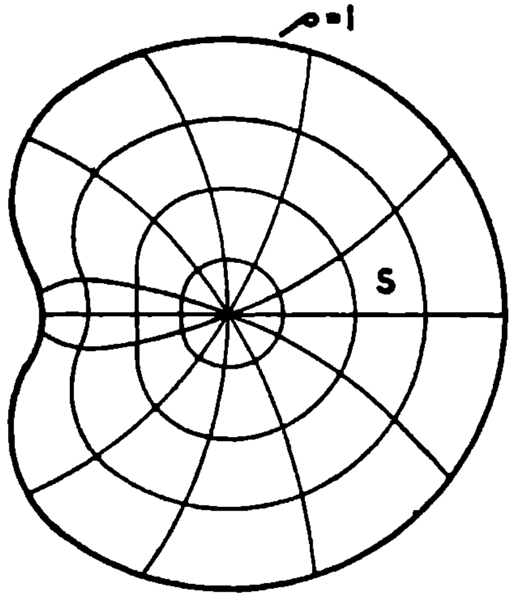


Fig. 22*a*.

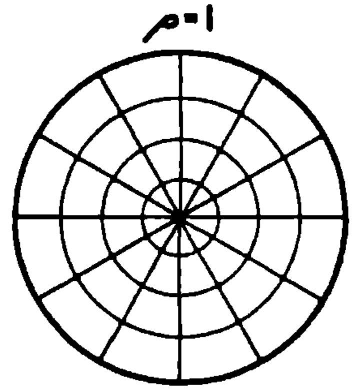


Fig. 22*b*.

the curves, corresponding to the circles $\rho =$ const. and the rays $\vartheta =$ const. of Fig. 22*b*. These curves are, of course, orthogonal.

3°. Epitrochoids.

Let
$$z = \omega(\zeta) = R(\zeta + m\zeta^n), \quad R > 0, \quad 0 \leq m \leq \frac{1}{n}, \tag{48.12}$$

where n is an integer larger than unity. Putting, as before, $z = x + iy$ and $\zeta = \rho e^{i\vartheta}$, one finds

$$x = R(\rho \cos \vartheta + m\rho^n \cos n\vartheta), \quad y = R(\rho \sin \vartheta + m\rho^n \sin n\vartheta). \tag{48.13}$$

The circle $|\zeta| = \rho = 1$ corresponds in the z plane to the curve L with the parametric representation

$$x = R(\cos \vartheta + m \cos n\vartheta), \qquad y = R(\sin \vartheta + m \sin n\vartheta). \tag{48.14}$$

These curves are *epitrochoids*. In fact, if a circle of radius r_1 rolls (in the z plane) on the outside of a circle with radius r_2, then a point M, lying at a fixed distance l from the centre of the moving circle and travelling with it, describes the curve

$$x = (r_1 + r_2) \cos \vartheta + l \cos n\vartheta, \quad y = r_1 + r_2 \sin \vartheta + l \sin n\vartheta, \tag{48.14'}$$

where ϑ denotes the polar angle of the point of contact of the circles and $n = (r_1 + r_2)/r_1$. Putting

$$r_1 = \overline{}, \quad r_2 = R \quad {}^{n-1} \qquad l = mR,$$

one finds that the curve (48.14′) agrees with the curve (48.14). Since, by assumption, $m \leq 1/n$, one has $l \leq r_1$. Hence, the point M lies inside the rolling circle and the curve does not intersect itself. In the limiting case $m = 1/n$ the point M lies on the circumference of the rolling circle and the curve becomes an *epicycloid* having $n - 1$ cusps. Fig. 23 shows the case $n = 1/m = 4$. On the basis of the theorem, stated in § 47, it is concluded that (48.12) maps the region inside the curve L on the region $|\zeta| < 1$. The circles $\rho = \text{const.}$ of the ζ plane correspond in the z plane to epitrochoids the parametric representation of which is given by (48.13).

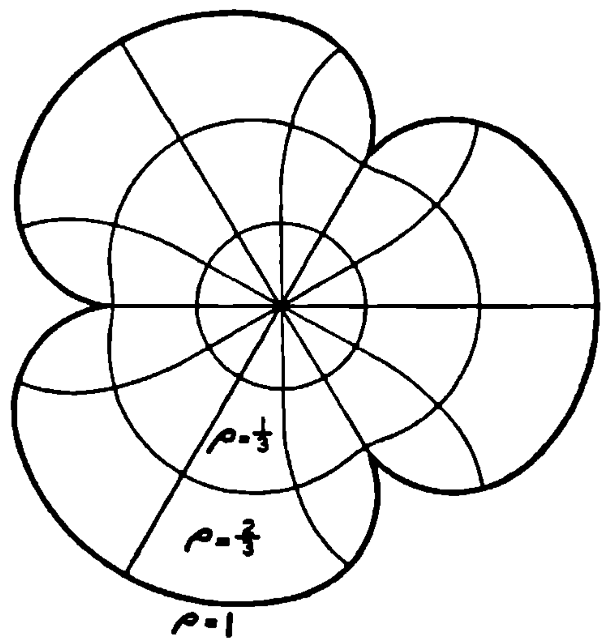


Fig. 23.

4° Hypotrochoids.

Let

$$= \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta^n}\right), \quad R > 0, \quad 0 \leq m \qquad {}_{n} \tag{48.15}$$

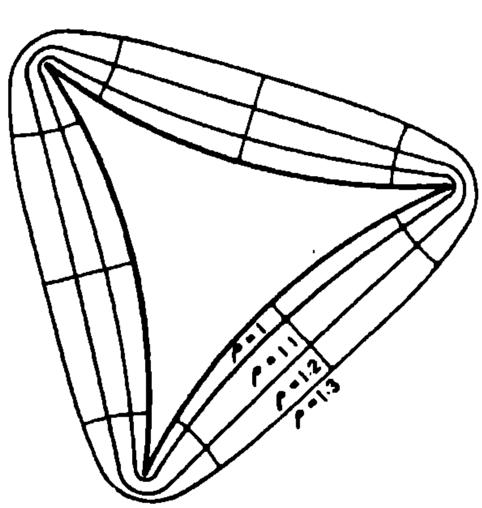
where n is a positive integer. In this case the curve L corresponding to $|\zeta| = 1$ is easily seen to be an hypotrochoid which does not intersect itself. It is described by the point M of a circle of radius r_1, rolling on the inside of a circle with radius r_2; if l is the distance of M from the centre of the moving circle, then

$$\begin{matrix} R \\ n \end{matrix} \qquad r_2 = R\,\frac{n+1}{n}, \quad l = mR.$$

It is easily seen that (48.15) maps the outside of L in the z plane on to the region $|\zeta| > 1$. Circles $|\zeta| = \rho = \text{const.} > 1$ in the ζ plane also

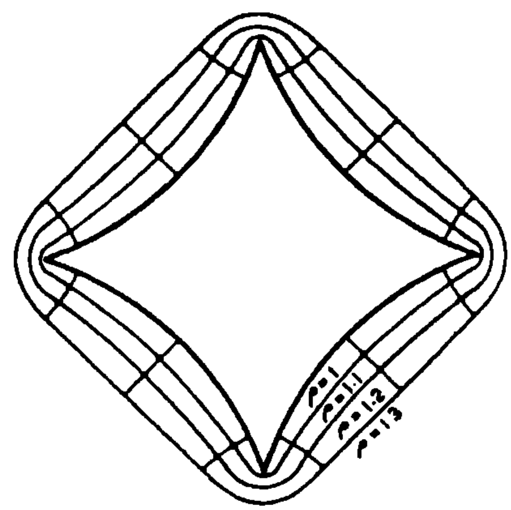
correspond to hypotrochoids in the z plane. If $n = 1$, the contour L will be an ellipse; this case will be considered in detail later on. For $m = 1/n$, the curve L becomes an hypocycloid with $n + 1$ cusps.

When $n = 1/m = 2$ or $n = 1/m = 3$, the corresponding contours have three or four cusps respectively, and they resemble in shape a triangle or square. Circles with radii $\rho > 1$ in the ζ plane correspond in the z plane to hypotrochoids which likewise for ρ near 1 resemble triangles or squares with rounded corners. In Figs. 24 and 25 the cases $n = 1/m = 2$ and $n = 1/m = 3$ are illustrated.



$n = \frac{1}{m} = 2$

Fig. 24.



$n = \frac{1}{m} = 3$

Fig. 25.

If in (48.15) ζ is replaced by $1/\zeta$, one obtains the transformation of the region inside the hypotrochoids on the unit circle; in this case

$$= \omega(\zeta) = R\left(\frac{1}{\zeta} + m\zeta^n\right). \tag{48.15'}$$

5°. Elliptic rings.

Let

$$z = \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right), \quad R > 0, \quad m \geq 0, \tag{48.16}$$

i.e., in the above notation,

$$x = R\left(\rho + \frac{m}{\rho}\right)\cos\vartheta, \quad y = R\left(\rho - \frac{m}{\rho}\right)\sin\vartheta. \tag{48.17}$$

Circles with radii ρ_1 correspond to ellipses in the z plane, their parametric

representation being

$$x = R\left(\rho_1 + \frac{m}{\rho_1}\right)\cos\vartheta, \quad y = R\left(\rho_1 - \frac{m}{\rho_1}\right)\sin\vartheta.$$

If $\rho_1^2 \geq m$, then the semi-axes of the ellipses will be

$$a_1 = R\left(\rho_1 + \frac{m}{\rho_1}\right), \quad b_1 = R\left(\rho_1 - \frac{m}{\rho_1}\right) \tag{48.18}$$

and the point z describes an ellipse in the z plane in an anti-clockwise direction, as the point ζ moves around the circle with radius ρ_1 in the ζ plane, likewise in an anti-clockwise direction.

Thus, if one selects in the ζ plane two circles γ_1, γ_2 with radii ρ_1, ρ_2, and if $\rho_2 > \rho_1 \geq \sqrt{m}$, then, by the theorem of § 47, (48.17) maps the region between the ellipses L_1 and L_2, corresponding to these circles, on the ring between them. The ellipses will be confocal, since by (48.18) the distance c of the foci of the ellipse L_1 from the origin is given by $c^2 = a_1^2 - b_1^2 = 4mR^2$, i.e., it is independent of ρ_1. Circles with radii ρ ($\rho_1 < \rho < \rho_2$) will become ellipses, lying between L_1 and L_2 and confocal with the latter. The rays $\vartheta = \text{const.}$ in the ζ plane will correspond to confocal hyperbolas, having the same foci as the ellipses. These ellipses and hyperbolas are, of course, orthogonal.

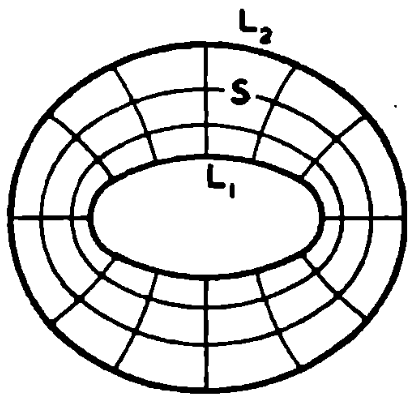


Fig. 26.

If one lets ρ_2 tend to infinity, one obtains in the z plane the infinite region consisting of the points outside the ellipse L_1; this region is transformed into the ζ plane with the circular opening γ_1. In this case, the circle $\rho_1 = 1$ will always be used, and hence one will have $m \leq 1$. For $m = 1$, the ellipse becomes a straight slit. For $m = 0$, one obtains a circle.

If one replaces in (48.16) ζ by $1/\zeta$, i.e., if one puts

$$z = \omega(\zeta) = R\left(\frac{1}{\zeta} + m\zeta\right), \quad R > 0,\ 0 \leq m < 1, \tag{48.16'}$$

one obtains the transformation of the plane with an elliptic hole into the unit circle $|\zeta| < 1$.

6. As has just been stated, the function

$$z_1 = x_1 + iy_1 = \omega_1(\zeta) = R\left(\frac{1}{\zeta} + m\zeta\right), \quad R > 0,\ 0 \leq m \leq 1 \tag{48.19}$$

transforms the infinite z_1 plane with an elliptic hole into the circle $|\zeta| < 1$. The equation of the boundary of the opening will be

$$\frac{}{R^2(1+m)^2} + \frac{y^2}{R^2(1-m)^2} = 1. \tag{48.20}$$

Let

$$z = \tag{48.21}$$

then, by (48.19),

$$\omega(\zeta) = \frac{\zeta}{R(1+m\zeta^2)} \tag{48.22}$$

which maps the finite region bounded by the lemniscate of Booth on the unit circle. When m is almost equal to unity, this region differs little from that produced by two contacting circles of equal radius. Fig. 27 shows the curve corresponding to $m = 0.8$.

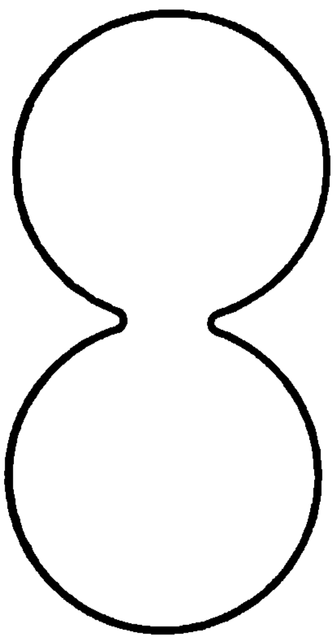


Fig. 27.

If one replaces the transformation (48.21) by

$$z - c = \frac{1}{\quad},$$

where the point c is outside the ellipse (48.20), then one is easily seen to obtain the transformation of some region, which for $m = 1$ becomes the infinite plane cut along the arc of a circle, into the circle $|\zeta| < 1$. (In fact, for $m = 1$, the ellipse becomes a straight slit, and hence it is transformed into the arc of a circle, because the bilinear transformation, of which the above is a special case, maps straight lines into circles).

§ 49. Curvilinear coordinates, connected with conformal transformations into circular regions. In the sequel, use will be made of conformal mapping of a given region S in the z plane on the region Σ of the ζ plane, where the latter will either be a circle, a circular ring or the infinite plane with a circular hole; the origin $\zeta = 0$ will always be chosen as centre. In all these cases it is natural to introduce polar coordinates ρ and ϑ in the ζ plane by putting $\zeta = \rho e^{i\vartheta}$. Circles $\rho =$ const. and radii $\vartheta =$ const. of the ζ plane willcorre spond to certain curves in the z plane which will be denoted by $\rho =$ const. and $\vartheta =$ const.

If S is a finite region bounded by one contour L and Σ the unit circle with centre at $\zeta = 0$, it can always be assumed that the points $z = 0$ and $\zeta = 0$ correspond to each other. Then the curves $\rho = \text{const}$ in the z plane will be simple contours, surrounding $z = 0$, while the curves $\vartheta = \text{const.}$ will pass through this point and end at the contour L which will correspond to $\rho = 1$.

If S is an infinite region bounded by a simple contour L and Σ the infinite plane with a circular hole, and if the points $\zeta = \infty$ and $z = \infty$ correspond to each other (and it is known that this can always be arranged), the curves $\rho = \text{const.}$ will be contours surrounding L and the curves $\vartheta = \text{const.}$ will start on L and go to infinity. Similar circumstances will prevail when the infinite region S is mapped on the circle $|\zeta| < 1$. Likewise it is easy to understand the distribution of the curves $\rho = \text{const.}$ and $\vartheta = \text{const.}$ in the case of a region S, bounded by two contours and mapped on the circular ring Σ.

The quantities ρ and ϑ may be considered as curvilinear coordinates of the point (x, y) of the z plane. They are related to x, y by the equation

$$x + iy = \omega(\zeta) = \omega(\rho e^{i\vartheta}); \tag{49.1}$$

the lines $\rho = \text{const.}$ and $\vartheta = \text{const.}$ will be the coordinate lines which, as a consequence of the conformal property of the transformation, will be orthogonal.

Let there be given some point of the z plane and draw through it the relevant lines

$$\rho = \text{const. and } \vartheta = \text{const.}$$

Let (ρ) denote the tangent to the line $\vartheta = \text{const}$, drawn to the side of increasing ρ. Let (ϑ) be the tangent to the line $\rho = \text{const}$, drawn to the side of increasing ϑ. These tangents will be called the axes of the curvilinear coordinates at the point (ρ, ϑ).

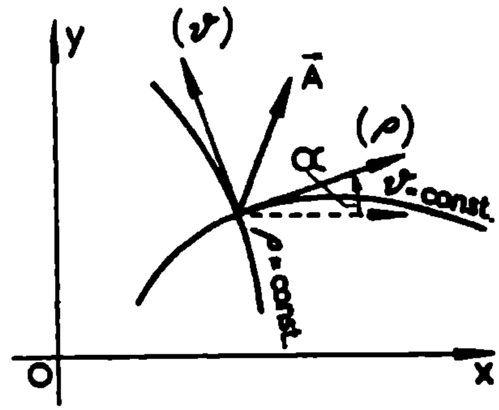


Fig. 28.

The system of axes (ρ), (ϑ) in the stated order is oriented as the system of axes Ox, Oy, i.e., moving in the positive direction of the axis (ρ), the axis (ϑ) is directed to the left. This follows already from the fact that a conformal transformation preserves the orientation of directions.

Let $\vec{A}$ be some vector in the z plane, starting from the point $z = \omega(\rho e^{i\vartheta})$

(Fig. 28). The projections of this vector on the axes Ox, Oy will be denoted by A_x, A_y, and on the axes (ρ), (ϑ) by A_ρ, A_ϑ. Obviously

$$A_\rho + iA_\vartheta = e^{-i\alpha}(A_x + iA_y), \tag{49.2}$$

where α is the angle between the axes (ρ) and Ox, measured from the latter anti-clockwise. If the point z be given a displacement dz in the tangential direction (ρ), the corresponding point ζ will undergo a displacement $d\zeta$ in the radial direction. Hence

$$dz = e^{i\alpha} \mid dz \mid, \quad d\zeta = e^{i\vartheta} \mid d\zeta \mid,$$

whence

$$\begin{gathered} e^{i\alpha} = \frac{dz}{\mid dz \mid} = \frac{\omega'(\zeta)d\zeta}{\mid \omega'(\zeta) \mid . \mid d\zeta \mid} = e^{i\vartheta}\frac{\omega'(\zeta)}{\mid \omega'(\zeta) \mid} = \frac{\zeta}{\rho}\frac{\omega'(\zeta)}{\mid \omega'(\zeta) \mid}, \\ e^{-i\alpha} = e^{-i\vartheta}\frac{\overline{\omega'(\zeta)}}{\mid \omega'(\zeta) \mid} = \frac{\bar{\zeta}}{\rho}\frac{\overline{\omega'(\zeta)}}{\mid \omega'(\zeta) \mid}. \end{gathered} \tag{49.3}$$

Hence, by (49.2),

$$A_\rho + iA_\vartheta = \frac{\bar{\zeta}}{\rho}\frac{\overline{\omega'(\zeta)}}{\mid \omega'(\zeta) \mid}(A_x + iA_y). \tag{49.4}$$

§ 50. Transformation of the formulae of the plane theory of elasticity. In the sequel, expressions will be required for the quantities $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$ (i.e., for the derivatives of the Airy function), the displacements and the stresses in terms of the new variable ζ, defined by

$$z = \omega(\zeta). \tag{50.1}$$

Denote by

$$\varphi_1(z), \quad \psi_1(z), \quad \Phi_1(z), \quad \Psi_1(z)$$

the functions which were earlier written as

$$\varphi(z), \quad \psi(z), \quad \Phi(z), \quad \Psi(z)$$

and introduce the new notation

$$\varphi(\zeta) = \varphi_1(z) = \varphi_1(\omega(\zeta)), \quad \psi(\zeta) = \psi_1(z) = \psi_1(\omega(\zeta)), \tag{50.2}$$

$$\Phi(\zeta) = \Phi_1(z) = \frac{d\varphi_1}{dz} = \frac{\varphi'(\zeta)}{\omega'(\zeta)}, \quad \Psi(\zeta) \quad \Psi_1(z) = \frac{\psi'(\zeta)}{\omega'(\zeta)}. \tag{50.3}$$

With this notation, the formulae (31.4) and (32.1) become

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \varphi(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi'(\zeta)} + \overline{\psi(\zeta)}, \tag{50.4}$$

and

$$2\mu(u + iv) = \varkappa\varphi(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi'(\zeta)} - \overline{\psi(\zeta)}, \tag{50.5}$$

respectively. The components v_ρ, v_ϑ of the displacements in terms of the curvilinear coordinates are, by (49.4),

$$v_\rho + iv_\vartheta = \frac{\bar{\zeta}}{\rho}\frac{\overline{\omega'(\zeta)}}{|\,\omega'(\zeta)\,|}(u + iv), \tag{50.6}$$

and hence

$$2\mu\,|\,\omega'(\zeta)\,|\,.\,(v_\rho + iv_\vartheta) = \frac{\bar{\zeta}}{\rho}\,\overline{\omega'(\zeta)}\left\{\varkappa\varphi(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi'(\zeta)} - \overline{\psi(\varphi)}\right\}. \tag{50.7}$$

Next, the components of stress will be found in the curvilinear coordinate systems. Denote these components by $\widehat{\rho\rho}$, $\widehat{\vartheta\vartheta}$, $\widehat{\rho\vartheta}$ so that, if the system $O'x'y'$ is placed in such a way that the axes $O'x'$ and $O'y'$ coincide with (ρ) and (ϑ) respectively, one has

$$\widehat{\rho\rho} = X'_{x'}, \quad \widehat{\vartheta\vartheta} = Y'_{y'}, \quad \widehat{\rho\vartheta} = X'_{y'}$$

(cf. § 39). Then, by (8.8),

$$\widehat{\rho\rho} + \widehat{\vartheta\vartheta} = X_x + Y_y, \quad \widehat{\vartheta\vartheta} - \widehat{\rho\rho} + 2i\widehat{\rho\vartheta} = (Y_y - X_x + 2iX_y)e^{2i\alpha}. \tag{50.8}$$

By (32.9), (32.10) and (49.3), the last giving

$$\begin{array}{cccc} e^{2i\alpha} & \zeta^2\;\; (\omega'(\zeta))^2 & \zeta^2\;\; (\omega'(\zeta))^2 & \zeta^2\;\; \omega'(\zeta) \\ & \omega'(\zeta) & \rho^2\;\; \omega'(\zeta)\overline{\omega'(\zeta)} & \rho^2\;\; \overline{\omega'(\zeta)} \end{array}$$

one easily finds

$$\rho\rho + \vartheta\vartheta = 4\Re\Phi(\zeta) = 2[\Phi(\zeta) + \Phi(\zeta)], \tag{50.9}$$

$$\vartheta\vartheta - \rho\rho + 2i\rho\vartheta = \frac{2\zeta^2}{\rho^2\overline{\omega'(\zeta)}}\{\omega(\zeta)\,\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)\}. \tag{50.10}$$

Subtracting (50.10) from (50.9), one obtains

$$\rho\rho - i\,\rho\vartheta = \Phi(\zeta) + \overline{\Phi(\zeta)} - \frac{\zeta^2}{\rho^2\overline{\omega'(\zeta)}}\{\overline{\omega(\zeta)}\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)\}, \tag{50.11}$$

giving the stresses acting on the contour $\rho =$ const. from the side where ρ increases. The formulae (50.7), (50.9) — (50.11) are analogous to those given by G. V. Kolosov [1, 2].

Finally, a formula will be deduced which relates to the case when an infinite region S is mapped on an infinite region Σ, so that the points $\zeta = \infty$, $z = \infty$ correspond to each other. Then for large $|z|$, by (36.4) and (36.5),

$$\begin{aligned} \varphi_1(z) &= -\frac{X+iY}{2\pi(1+\varkappa)}\log z + \Gamma z + \varphi_1^0(z), \\ \psi_1(z) &= \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\log z + \Gamma' z + \psi_1^0(z), \end{aligned} \tag{50.12}$$

where $\varphi_1^0(z)$, $\psi_1^0(z)$ are functions holomorphic at $z = \infty$. Further, for sufficiently large $|\zeta|$ and $|z|$ [cf. (47.2′)],

$$\omega(\zeta) = R\zeta + c_0 + \frac{c_1}{\zeta} + \frac{c_2}{\zeta^2} + \ldots \tag{50.13}$$

Hence, by (50.12),

$$\varphi(\zeta) = -\frac{X+iY}{2\pi(1+\varkappa)}\log\zeta + R\Gamma\zeta + \varphi_0(\zeta), \tag{50.14}$$

$$\psi(\zeta) = \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\log\zeta + R\Gamma'\zeta + \psi_0(\zeta), \tag{50.15}$$

where $\varphi_0(\zeta)$, $\psi_0(\zeta)$ are functions, holomorphic for $\zeta = \infty$.

§ 51. Boundary conditions in the image regions. First consider the case when the (finite or infinite) region S is bounded by one simple contour L. Map this region on the unit circle or on the infinite region outside this circle (there being really no difference, but, generally speaking, it will be more convenient in practical problems to map finite and infinite regions on similar types of regions).

The boundary condition of the first fundamental problem, i.e., when the external stresses acting on the boundary are given, may be expressed in two ways. Firstly, by starting from (41.5), which becomes in the new notation

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \varphi_1(z) + z\overline{\varphi_1'(z)} + \overline{\psi_1(z)} = f_1 + if_2 + \text{const. on L.}$$

Introducing ζ by the relation $z = \omega(\zeta)$ and denoting by $\sigma = e^{i\vartheta}$ points

of the circle γ, corresponding to the contour L, this condition takes the form [cf. (50.4)]

$$\varphi(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = f_1 + if_2 + \text{const.} \quad \text{on } \gamma. \tag{51.1}$$

In this formula $f_1 + if_2$ is a known quantity defined by (41.3), i.e.,

$$f_1 + if_2 = i\int_0^s (X_n + iY_n)ds, \tag{51.2}$$

where the integral on the right-hand side is taken with respect to the arc coordinate s of the contour L; thus $f_1 + if_2$ is a definite function of s. But s is also a function of the arc ϑ of the contour γ, and hence $f_1 + if_2$ in (51.1) may be considered as a known function of ϑ.

The boundary condition of the first fundamental problem can also be expressed in terms of the functions Φ and Ψ, if one makes use of (50.11) which gives (for $\rho = 1$)

$$\Phi(\sigma) + \overline{\Phi(\sigma)} - \frac{\sigma^2}{\overline{\omega'(\sigma)}}\{\overline{\omega(\sigma)}\,\Phi'(\sigma) + \omega'(\sigma)\Psi(\sigma)\} = \widehat{\rho\rho} - i\widehat{\rho\vartheta} \text{ on } \gamma, \tag{51.3}$$

where $\widehat{\rho\rho}$ and $\widehat{\vartheta\vartheta}$ must be understood as known functions of ϑ.

The boundary condition of the second fundamental problem may be written, using (50.5),

$$\varkappa\varphi(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \overline{\varphi'(\sigma)} - \overline{\psi(\sigma)} = 2\mu(g_1 + ig_2) \quad \text{on } \gamma, \tag{51.4}$$

where g_1, g_2 are the boundary values of the displacement components u and v (referred to the *old* coordinate axes Ox, Oy), but referring to the angle ϑ.

An analogous procedure may be used in the case of a doubly connected region, bounded by two simple contours L_1 and L_2, after mapping it on a circular ring (cf. § 41).

PART III

SOLUTION OF SEVERAL PROBLEMS OF THE PLANE THEORY OF ELASTICITY BY MEANS OF POWER SERIES

Several simple boundary value problems of the plane theory of elasticity will be solved by use of power series. This method of solution is directly applicable to regions bounded by one or two concentric circles. However, conformal transformation permits extension of the method to regions of more general shape.

CHAPTER 8

ON FOURIER SERIES

§ 52. On Fourier series in complex form. In the subsequent sections use will be made of the expansion of given functions in Fourier series and it will be more convenient to represent them in complex form; some remarks will now be made about this.

Let $f(\vartheta)$ be a real function, given in an interval $0 \leqslant \vartheta \leqslant 2\pi$. Under well-known, very general conditions, such a function may be represented in the form of a Fourier series

$$f(\vartheta) = \tfrac{1}{2}\alpha_0 + \sum_{k=1} (\alpha_k \cos k\vartheta + \beta_k \sin k\vartheta), \tag{52.1}$$

where

$$\alpha_k = \frac{1}{\pi} \int^{2\pi} f(\vartheta) \cos k\vartheta\, d\vartheta, \quad \beta_k = \frac{1}{\pi} \int^{2\pi} f(\vartheta) \sin k\vartheta\, d\vartheta \quad (k = 0, 1, 2, \ldots). \tag{52.2}$$

In order that the function $f(\vartheta)$ may be developed in a Fourier series, it is sufficient, for example, that it satisfy in the interval $(0,2\pi)$ under consideration the so-called *Dirichlet condition* which consists of the following: The function is continuous in the interval, with the possible exclusion of a finite number of first order discontinuities, and has a finite number of maxima and minima. A discontinuity of first order is such that, if ϑ_0 be the point of discontinuity and if the argument ϑ tends to ϑ_0 from the left or from the right, the function $f(\vartheta)$ tends to (different) finite limits ("limit from the left" or "limit from the right") which are usually denoted by $f(\vartheta_0 - 0)$ and $f(\vartheta_0 + 0)$. The Dirichlet condition further assumes that, when ϑ approaches the ends 0 and 2π of the interval, the function $f(\vartheta)$ tends to definite limits which are denoted by $f(+0)$ and $f(2\pi - 0)$.

If the Dirichlet condition is satisfied, the Fourier series (52.1) converges at all points of the interval $(0,2\pi)$. However, at points of discontinuity, it does not give the value $f(\vartheta_0)$, but

$$\frac{f(\vartheta_0 - 0) + f(\vartheta_0 + 0)}{2};$$

at the ends 0 and 2π of the interval the series gives

$$f(+0) + f(2\pi - 0)$$

If $f(\vartheta)$ does not only satisfy the Dirichlet condition, but is also continuous throughout the interval $0 \leq \vartheta \leq 2\pi$, and if further $f(0) = f(2\pi)$, the Fourier series gives the values of $f(\vartheta)$ in the whole interval, including the ends; in this case the series converges uniformly.

Finally note that functions, satisfying the Dirichlet condition, are particular cases of so-called "functions with bounded variation". All that has been said here and later on will remain true, if the Dirichlet condition is replaced by the less strict requirement that the functions are of bounded variation.

Substituting in (52.1) the known expressions

$$\cos k\vartheta = \frac{e^{ik\vartheta} + e^{-ik\vartheta}}{2}, \quad \sin k\vartheta = \frac{e^{ik\vartheta} - e^{-ik\vartheta}}{2i},$$

one finds the expansion

$$f(\vartheta) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \left\{ \frac{\alpha_k - i\beta_k}{2} e^{ik\vartheta} + \frac{\alpha_k + i\beta_k}{2} e^{-ik\vartheta} \right\} \tag{52.1'}$$

which, with

$$\frac{\alpha_0}{2} = a_0, \quad \frac{\alpha_k - i\beta_k}{2} = a_k, \quad \frac{\alpha_k + i\beta_k}{2} = a_{-k}, \tag{52.2'}$$

gives

$$f(\vartheta) = a_0 + \sum_{k=1}^{\infty} (a_k e^{ik\vartheta} + a_{-k} e^{-ik\vartheta}). \tag{52.3}$$

This formula may, obviously, be written

$$f(\vartheta) = \sum_{-\infty}^{+\infty} a_k e^{ik\vartheta}, \tag{52.4}$$

where summation extends over all integers from $-\infty$ to $+\infty$.

In order to deduce expressions for the coefficients a_k, note that

$$\int^{2\pi} e^{in\vartheta} \, d\vartheta = \begin{cases} 0 \;, & \text{if } n \text{ is an integer}, \quad n \neq 0, \\ 2\pi, & \text{if } n = 0. \end{cases} \tag{52.5}$$

Multiplying both sides of (52.4) by $e^{-in\vartheta}$, where n is any integer or zero, and integrating with respect to ϑ from 0 to 2π, one obtains

$$\int^{2\pi} e^{-in\vartheta} f(\vartheta) d\vartheta = \sum_{k=-\infty}^{+\infty} a_k \int^{2\pi} e^{i(k-n)\vartheta} \, d\vartheta.$$

But, by (52.5), the only non-zero term on the right-hand side is obtained for $k = n$ and it is equal to $2\pi a_n$. Hence

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\vartheta)e^{-in\vartheta}\,d\vartheta. \tag{52.6}$$

The above deduction would be quite correct, if the series on the right-hand side were uniformly convergent, because in that case the integration is permissible. But the result (52.6) holds also true when the function $f(\vartheta)$ is an ordinary Fourier series. In order to verify this, it is sufficient to note that (52.6) may be obtained indirectly by replacing in (52.2′) α_k and β_k by their expressions (52.2).

Consider now an expression of the form $f_1(\vartheta) + if_2(\vartheta)$, where f_1 and f_2 are real functions which may be represented in the interval $(0,2\pi)$ by ordinary Fourier series, and hence by series of the form (52.4). Adding these series, after multiplying the second one by i, one obviously obtains a series expansion of the form

$$f_1(\vartheta) + if_2(\vartheta) = \sum_{-\infty}^{+\infty} a_k e^{ik\vartheta}, \tag{52.7}$$

where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} (f_1 + if_2)e^{-in\vartheta}\,d\vartheta, \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{52.8}$$

The only difference from the preceding cases is that there the quantities a_n, a_{-n} are conjugate complex numbers, as follows from (52.2′) or (52.6), whereas here a_n and a_{-n} will not, generally speaking, be conjugate.

NOTE. Separating real and imaginary parts, one may, conversely, find from (52.7) the common Fourier series for the functions $f_1(\vartheta)$ and $f_2(\vartheta)$. In fact, putting $a_k = \alpha_k + i\beta_k$ (where α_k, β_k are real), one finds

$$\begin{aligned}
f_1 + if_2 &= \sum_{-\infty}^{+\infty} (\alpha_k + i\beta_k)(\cos k\vartheta + i\sin k\vartheta) = \\
&= \sum_{-\infty}^{+\infty} (\alpha_k \cos k\vartheta - \beta_k \sin k\vartheta) + i\sum_{-\infty}^{+\infty} (\beta_k \cos k\vartheta + \alpha_k \sin k\vartheta) = \\
&= \alpha_0 + \sum_{k=1}^{\infty} \{(\alpha_k + \alpha_{-k})\cos k\vartheta - (\beta_k - \beta_{-k})\sin k\vartheta\} + \\
&\qquad + i\beta_0 + i\sum_{k=1} \{(\beta_k + \beta_{-k})\cos k\vartheta + (\alpha_k - \alpha_{-k})\sin k\vartheta\}.
\end{aligned}$$

Hence

$$f_1(\vartheta) = \tfrac{1}{2}A_0 + \sum_{k=1}^{\infty} (A_k \cos k\vartheta + B_k \sin k\vartheta),$$

$$f_2(\vartheta) = \tfrac{1}{2}A_0' + \sum_{k=1}^{\infty} (A_k' \cos k\vartheta + B_k' \sin k\vartheta),$$

where

$$\tfrac{1}{2}A_0 = \alpha_0, \quad A_k = \alpha_k + \alpha_{-k}, \quad B_k = -\beta_k + \beta_{-k},$$

$$\tfrac{1}{2}A_0' = \beta_0, \quad A_k' = \beta_k + \beta_{-k}, \quad B_k' = \alpha_k - \alpha_{-k} \qquad (k = 1, 2, 3, \ldots).$$

Incidentally, it follows from the foregoing that the expansion of the form (52.7) is unique, because this is known to be true for ordinary Fourier series.

§ 53. On the convergence of Fourier series. If a function $f(\vartheta)$ is continuous and has in the interval $0 \leqslant \vartheta \leqslant 2\pi$ continuous derivatives of order up to and including $\nu - 1$, and if, further, the derivative of order ν satisfies the Dirichlet condition in that interval, the coefficients α_k, β_k of the Fourier series (52.1) satisfy inequalities of the form

$$|\alpha_k| < \frac{C}{k^{\nu+1}}, \quad |\beta_k| < \frac{C}{k^{\nu+1}} \quad (k = 1, 2, \ldots), \tag{53.1}$$

where C is a positive constant.

The above statement that a function is continuous in $0 \leq \vartheta \leq 2\pi$ will be understood to mean that the function is not only continuous in this interval, but also that *its values at the end of the interval are equal to each other.* The inequalities (53.1) will also be true, if one assumes that the νth derivative is of bounded variation.

It follows from (53.1) that the coefficients of the complex Fourier series (52.7) satisfy inequalities of the form

$$|a_k| < \frac{C}{|k|^{\nu+1}} \quad (k = \pm 1, \pm 2, \ldots), \tag{53.2}$$

provided $f_1(\vartheta)$ and $f_2(\vartheta)$ satisfy the conditions stated above for $f(\vartheta)$.

If $\nu = 1$, i.e., in the case, when the function has a first derivative satisfying the Dirichlet condition, one will have

$$|\alpha_k| < \frac{C}{k^2}, \quad |\beta_k| < \frac{C}{k^2},$$

from which it follows that the Fourier series for $f(\vartheta)$ will be uniformly

and absolutely convergent. (Uniform convergence actually ensures continuity of $f(\vartheta)$ and bounded variation, or, in particular, fulfilment of the Dirichlet condition.) In fact, one has

$$|\alpha_k \cos k\vartheta + \beta_k \sin k\vartheta| \leqslant |\alpha_k| + |\beta_k| < \frac{2C}{k^2};$$

thus the terms of the series (52.1) are less in absolute value than the terms of the convergent series

$$\sum_{k=1}^{\infty} \frac{2C}{k^2} = 2C \sum_{k=1}^{\infty} \frac{1}{k^2},$$

with positive terms, which do not depend on ϑ.

CHAPTER 9

SOLUTION FOR REGIONS, BOUNDED BY A CIRCLE

§ 54. Solution of the first fundamental problem for the circle. Solutions of this problem have been given by many authors. A simpler, but less elementary solution is given in § 80.

Let the origin of coordinates be at the centre of the circle with radius R. Let X_n, Y_n be the known components of the external stresses, acting on the circumference L of this circle. They will be assumed to be continuous and single-valued on L and varying with the polar angle ϑ, measured like the arc coordinate s from the positive Ox axis.

By (41.3),

$$f_1 + if_2 = i\int_0^s (X_n + iY_n)ds = iR\int_0^\vartheta (X_n + iY_n)d\vartheta. \qquad (54.1)$$

It is known that for the existence of a regular solution the functions f_1 and f_2 must be continuous and single-valued on L (§ 42), i.e., one must have

$$(X_n + iY_n)d\vartheta = 0 \qquad (54.2)$$

(which means that the resultant vector must vanish). Further, the condition of zero resultant moment (§ 42)

$$(f_1 dx + f_2 dy) = 0$$

here takes the form

$$\int_0^{2\pi} (-f_1 \sin\vartheta + f_2 \cos\vartheta)d\vartheta = 0. \qquad (54.3)$$

The conditions (54.2) and (54.3) will be assumed satisfied.

The boundary condition (41.5) may be written (putting const. = 0)

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = f_1 + if_2 \text{ on } L. \tag{54.4}$$

The expression $f_1 + if_2$ may be represented by the series

$$f_1 + if_2 = \sum^{+\infty} A_n e^{in\vartheta} \tag{54.5}$$

the coefficients of which may be calculated by the method of § 52; hence these coefficients will be assumed known.

It is known that the functions $\varphi(z)$, $\psi(z)$ must be holomorphic inside L and that, by § 41, one may assume $\varphi(0) = 0$. Thus $\varphi(z)$ and $\psi(z)$ may be developed for $|z| < R$ in power series of the form

$$\varphi(z) = \sum_{k=1}^{\infty} a_k z^k, \qquad \psi(z) = \sum_{k=0}^{\infty} a_k' z^k, \tag{54.6}$$

where in the first series the constant term is absent, because of the condition $\varphi(0) = 0$. Further, one has

$$\overline{\varphi'(z)} = \Sigma\, k\bar{a}_k \bar{z} \qquad \overline{\psi(z)} = \Sigma\, \bar{a}_k' \bar{z}^k. \tag{54.6'}$$

Assuming these series to converge, not only in the interior, but also on L, and substituting them in (54.4), one finds

$$\sum_{k=1}^{\infty} a_k z^k + z \sum_{k=1}^{\infty} k\bar{a}_k \bar{z}^{k-1} + \sum_{k=0}^{\infty} \bar{a}_k' \bar{z}^k = f_1 + if_2 \text{ on } L.$$

But on L: $z = Re^{i\vartheta}$, $\bar{z} = Re^{-i\vartheta}$. Noting also that

$$z . \sum_{k=1}^{\infty} k\bar{a}_k \bar{z}^{k-1} = \sum_{k=1}^{\infty} k\bar{a}_k R^k e^{(k-2)i\vartheta} = \bar{a}_1 Re^{i\vartheta} + \sum_{k=1}^{\infty} (k+2)\bar{a}_{k+2} R^{k+2} e^{-ki\vartheta},$$

one finds from the preceding formula, using (54.5),

$$\sum_{k=1}^{\infty} a_k R^k e^{ik\vartheta} + \bar{a}_1 Re^{i\vartheta} + \sum_{k=0}^{\infty} (k+2)\bar{a}_{k+2} R^{k+2} e^{-ik\vartheta} + \sum_{k=0}^{\infty} \bar{a}_k' R^k e^{-ik\vartheta} = \sum_{-\infty}^{+\infty} A_k e^{ik\vartheta}.$$

Comparing coefficients of $e^{i\vartheta}$, one obtains

$$a_1 R + \bar{a}_1 R = A_1, \text{ i.e., } a_1 + \bar{a}_1 = \frac{A_1}{R}. \tag{54.7}$$

Similarly, one has for $e^{in\vartheta}$ $(n > 1)$

$$a_n R^n = A_n \quad (n > 1). \tag{54.7'}$$

Finally, $e^{-in\vartheta}$ $(n \geqslant 0)$ gives

$$(n+2)\bar{a}_{n+2}R^{n+2} + R^n\bar{a}_n' = A_{-n} \quad (n \geqslant 0). \tag{54.8}$$

The equality (54.7) is only possible, if A_1 is real, since $a + \bar{a}_1 = 2\alpha_1$, where α_1 is the real part of a_1. Hence, in order that the problem may be possible, one must have

$$A_1 = \text{a real quantity.} \tag{54.9}$$

The meaning of this condition will be explained below. If it is satisfied, the real part α_1 of the coefficient a_1 is given by

$$\Re a_1 = \alpha_1 = \frac{A_1}{2R}. \tag{54.10}$$

As was to be expected, the imaginary part of a_1 remains indeterminate, because it is the imaginary part of $\varphi'(0)$ which may be fixed arbitrarily (§ 41), for example, by putting it equal to zero.

Further, the coefficients a_n $(n > 1)$ are given by (54.7′) as

$$a_n = \frac{A_n}{R^n} \quad (n > 1), \tag{54.11}$$

and, finally, one obtains for a_n' $(n \geqslant 0)$ from (54.8) [replacing all quantities by their conjugate complex values]

$$\frac{\bar{A}_{-n}}{R^n} \qquad (n+2)a_{n+2}R^2 = \frac{\bar{A}_{-n}}{R^n} - (n+2)\frac{A_{n+2}}{R^n} \quad (n \geqslant 0). \tag{54.12}$$

Thus all coefficients of (54.6) have been determined and the problem could be considered solved, once it has been proved that the series for $\varphi(z)$ and $\psi(z)$ actually satisfy the conditions of the problem. This question will now be studied, but first the condition (54.9) will be explained. One has (§ 52)

$$2\pi A_1 = \int_0^{2\pi} (f_1 + if_2)e^{-i\vartheta}d\vartheta =$$

$$= \int_0^{2\pi} (f_1 \cos\vartheta + f_2 \sin\vartheta)d\vartheta + i\int_0^{2\pi} (f_2\cos\vartheta - f_1 \sin\vartheta)d\vartheta,$$

i.e., (54.9) leads to (54.3) which expressed that the resultant moment of the external forces vanishes.

As regards the question posed above with respect to the series for $\varphi(z)$ and $\psi(z)$, consideration will be limited to the simple case, when not only the functions X_n and Y_n are continuous, but when also their first order derivatives satisfy the Dirichlet condition. (Actually, it is not difficult to prove the correctness of the solution for more general conditions, but this will not be done here.) It is easily shown that under the above conditions the series

$$\varphi(z) = \sum_{k=1}^{\infty} a_k z^k, \quad \varphi'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}, \quad \psi(z) = \sum_{k=1}^{\infty} a_k' z^k$$

are absolutely and uniformly convergent on the circle L, and hence also inside L. Thus φ, ψ, φ' will be continuous up to the boundary and the solution is regular.

To prove the convergence of these series on L, consider the series, formed by the moduli of the terms of the former when $|z| = R$,

$$\Sigma \,|\, a_k \,| R^k, \quad \Sigma\, k \,|\, a_k \,| R^{k-1}, \quad \Sigma \,|\, a_k' \,| R^k. \tag{a}$$

Since X_n, Y_n have first order derivatives, satisfying the Dirichlet condition, the functions f_1 and f_2 have derivatives of second order, having the same property. Hence, by what has been said in § 53,

$$|\, A_k \,| < \frac{C}{k^3}, \quad |\, A \qquad < \frac{C}{k^3} \quad (k = 1, 2, \ldots),$$

where C is some constant and, by (54.11) and (54.12),

$$a_k \,|\, R^k < \frac{C}{k^3}, \quad k \,|\, a_k \,|\, R^{k-1} < \frac{C'}{k^2}, \quad | \qquad R^k < \frac{C''}{k^2},$$

where C', C'' are some other constants. From this follows immediately the convergence of the series (a), and consequently the uniform and absolute convergence of the series for φ, φ' and ψ.

NOTE. The problem has been solved using the boundary conditions in the form (41.5). One could also have used the conditions in the form (41.9). This alternative proof will be left to the reader (cf. § 56, where an analogous problem is solved by this method).

§ 55. Solution of the second fundamental problem for the circle *). This solution is quite analogous to the preceding one. In fact, the condition (41.1) gives

$$\varkappa\varphi(z) - z\,\overline{\varphi'(z)} - \overline{\psi(z)} \qquad 2\mu(g_1 + ig_2) \text{ on } L. \tag{55.1}$$

*) Another solution is given in § 81.

Developing the given expression $2\mu(g_1 + ig_2)$ into a complex Fourier series

$$2\mu(g_1 + ig_2) = \Sigma A_k e^{ik\vartheta} \tag{55.2}$$

and substituting the series (54.6) in (55.1), one finds, as before,

$$\varkappa \sum_{k=1} a_k R^k e^{ik\vartheta} - \bar{a}_1 R e^{i\vartheta} - \sum_{k=0} (k+2)\bar{a}_{k+2} R^{k+2} e^{-ik\vartheta} - \sum_{k=0} \bar{a}'_k R^k e^{-ik\vartheta} = \sum_{-\infty} A_k e^{ik\vartheta},$$

and hence

$$R(\varkappa a_1 - \bar{a}_1) = A_1, \tag{55.3}$$

$$\varkappa a_n R^n = A_n \ (n \quad 1), \quad -(n+2)\bar{a}_{n+2} R^{n+2} - \bar{a}'_n R^n = A_{-n} \ (n \geqslant 0). \tag{55.4}$$

All coefficients are determined by these formulae, i.e., in contrast to the case of the last section, a_1 is also completely determined by (55.3), as was to be expected, since in the present problem it is impossible to fix arbitrarily the imaginary part of $\varphi'(0)$. In fact, equation (55.3) and its conjugate equation give

$$\varkappa \alpha_1 - a_1 = \frac{A_1}{R}, \quad \varkappa \bar{a}_1 - a_1 = \frac{\bar{A}_1}{R},$$

and hence

$$a_1 = \frac{\varkappa A_1 + \bar{A}_1}{(\varkappa^2 - 1)R}$$

(remembering that always $\varkappa > 1$).

As in § 54, it is easily proved that these series actually satisfy the conditions of the problem, if, for example, g_1 and g_2 have second order derivatives satisfying the Dirichlet condition.

§ 56. Solution of the first fundamental problem for the infinite plane with a circular hole *). This problem may be solved by a method quite similar to that of § 54. However, as demonstration, use will be made of the boundary condition in the form (41.9). Let the origin of coordinates be at the centre of the hole of radius R. One has then, in the notation of § 39,

$$\widehat{rr} - i\,\widehat{r\vartheta} = N - iT \text{ on the circle } L, \tag{56.1}$$

where N and T (cf. § 41) are the components of the external stresses

*) This problem will be solved by another method in § 82 for the more general case of an elliptic hole. Cf. also § 87*a*.

acting on the circumference of L in the direction of the normal n, outward with respect to the body (i.e., directed towards the origin), and of the tangent t, directed to the left of the normal n.

The correctness of (56.1) is easily verified, i.e., the truth of the relations

$$\widehat{rr} = N, \ \widehat{r\vartheta} = T.$$

The definitions of $\widehat{rr}$ and $\widehat{r\vartheta}$ are given in § 39. It should not be overlooked that the axes (r) and (ϑ) of § 39 are now in the opposite directions of n and t, while $\widehat{rr}$ and $\widehat{r\vartheta}$ refer to stresses, acting on the sides of elements opposite to the direction of n.

The condition (41.9) may be obtained directly from (39.5) which gives

$$\Phi(z) + \overline{\Phi(z)} - e^{2i\vartheta}[z\Phi'(\bar{z}) + \Psi(z)] = N - iT \text{ on } L. \tag{56.2}$$

[In (41.9) one should have $e^{2i\alpha}$ instead of $e^{2i\vartheta}$, where $\alpha = \vartheta \pm \pi$ is the angle between the normal n and the axis Ox. But $e^{2i\vartheta} = e^{2i\alpha}$, since $e^{\pm 2\pi i} = 1$.]

Consider now the formulae (36.4), (36.5) and (36.7) and note that in the present case the expansions (36.7) hold true in the entire region S, i.e., outside the circle L (cf. Note at end of § 36.) Differentiating the above-mentioned formulae, one finds for $\Phi(z) = \varphi'(z)$ and $\Psi(z) = \psi'(z)$ expansions of the form

$$\Phi(z) = \sum_{k=0}^{\infty} a_k z^{-k}, \quad \Psi(z) = \sum_{k=0}^{\infty} a_k' z^{-k}, \tag{56.3}$$

where the notation for the coefficients is different from that of § 36. In particular, the coefficients a_0, a_0', a_1, a_1' in (56.3) have the values

$$a_0 = \Gamma = B, \quad a_0' = \Gamma' = B' + iC', \tag{56.4}$$

(remembering that it had been agreed in § 40 to assume $C = 0$),

$$a_1 = \frac{X + iY}{2\pi(1 + \varkappa)} \qquad a_1' = -\frac{\varkappa(X - iY)}{2\pi(1 + \varkappa)}. \tag{56.5}$$

The formulae (56.5) are not necessary for the solution of the problem. One has, of course, only to use the condition of single-valuedness of the displacements which in the present case may be expressed as

$$\varkappa a_1 + \bar{a}_1' = 0 \tag{56.6}$$

[cf. (35.7), where the quantities γ_k' and γ_k refer to the contour L_k; but here $L = L_1$, $\gamma_k = a_1$, $\gamma_k' = a_1'$.] Substituting (56.3) in (56.2) and assuming

the series to converge on the circle L, one finds (cf. § 54)

$$\sum_0^\infty \frac{1+k}{R^k} a_k e^{-ik\vartheta} + \sum_0^\infty \frac{\bar{a}_k}{R^k} e^{ik\vartheta} - a_0' e^{2i\vartheta} - \frac{a_1'}{R} e^{i\vartheta} - \sum_0^\infty \frac{a_{k+2}'}{R^{k+2}} e^{-ik\vartheta} =$$
$$= N - iT \text{ on } L. \quad (56.7)$$

Expand the function $N - iT$, given on L, in a complex Fourier series

$$N - iT = \sum_{-\infty}^{+\infty} A_k e^{ik\vartheta} \quad (56.8)$$

and compare coefficients of $e^{ik\vartheta}$, after introducing (56.8) into (56.7). Then one obtains from the constant term and from those involving $e^{i\vartheta}$ and $e^{2i\vartheta}$ respectively

$$2a_0 - \frac{a_2'}{R^2} = A_0, \quad \frac{\bar{a}_1}{R} - \frac{a_1'}{R} = A_1, \quad \frac{\bar{a}_2}{R^2} - a_0' = A_2. \quad (56.9)$$

For $e^{in\vartheta}$ $(n \geqslant 3)$, one finds

$$\frac{\bar{a}_n}{R^n} = A_n \quad (n \geqslant 3), \quad (56.10)$$

while $e^{-in\vartheta}$ $(n \geqslant 1)$ gives

$$\frac{1+n}{R^n} a_n - \frac{a_{n+2}'}{R^{n+2}} = A_{-n} \quad (n \geqslant 1). \quad (56.11)$$

From (56.10) one finds

$$\bar{a}_n = A_n R^n \qquad (n \geqslant 3). \quad (56.12)$$

Further, it is known that

$$a_0 = \Gamma, \quad a_0' = \Gamma', \quad (56.4)$$

where Γ, Γ' are known quantities, specifying the stress distribution at infinity. Hence, by the last of the formulae (56.9),

$$a_2 = \bar{\Gamma}' R^2 + \bar{A}_2 R^2. \quad (56.13)$$

In order to find expressions for a_1 and a_1', it is necessary to refer to the condition (56.6) for single-valuedness of the displacements which in combination with the second relation of (56.9) gives

$$a_1 = \frac{\bar{A}_1 R}{1+\varkappa}, \quad a_1' = -\frac{\varkappa A_1 R}{1+\varkappa}. \quad (56.14)$$

The first formula of (56.9) leads to

$$a_2' = 2\Gamma R^2 - A_0 R^2 \tag{56.15}$$

and, finally, (56.11) determines all coefficients a_n' for $n \geqslant 3$:

$$a_n' = (n-1)R^2 a_{n-2} \cdot R^n A_{-n+2} \quad (n \geqslant 3). \tag{56.16}$$

Thus the problem of determining the coefficients has been solved.

It is easily shown by elementary arguments of the type used in § 54 that, if N and T have second order derivatives satisfying the Dirichlet condition, the series for $\Phi(z)$, $\Phi'(z)$ and $\Psi(z)$ will be uniformly and absolutely convergent on L (and consequently also outside L); it follows from this that they are solutions of the problem.

NOTE. If one had started from the boundary condition (41.5) instead of from (56.2), one would have obtained for $\varphi(z)$, $\psi(z)$ series, for which one could have proved by the method of § 54 that they solve the problem, provided X_n and Y_n have first order derivatives satisfying the Dirichlet condition. Thus, by applying the boundary condition (41.9) [i.e., condition (56.2)], one has been forced to impose more restrictive conditions than would have been necessary with the condition (41.5). However, it is easily seen that these additional limitations are not due to the problem, but to the elementary method used in proving the correctness of the solutions. In fact, it is almost obvious (and this is easily verified directly) that, starting from (41.5), one would find for $\varphi(z)$, $\psi(z)$ series which could have been obtained by differentiating those found above for $\Phi(z)$, $\Psi(z)$. But as $\varphi(z)$, $\psi(z)$ satisfy the conditions of the problem, obviously $\Phi(z) = \varphi'(z)$, $\Psi(z) = \Psi'(z)$ will also solve the problem.

§ 56a. Examples.

1°. Uni-directional tension of a plate, weakened by a circular hole.

Let the edges of the hole be free from external stresses and let at infinity

$$X_x^{(\infty)} = p, \quad Y_y^{(\infty)} = X_y^{(\infty)} = 0,$$

where p is a constant, (i.e., tension in the direction Ox which is equal to p at infinity). Then, as is shown by (36.10) (remembering that, by supposition, $C = 0$),

$$\Gamma = {}^{p} \qquad \Gamma' \qquad {}^{p} \tag{56.1a}$$

Further, since on the contour $N - iT = 0$, one must put in the formulae of § 56 $A_k = 0$ for all k. Under these circumstances (56.12) and (56.16) give

$$a_n = 0 \quad (n \geqslant 3), \quad a_n' = 0 \quad (n \geqslant 5).$$

Also, from (56.4), (56.14), (56.13), (56.15) and (56.16),

$$a_0 = \frac{p}{4}, \quad a_0' = -\frac{p}{2}, \quad a_1 = a_1' = 0, \quad a_2 = -\frac{p}{2} R^2,$$

$$a_2' = \frac{p}{2} R^2, \quad a_3' = 0, \quad a_4' = -\frac{3pR^4}{2},$$

and hence, finally,

$$\Phi(z) = \frac{p}{4}\left(1 - \frac{2R^2}{z^2}\right), \quad \Psi(z) = -\frac{p}{2}\left(1 - \frac{R^2}{z^2} + \frac{3R^4}{z^4}\right). \tag{56.2a}$$

Next determine the corresponding components of stress in polar coordinates. By (39.4), putting $z = re^{i\vartheta}$,

$$\widehat{rr} + \widehat{\vartheta\vartheta} = 4\Re\,\Phi(z) = p\Re\left(1 - \frac{2R^2}{r^2} e^{-2i\vartheta}\right) =$$

$$= p\left(1 - \frac{2R^2}{r^2}\cos 2\vartheta\right), \tag{56.3a}$$

$$\widehat{\vartheta\vartheta} - \widehat{rr} + 2i\widehat{r\vartheta} = 2[\bar{z}\Phi'(z) + \Psi(z)]e^{2i\vartheta} =$$

$$= p\left\{\frac{2R^2}{r^2} e^{-2i\vartheta} - e^{2i\vartheta} + \frac{R^2}{r^2} - \frac{3R^4}{r^4} e^{-2i\vartheta}\right\},$$

whence, separating real and imaginary parts and solving for $\widehat{rr}$, $\widehat{\vartheta\vartheta}$ and $\widehat{r\vartheta}$, one finds

$$\widehat{rr} = \frac{p}{2}\left(1 - \frac{R^2}{r^2}\right) + \frac{p}{2}\left(1 - \frac{4R^2}{r^2} + \frac{3R^4}{r^4}\right)\cos 2\vartheta,$$

$$\widehat{\vartheta\vartheta} = \frac{p}{2}\left(1 + \frac{R^2}{r^2}\right) - \frac{p}{2}\left(1 + \frac{3R^4}{r^4}\right)\cos 2\vartheta, \tag{56.4a}$$

$$\widehat{r\vartheta} = -\frac{p}{2}\left(1 + \frac{2R^2}{r^2} - \frac{3R^4}{r^4}\right)\sin 2\vartheta.$$

At the internal boundary (i.e., for $r = R$), as was to be expected, one has

$$\widehat{rr} = \widehat{r\vartheta} = 0,$$

while the value of $\widehat{\vartheta\vartheta}$ there is given by

$$\widehat{\vartheta\vartheta} = p(1 - 2\cos 2\vartheta) \quad \text{on} \quad L.$$

The maximum value of $\widehat{\vartheta\vartheta}$ thus occurs for $\cos 2\vartheta = -1$, i.e., for

$$\vartheta = \pm \frac{\pi}{2},$$

where

$$\widehat{\vartheta\vartheta}_{max} = 3p,$$

so that *the value of the tensile stress is increased.*

This problem was first solved by G. Kirsch (Zeitschrift des Ver. d. Ing., 1898) in a quite different way. Cf. also the solution by G. V. Kolosov [1], pp. 20—24. The solution for the case of an elliptic hole will be given in § 82*a*.

In order to find the displacements, calculate the functions

$$\varphi(z) = \int \Phi(z)dz, \quad \psi(z) = \int \Psi(z)dz.$$

One obtains, omitting unimportant constants,

$$\varphi(z) = \frac{p}{4}\left(z + \frac{2R^2}{z}\right), \quad \psi(z) = -\frac{p}{2}\left(z + \frac{R^2}{z} - \frac{R^4}{z^3}\right). \quad (56.2'a)$$

Then, by (39.3), one has

$$2\mu(v_r + iv_\vartheta) = e^{-i\vartheta}\{\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\} =$$

$$= \frac{p}{4}\left\{(\varkappa - 1)r + \varkappa\frac{2R^2}{r}e^{-2i\vartheta} + \frac{2R^2}{r}e^{2i\vartheta} + 2re^{-2i\vartheta} + \frac{2R^2}{r} - \frac{2R^4}{r^3}e^{2i\vartheta}\right\},$$

whence, separating real and imaginary parts,

$$v_r = \frac{p}{8\mu r}\left\{(\varkappa - 1)r^2 + 2R^2 + 2\left[R^2(\varkappa + 1) + r^2 - \frac{R^4}{r^2}\right]\cos 2\vartheta\right\},$$

$$v_\vartheta = -\frac{p}{4\mu r}\left\{R^2(\varkappa - 1) + r^2 - \frac{R^2}{r_2}\right\}\sin 2\vartheta.$$

2°. Bi-axial tension.

The problem of bi-axial tension of a plate with a circular hole is solved still more easily; in that case one has at infinity

$$X_x^{(\infty)} = Y_y^{(\infty)} = p, \quad X_y^{(\infty)} = 0.$$

By (36.8)

$$\Gamma = \frac{p}{2}, \quad \Gamma' = 0,$$

and, similarly as before, one finds

$$a_0 = \frac{p}{2}, \quad a_2' = pR^2,$$

while all other coefficients of the series for $\Phi(z)$, $\Psi(z)$ vanish. Hence

$$\Phi(z) = \frac{p}{2}, \quad \Psi(z) = \frac{pR^2}{z^2} \tag{56.5a}$$

and

$$\varphi(z) = \frac{p}{2} z, \quad \psi(z) = -\frac{pR^2}{z}. \tag{56.5'a}$$

The stresses and displacements can be calculated, using, as before, the formulae (39.4) and (39.3) which give

$$\widehat{rr} = p\left(1 - \frac{R^2}{r^2}\right), \quad \widehat{\vartheta\vartheta} = p\left(1 + \frac{R^2}{r^2}\right), \quad \widehat{r\vartheta} = 0, \tag{56.6a}$$

$$v_r = \frac{p}{4\mu r}\left[(\varkappa - 1)r^2 + 2R^2\right], \quad v_\vartheta = 0. \tag{56.6'a}$$

This solution could have been obtained directly from the solution of problem 1° by superimposing two uni-directional stress distributions along the axes Ox and Oy respectively.

3°. Uniform normal pressure, applied to the edge of a circular hole.

Consider now the case when the edge of the hole is subject to uniform normal pressure P and when the stresses vanish at infinity. Then

$$N = -P, \quad T = 0, \quad \Gamma = \Gamma' = 0.$$

In (56.8)

$$A_0 = -P, \quad A_k = 0 \qquad (k \neq 0);$$

hence, on the basis of the formulae of § 56, one finds that

$$a_2' = PR^2$$

and that all other coefficients of the expansions for $\Phi(z)$ and $\Psi(z)$ are zero. Thus

$$\Phi(z) = 0, \quad \Psi(z) = \frac{PR^2}{z^2}, \quad \varphi(z) = 0, \quad \psi(z) = -\frac{PR^2}{z} \tag{56.7a}$$

and

$$\widehat{rr} = -\frac{PR^2}{r^2}, \quad \widehat{\vartheta\vartheta} = \frac{PR^2}{r^2}, \quad \widehat{r\vartheta} = 0,$$

$$v_r = \frac{PR^2}{2\mu r}, \quad v_\vartheta = 0. \tag{56.7'a}$$

4°. A concentrated force, applied at a point of the infinite plane.

Let the stresses at infinity be zero ($\Gamma = \Gamma' = 0$) and the stress, applied to the edge of the circular hole, have constant magnitude and direction:

$$X_n = \frac{X}{2\pi R}, \quad Y_n = \frac{Y}{2\pi R} \tag{56.8a}$$

where X, Y are constants. Obviously (X, Y) is the resultant vector of the external forces.

Under these conditions the normal and tangential stresses N, T are given by

$$N = -\frac{1}{2\pi R}(X\cos\vartheta + Y\sin\vartheta), \quad T = -\frac{1}{2\pi R}(-X\sin\vartheta + Y\cos\vartheta),$$

whence

$$\widehat{rr} - i\widehat{r\vartheta} = N - iT = -\frac{1}{2\pi R}(X - iY)e^{i\vartheta} \text{ on the contour.}$$

Hence only one of the coefficients in (56.8) does not vanish, i.e.,

$$A_1 = -\frac{X - iY}{2\pi R}$$

and, by (56.14) and (56.16) for $n = 3$,

$$a_{-1} = -\frac{X + iY}{2\pi(1+\varkappa)}, \quad a_1 = \frac{\varkappa(X - iY)}{2\pi(1+\varkappa)}, \quad a_3' = -R^2\frac{X + iY}{\pi(1+\varkappa)},$$

while the remaining a_n and a_n' are zero.

Thus the problem is solved by the functions

$$\Phi(z) = -\frac{X+iY}{2\pi(1+\varkappa)}\frac{1}{z}, \quad \Psi(z) = \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\frac{1}{z} - \frac{X+iY}{\pi(1+\varkappa)}\frac{R^2}{z^3}.$$

Let it now be assumed that the radius of the hole tends to zero and that the stress (X_n, Y_n) increases beyond all bounds, so that the resultant vector (X, Y) remains unchanged. Then the preceding formulae give

$$\Phi(z) = -\frac{X+iY}{2\pi(1+\varkappa)}\frac{1}{z}, \quad \Psi(z) = \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\cdot\frac{1}{z}. \qquad (56.9a)$$

Under the stated circumstances it will be said that *a concentrated force* (X, Y) acts at O. The state of stress, caused by a concentrated force, is determined by the functions Φ, Ψ of (56.9a). The determination of the components of stress and displacement does not offer any difficulties. For example, the stress components in polar coordinates are given by

$$\widehat{rr} = -\frac{\varkappa+3}{2\pi(\varkappa+1)}\frac{X\cos\vartheta + Y\sin\vartheta}{r},$$

$$\widehat{\vartheta\vartheta} = \frac{\varkappa-1}{2\pi(\varkappa+1)}\frac{X\cos\vartheta + Y\sin\vartheta}{r}, \qquad (56.9'a)$$

$$\widehat{r\vartheta} = \frac{\varkappa-1}{2\pi(\varkappa+1)}\frac{X\sin\vartheta - Y\cos\vartheta}{r}.$$

NOTE. When considering thin plates ("generalized plane stress"), the constant $\varkappa$ in the preceding formulae must be replaced by

$$\varkappa^* = \frac{3-\sigma}{1+\sigma}$$

(cf. § 32), and the quantities X, Y by

$$\frac{X^0}{2h}, \quad \frac{Y^0}{2h},$$

where X^0, Y^0 are the components of the concentrated force, applied to the plate of thickness $2h$. In fact, it must not be forgotten that X and Y are distributed over the thickness of the plate.

5°. Concentrated couple

Consider now the case, when a constant tangential force T is applied

to the edge of the hole. Let the stresses vanish at infinity. Then

$$\widehat{rr} = 0, \quad \widehat{r\vartheta} = T \text{ on the contour,}$$

and only the coefficient $A_0 = -iT$ in the series (56.8) will be different from zero.

The formulae of § 56 give

$$a_2' = -A_0R^2 = iTR^2;$$

all other quantities a_n, a_n' vanish. Hence, putting

$$TR^2 = -\frac{M}{2\pi},$$

one has

$$\Phi(z) = 0, \quad \Psi(z) = -\frac{iM}{2\pi}\frac{1}{z^2}, \tag{56.10a}$$

where M obviously denotes the resultant moment about the centre of the external forces, applied to the boundary. These formulae remain valid also in the limiting case, when R decreases and T increases in such a way that M remains constant. Then (56.10a) leads to what will be called the effect of a *concentrated couple*, with moment M about the origin, on the infinite plane. The stress components are easily found to be

$$\widehat{rr} = \widehat{\vartheta\vartheta} = 0, \quad \widehat{r\vartheta} = -\frac{M}{2\pi r^2}. \tag{56.11a}$$

(Cf. also the Note preceding this example).

§ 57. On the general problem of concentrated forces. In § 56a, 4°, expressions have been found for the functions Φ and Ψ, corresponding to concentrated forces acting at the origin of coordinates on an unbounded body. Now let the region S be arbitrary in shape and, in addition to ordinary forces corresponding to the functions Φ and Ψ holomorphic in S, let a concentrated force (X, Y) be applied to the body, say at the point $z = 0$. The effect of this concentrated force may be superimposed on that of the ordinary forces, and therefore the functions Φ and Ψ will have the form

$$\Phi(z) = -\frac{X+iY}{2\pi(1+\varkappa)}\frac{1}{z} + \Phi_0(z), \quad \Psi(z) = \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\frac{1}{z} + \Psi_0(z) \tag{57.1}$$

near $z = 0$ [cf. (56.9a)], where Φ_0 and Ψ_0 are functions holomorphic

in the neighbourhood of that point. If the concentrated force is applied at some arbitrary point $z = z_0$, instead of at $z = 0$, then, using z_0 as origin of an auxiliary coordinate system, (57.1) will take the form

$$\Phi_1(z_1) = -\frac{X+iY}{2\pi(1+\varkappa)}\frac{1}{z_1} + \Phi_1^0(z_1), \quad \Psi_1(z_1) = \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\frac{1}{z_1} + \Psi_1^0(z_1),$$

where $z_1 = z - z_0$. Reverting to the old system, one finds from (38.3) and (38.4)

$$\begin{aligned}\Phi(z) &= -\frac{X+iY}{2\pi(1+\varkappa)}\frac{1}{z-z_0} + \Phi_0,\\ \Psi(z) &= \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\frac{1}{z-z_0} - \frac{\bar{z}_0(X+iY)}{2\pi(1+\varkappa)}\frac{1}{(z-z_0)^2} + \Psi_0.\end{aligned} \tag{57.2}$$

The index 0 on the symbol indicates that the function is holomorphic near the point $z = z_0$. Integrating one obtains for φ and ψ

$$\begin{aligned}\varphi(z) &= -\frac{X+iY}{2\pi(1+\varkappa)}\log(z-z_0) + \varphi_0,\\ \psi(z) &= \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\log(z-z_0) + \frac{\bar{z}_0(X+iY)}{2\pi(1+\varkappa)}\frac{1}{z-z_0} + \psi_0.\end{aligned} \tag{57.3}$$

In an analogous manner one finds for a concentrated couple M, applied at $z = z_0$,

$$\Phi(z) = \Phi_0(z), \quad \Psi(z) = -\frac{iM}{2\pi}\frac{1}{(z-z_0)^2} + \Psi_0(z), \tag{57.4}$$

and

$$\varphi(z) = \varphi_0(z), \quad \psi(z) = \frac{iM}{2\pi}\frac{1}{(z-z_0)} + \psi_0(z) \tag{57.5}$$

[cf. (56.10*a*)].

It is thus seen that the point of application of a concentrated force or couple is an isolated singular point of the functions φ, ψ, Φ, Ψ. Conversely, every isolated singular point $z_0 = x_0 + iy_0$ of these functions (if the existence of such points is admitted) may be considered the point of application of concentrated forces or moments. In order to determine the analytic character of the functions φ and ψ near these points, it is sufficient to apply the reasoning of § 35 by surrounding the point z_0 by a sufficiently small contour L_0 and considering this contour

as one of the boundaries of S. Then, by § 35, one has near $z = z_0$

$$
\begin{aligned}
\varphi(z) &= -\frac{X + iY}{2\pi(1 + \varkappa)} \log (z - z_0) + \varphi^*(z), \\
\psi(z) &= \frac{\varkappa(X - iY)}{2\pi(1 + \varkappa)} \log (z - z_0) + \psi^*(z),
\end{aligned}
\tag{57.6}
$$

where φ^*, ψ^* are single-valued near z_0, X and Y are the components of the resultant vector of the external forces, applied to L_0 (or to any other contour, surrounding z_0).

The functions φ^*, ψ^*, which are single-valued in the neighbourhood of the isolated singular point z_0, may be represented by the Laurent series

$$
\varphi^* = \sum^{+\infty} (\alpha_n + i\beta_n)(z - z_0)^n, \quad \psi^* = \sum^{+\infty} (\alpha_n' + i\beta_n')(z - z_0)^n. \tag{57.7}
$$

Simple reasoning, based on (33.3), shows that the resultant moment about the origin of the forces, applied to L_0 from the inside, is given by

$$
M_0 = 2\pi\beta_{-1}' + \frac{\varkappa(x_0 Y - y_0 X)}{1 + \varkappa} = 2\pi\beta_{-1}' - \frac{x_0 Y - y_0 X}{1 + \varkappa} + x_0 Y - y_0 X, \tag{57.8}
$$

(provided the contour is infinitely small). Taking into consideration that the resultant vector of these forces is (X, Y), one obtains for the resultant moment M about the point z_0

$$
M = M_0 - (x_0 Y - y_0 X) \quad 2\pi\beta_{-1}' - \frac{x_0 Y - y_0 X}{1 + \varkappa}. \tag{57.9}
$$

Thus, z_0 is the point of application of the concentrated force (X, Y) and of the concentrated couple with moment M.

However, it is seen, that the knowledge of X, Y and M does not yet determine the singularities of φ and ψ. In fact, the coefficients of the negative powers of $(z - z_0)$ in (57.7), which characterize the singularities of φ and ψ, are arbitrary (within the limits of convergence), with the exception of the imaginary part of the coefficient $\alpha_{-1}' + i\beta_{-1}'$ which is determined by (57.9). Thus, the nature of the singularities, caused by concentrated forces and couples, remains to a large measure undecided, unless additional conditions are introduced. It was only possible to obtain completely defined expressions for these singularities [cf. (57.2)—(57.5)], because the concentrated forces and couple were introduced by means of a definite limiting process.

Exactly the same expressions would have been found by a number of other limiting processes. One of the simplest examples, which more or less accurately reproduces the special conditions of the application of "concentrated" forces and couples, will now be stated. Imagine that a rigid disc be introduced into a circular hole in an infinite plate, and let this disc, which has the same radius as the hole, be joined to the plate along its circumference. Further, let some force and couple (in the plane of the plate) act on this disc. The solution of the problem of elastic equilibrium of plates under these conditions will be given below (cf. § 83*a*, examples 3°. and 4°., where the solution of the more general case of an elliptic disc is given). If one now allows the radius of the disc to tend to zero, leaving the force and couple unchanged, the above-mentioned solution gives in the limit a result which agrees exactly with those obtained above.

In the sequel, when speaking of concentrated forces and couples applied to internal points of bodies, it will be assumed that the corresponding singularities are given by the formulae (57.2)—(57.5).

§ 58. Some cases of equilibrium of infinite plates containing circular discs of different material. By means of a simple modification of the formulae of § 56*a*, a number of problems are easily solved which are important from the point of view of application. These problems refer to the equilibrium of infinite plates with circular holes into which discs of the same or other (likewise isotropic and homogeneous) material have been inserted.

For the solution of some of these problems use will be made of the solution of the problem of equilibrium of elastic circular (continuous) discs under the influence of uniform normal pressure, applied to their edges. This solution was already stated in § 41 for discs of arbitrary shape; in the present case it may, of course, be obtained directly from the formulae of § 54, but it is simplest to utilize the fact that the conditions of the problem will obviously be fulfilled by putting

$$X_x = -P, \quad Y_y = -P, \quad X_y = 0 \tag{58.1}$$

in the entire disc, where P denotes the magnitude of the constant pressure, applied to the edge.

In fact, for such a stress distribution, the stress on an arbitrarily oriented element reduces to the normal pressure P; this follows immediately from (8.8). Thus, in particular, the edge will be subject to the normal pressure P. This will likewise remain true in the case of discs of arbitrary shape.

It is easily verified that the functions $\Phi(z)$, $\Psi(z)$, $\varphi(z)$, $\psi(z)$, corresponding to the state of stress (58.1), have the form

$$\Phi(z) = -\frac{P}{2}, \quad \Psi(z) = 0, \quad \varphi(z) = -\frac{Pz}{2}, \quad \psi(z) = 0 \tag{58.2}$$

(omitting unessential arbitrary terms which only effect the rigid body displacements). The polar components of stress and displacement follow then from (39.4) and (39.3):

$$\widehat{rr} = \widehat{\vartheta\vartheta} = -P, \qquad \widehat{r\vartheta} = 0, \tag{58.3}$$

$$v_r = -\frac{P(\varkappa - 1)r}{4\mu}, \qquad v_\vartheta = 0. \tag{58.3'}$$

Next, a number of selected problems will be solved.

1°. Infinite plate with a circular hole into which an elastic circular disc with an originally larger radius has been inserted.

It will be assumed that there is no friction between the disc and the plate, so that the interaction of these bodies reduces to normal pressure on the edges of the disc and the plate. In view of the complete symmetry, this pressure will be constant along the boundaries. Therefore it is obvious that the solution of this problem may be constructed from the solution of problem 3° of § 56*a* — for a plate with a hole — and from (58.3) and (58.3′) — for the disc —, if one wants to calculate the magnitude of the pressure P acting between the plate and the disc.

Let the radius of the undeformed disc be $R + \varepsilon$, where R is the radius of the hole in the plate before deformation (and ε is, of course, assumed to be small, i.e., of the same order as the admissible displacements). All terms (elastic constants, components of stress etc.) referring to the disc will be marked with an index 0. For example, v_r^0 will denote the radial displacements of points of the disc, while v_r will refer to those of points of the surrounding plate.

It follows from the conditions of the problem that, after insertion of the disc into the hole in the plate, one must have along the common boundary of the disc and plate

$$v_r - v_r^0 = \varepsilon. \tag{58.4}$$

The radial displacement v_r^0 of a point of the rim of the disc may be considered to consist of the radial displacement $(-\varepsilon)$, necessary to reduce the radius of the

disc to R, and of the displacement v_r, which it undergoes together with the point of the edge of the hole with which it is in contact. Thus $v_r^0 = -\varepsilon + v_r$, whence follows (58.4).

But by (56.7′*a*) and (58.3′)

$$v_r = \frac{PR^2}{2\mu r}, \quad v_r^0 = -\frac{P(\varkappa_0 - 1)r}{4\mu_0}.$$

Putting in these expressions $r = R$ and substituting them in (58.4), one finds

$$\frac{PR}{2\mu} + \frac{P(\varkappa_0 - 1)R}{4\mu_0} = \varepsilon,$$

whence

$$P = \frac{4\varepsilon\mu\mu_0}{R[2\mu_0 + \mu(\varkappa_0 - 1)]}, \tag{58.5}$$

and the problem is solved. (Note that in actual fact one should have put $r = R + \varepsilon$ for the points of the disc, but in view of the magnitude of ε this is of no importance).

In the case of an *absolutely rigid disc* one will have, instead of (58.4),

$$v_r = \varepsilon, \tag{58.4′}$$

and hence, proceeding as before,

$$P = \frac{2\mu\varepsilon}{R}. \tag{58.5′}$$

The same value of P would have been obtained by putting in (58.5) $\mu_0 = \infty$ and assuming $\varkappa_0$ to be finite.

In this case (56.7′*a*) gives for the plate

$$\begin{gathered} \widehat{rr} = -\frac{2\mu R\varepsilon}{r^2}, \quad \widehat{\vartheta\vartheta} = \frac{2\mu R\varepsilon}{r^2}, \quad \widehat{r\vartheta} = 0, \\ v_r = \frac{\varepsilon R}{r}, \quad v_\vartheta = 0. \end{gathered} \tag{58.6}$$

2°. Stretching of plates with inserted or attached rigid discs.

In § 56*a* (example 1°), the solution was obtained of the problem of a plate with a circular hole of radius R under uni-directional tension. The

functions $\varphi(z)$, $\psi(z)$, giving the solution of this problem, may be rewritten

$$\varphi(z) = \frac{p}{4}\left(z + \frac{\beta R^2}{z}\right), \quad \psi(z) = -\frac{p}{2}\left(z + \frac{\gamma R^2}{z} + \frac{\delta R^4}{z^3}\right), \tag{58.7}$$

where

$$\beta = 2, \quad \gamma = 1, \quad \delta = -1. \tag{58.8}$$

The stresses and displacements, corresponding to these functions $\varphi(z)$ and $\psi(z)$, whatever may be the real constants β, γ, δ, are easily calculated on the basis of (39.4) and (39.3) which give (cf. § 56*a*)

$$\begin{aligned} \widehat{rr} &= \frac{p}{2}\left[1 - \frac{\gamma R^2}{r^2} + \left(1 - \frac{2\beta R^2}{r^2} - \frac{3\delta R^4}{r^4}\right)\cos 2\vartheta\right], \\ \widehat{\vartheta\vartheta} &= \frac{p}{2}\left[1 + \frac{\gamma R^2}{r^2} - \left(1 - \frac{3\delta R^4}{r^4}\right)\cos 2\vartheta\right], \\ \widehat{r\vartheta} &= -\frac{p}{2}\left(1 + \frac{\beta R^2}{r^2} + \frac{3\delta R^4}{r^4}\right)\sin 2\vartheta, \end{aligned} \tag{58.9}$$

and

$$\begin{aligned} v_r &= \frac{p}{8\mu r}\left\{(\varkappa - 1)r^2 + 2\gamma R^2 + \left[\beta(\varkappa + 1)R^2 + 2r^2 + \frac{2\delta R^4}{r^2}\right]\cos 2\vartheta\right\}, \\ v_\vartheta &= -\frac{p}{8\mu r}\left\{\beta(\varkappa - 1)R^2 + 2r^2 - \frac{2\delta R^4}{r^2}\right\}\sin 2\vartheta. \end{aligned} \tag{58.10}$$

If the constants β, γ, δ have the values (58.8), one obtains the earlier solutions of the problem of tension of a plate with a circular hole. By allotting these constants other (real) values, one may solve some problems which are of equal interest. Thus, for example, it is easy to deduce the solution of the problem of the stretching of a plate with a circular opening, cut before deformation and filled with a perfectly rigid disc of the same radius R.

First, suppose that the rigid disc is *joined* to the surrounding plate along its edge. It may be assumed that the rigid disc is not displaced during the stretching of the plate; otherwise it would be sufficient to subject the entire system to a rigid displacement, in order to return the disc to its original position. Hence the conditions of the problem are

$$v_r = 0, \quad v_\vartheta = 0 \qquad \text{for } r = R. \tag{58.11}$$

The problem will be solved, if one succeeds in choosing the constants

β, γ, δ, figuring in (58.9) and (58.10), so that (58.11) is satisfied. By (58.10), the conditions (58.11) give

$$\varkappa - 1 + 2\gamma = 0, \quad (\varkappa + 1)\beta + 2 + 2\delta = 0, \quad (\varkappa - 1)\beta + 2 - 2\delta = 0,$$

whence one finds

$$\beta = -\frac{2}{\varkappa}, \quad \gamma = -\frac{\varkappa - 1}{2}, \quad \delta = \frac{1}{\varkappa}, \tag{58.12}$$

or, remembering that $\varkappa = (\lambda + 3\mu)/(\lambda + \mu)$,

$$\beta = -\frac{2(\lambda + \mu)}{\lambda + 3\mu}, \quad \gamma = -\frac{\mu}{\lambda + \mu}, \quad \delta = \frac{\lambda + \mu}{\lambda + 3\mu}. \tag{58.12'}$$

(This problem can also be solved for the case when, in addition, arbitrarily given forces and couples act on the rigid disc; cf. § 83*a*).

Next consider the problem when the disc is not joined to the plate, but only inserted into the opening, under the assumption that there is no friction between the disc and the surrounding plate. Instead of (58.11), one has now the conditions

$$v_r = 0, \; \widehat{r\vartheta} = 0 \qquad \text{for } r = R, \tag{58.13}$$

since it may no longer be postulated that $v_\vartheta = 0$ at the edge of the plate, because points of the plate there are free to slide on the rim of the disc. As in the last example, it is easily verified that (58.13) will be satisfied, if one puts in (58.9) and (58.10)

$$\beta = -\frac{4}{3\varkappa + 1}, \quad \gamma = -\frac{\varkappa - 1}{2}, \quad \delta = -\frac{\varkappa - 1}{3\varkappa + 1}, \tag{58.14}$$

or

$$\beta = -\frac{2(\lambda + \mu)}{2\lambda + 5\mu}, \quad \gamma = -\frac{\mu}{\lambda + \mu}, \quad \delta = -\frac{\mu}{2\lambda + 5\mu}. \tag{58.14'}$$

However, it must be noted that the first of the conditions (58.13) assumes that the material of the plate is in close contact with the disc all along the common boundary; if this assumption is modified, the problem becomes considerably more difficult. It is easily verified that for the values of β, γ, δ, given by (58.14) and determined under the above assumption, the normal stress $\widehat{rr}$ becomes positive over certain parts of the common boundary, i.e., the disc does not press on the surrounding material, but pulls it away. However, this is physically impossible, because the disc and the plate are not joined to each other. In order to make the problem physically possible, it is sufficient, for example, to

suppose that the radius of the rigid disc is somewhat larger than was the radius of the opening before the stretching of the plate and before the disc was inserted. The solution, corresponding to this supposition, is obtained by superimposing the preceding solution on that given by (58.6). One has, of course, to take ε so large that the composite solution has $\widehat{rr} \leqslant 0$ along the common boundary.

3°. Stretching of plates with inserted or attached elastic discs. The preceding results will now be generalized to the case when the disc, inserted into the opening of the plate, is also elastic, but not of the same material as the plate.

An attempt will be made to satisfy the conditions of the problem, assuming that in the region occupied by the plate (i.e., for $r > R$) the elastic equilibrium is determined as before by the formulae (58.7), viz.,

$$\varphi(z) = \left(z + \frac{\beta R^2}{z}\right), \quad \psi(z) = \frac{p}{2}\left(z + \gamma R^2 + \frac{\delta R^4}{z^3}\right),$$

and that the equilibrium in the region, occupied by the disc (i.e., for $r < R$) is governed by

$$\varphi_0(z) = \frac{p}{4}\left(\beta_0 z + \frac{\gamma_0 z^3}{R^2}\right), \quad \psi_0(z) = -\frac{p}{2}\delta_0 z, \tag{58.15}$$

where $\beta, \gamma, \delta, \beta_0, \gamma_0, \delta_0$ are real constants, subject to definition.

This method of solution of the problem displays (outwardly) an artificial character, since the form of the solutions has been partially guessed beforehand. One could, of course, have eliminated any artificiality by using infinite series instead of (58.7) and (58.15). In that case one would have found for the solution of the problem that all coefficients of the series, with the exception of those retained above, must vanish. This observation also applies to the other problems treated in this section.

The stresses and displacements, corresponding to the functions $\varphi(z)$ and $\psi(z)$, are given by (58.9) and (58.10). Those corresponding to $\varphi_0(z)$ and $\psi_0(z)$ must be calculated from the formulae of § 39 which give

$$\begin{aligned} \widehat{rr}^0 \quad & \frac{p}{2}[\beta_0 + \delta_0 \cos 2\vartheta], \\ \widehat{\vartheta\vartheta}^0 &= \frac{p}{2}\left[\beta_0 + \left(\frac{6\gamma_0}{R^2} r^2 - \delta_0\right)\cos 2\vartheta\right], \\ \widehat{r\vartheta}^0 &= \frac{p}{2}\left(\frac{3\gamma_0}{R^2} r^2 - \delta_0\right)\sin 2\vartheta, \end{aligned} \tag{58.16}$$

and

$$v_r^0 = \frac{pr}{8\mu_0}\left\{\beta_0(\varkappa_0 - 1) + \left[\frac{\gamma_0(\varkappa_0 - 3)}{R^2} r^2 + 2\delta_0\right]\cos 2\vartheta\right\},$$
$$v_\vartheta^0 = \frac{pr}{8\mu_0}\left\{\frac{\gamma_0(\varkappa_0 + 3)}{R^2} r^2 - 2\delta_0\right\}\sin 2\vartheta. \tag{58.17}$$

Let it first be assumed that the disc has been welded into the hole and that the radii of the disc and the hole were equal before deformation. Then the following boundary conditions must be satisfied:

$$\widehat{rr}^0 = \widehat{rr}, \quad \widehat{r\vartheta}^0 = \widehat{r\vartheta}, \quad v_r^0 = v_r, \quad v_\vartheta^0 = v_\vartheta \quad \text{for } r = R. \tag{58.18}$$

Substituting in (58.18) the expressions (58.9), (58.10), (58.16) and (58.17), one finds the following equations for the determination of β, γ, δ, β_0, γ_0, δ_0:

$$\beta_0 = 1 - \gamma, \quad \delta_0 = 1 - 2\beta - 3\delta, \quad 3\gamma_0 - \delta_0 = -1 - \beta - 3\delta,$$
$$\frac{\beta_0(\varkappa_0 - 1)}{\mu_0} = \frac{\varkappa - 1 + 2\gamma}{\mu}, \quad \frac{\gamma_0(\varkappa_0 - 3) + 2\delta_0}{\mu_0} = \frac{(\varkappa + 1)\beta + 2 + 2\delta}{\mu},$$
$$\frac{\gamma_0(\varkappa_0 + 3) - 2\delta_0}{\mu_0} = \frac{(\varkappa - 1)\beta + 2 - 2\delta}{\mu}.$$

Solving this system of equations, one obtains

$$\beta = -\frac{2(\mu_0 - \mu)}{\mu + \mu_0\varkappa}, \quad \gamma = \frac{\mu(\varkappa_0 - 1) - \mu_0(\varkappa - 1)}{2\mu_0 + \mu(\varkappa_0 - 1)}, \quad \delta = \frac{\mu_0 - \mu}{\mu + \mu_0\varkappa},$$
$$\beta_0 = \frac{\mu_0(\varkappa + 1)}{2\mu_0 + \mu(\varkappa_0 - 1)}, \quad \gamma_0 = 0, \quad \delta_0 = \frac{\mu_0(\varkappa + 1)}{\mu + \mu_0\varkappa}. \tag{58.19}$$

Consider now the circumstance that, because of the relation $\gamma_0 = 0$, the functions $\varphi_0(z)$ and $\psi_0(z)$ characterizing the elastic equilibrium of the disc are linear:

$$\varphi_0(z) = \frac{p}{4}\beta_0 z, \quad \psi_0(z) = -\frac{p}{2}\delta_0 z; \tag{58.15'}$$

this means that the disc undergoes homogeneous deformation. In rectangular coordinates the stress components will be constant; in fact, it is easily verified that

$$X_x^0 = p\,\frac{\beta_0 + \delta_0}{2}, \quad Y_y^0 = p\,\frac{\beta_0 - \delta_0}{2}, \quad X_y^0 = 0. \tag{58.20}$$

In the direction of the Ox axis the disc is subject to tension, while in the y direction it suffers tension or compression, depending on the sign of $\beta_0 - \delta_0$.

In the limiting case $\mu_0 = \infty$ (perfectly rigid disc), one obtains for β, γ, δ the values (58.12); in the limiting case $\mu_0 = 0$ (no disc), one obtains for these constants the values (58.8). Finally, if $\mu = \mu_0$, $\varkappa = \varkappa_0$, one is dealing with a continuous homogeneous plate. In this case (58.19) shows that $\beta = \gamma = \delta = \gamma_0 = 0$, $\beta_0 = \delta_0 = 1$ and that the functions $\varphi(z)$, $\psi(z)$, characterizing the equilibrium of the plate as well as of the disc, are given by

$$\varphi(z) = \frac{p}{4} z, \quad \psi(z) = -\frac{p}{2} z, \tag{58.15''}$$

as had, of course, to be expected.

Next consider the case when the disc has been *inserted* into the opening in the plate, assuming that the radii of the disc and of the hole were the same before deformation and that no friction is present. Obviously the boundary conditions have the form

$$\widehat{rr}^0 = \widehat{rr}, \quad \widehat{r\vartheta}^0 = 0, \quad \widehat{r\vartheta} = 0, \quad v_r^0 = v_r \qquad \text{for } r = R. \tag{58.21}$$

Substituting in (58.21) from (58.16), (58.17), (58.9) and (58.10), one finds

$$\beta_0 = 1 - \gamma, \quad \delta_0 = 1 - 2\beta - 3\delta, \quad 3\gamma_0 - \delta_0 = 0, \quad 1 + \beta + 3\delta = 0,$$

$$\frac{\beta_0(\varkappa_0 - 1)}{\mu_0} = \frac{\varkappa - 1 + 2\gamma}{\mu}, \quad \frac{\gamma_0(\varkappa_0 - 3) + 2\delta_0}{\mu_0} = \frac{\beta(\varkappa + 1) + 2 + 2\delta}{\mu}.$$

Solving these equations, one obtains

$$\beta = 2\frac{\mu(\varkappa_0 + 3) - 2\mu_0}{\mu(\varkappa_0 + 3) + \mu_0(3\varkappa + 1)}, \qquad \gamma = -\frac{\mu(\varkappa_0 - 1) - \mu_0(\varkappa - 1)}{2\mu_0 + \mu(\varkappa_0 - 1)},$$

$$\delta = -\frac{\mu(\varkappa_0 + 3) + \mu_0(\varkappa - 1)}{\mu(\varkappa_0 + 3) + \mu_0(3\varkappa + 1)}, \qquad \beta_0 = \frac{\mu_0(\varkappa + 1)}{2\mu_0 + \mu(\varkappa_0 - 1)}, \tag{58.22}$$

$$\gamma_0 = \frac{2\mu_0(\varkappa + 1)}{\mu(\varkappa_0 + 3) + \mu_0(3\varkappa + 1)}, \qquad \delta_0 = \frac{6\mu_0(\varkappa + 1)}{\mu(\varkappa_0 + 3) + \mu_0(3\varkappa + 1)}.$$

As is easily seen, the values of $\widehat{rr}^0$ and $\widehat{rr}$ will be positive on parts of the common boundary, and this is physically impossible. The problem can be made physically possible by superimposing on the present solution that of problem 1°.

By putting in (58.22) $\mu_0 = 0$ or $\mu_0 = \infty$, one finds for β, γ, δ the values (58.8) or (58.14) respectively.

CHAPTER 10

THE CIRCULAR RING

§ 59. Solution of the first fundamental problem for the circular ring.

A solution, using definite integrals and differing from the one to be deduced here, was published by G. V. Kolosov [5]. S. G. Mikhlin [8] (using power series) solved the somewhat more general problem where the ring consists of two concentric rings with different elastic constants, under the supposition that they are joined along the common boundary. In particular, the inner ring may be a continuous disc.

Consider the case when the region S occupied by the body is a circular ring, bounded by two concentric circles L_1 and L_2 with radii R_1 and R_2 ($R_1 < R_2$) and centre at the origin. Let the external stresses acting on L_1 and L_2 be given, i.e., the values of $\widehat{rr} - i\widehat{r\vartheta}$ on L_1 and L_2 as functions of the angle ϑ. Expanding this expression for L_1, as well as for L_2, in complex Fourier series, one will have

$$\begin{aligned} \widehat{rr} - i\widehat{r\vartheta} &= \sum_{-\infty}^{+\infty} A_k' e^{ik\vartheta} \text{ on } L_1, \\ \widehat{rr} - i\widehat{r\vartheta} &= \sum_{-\infty}^{+\infty} A_k'' e^{ik\vartheta} \text{ on } L_2. \end{aligned} \tag{59.1}$$

The boundary conditions may then be written (cf. § 56)

$$\Phi(z) + \Phi(z) - e^{2i\vartheta}[\bar{z}\Phi'(z) + \Psi(z)] \quad \begin{matrix} \sum\limits^{+\infty} A_k' e^{ik\vartheta} & \text{for} & r = R_1, \\ \sum\limits^{+\infty} A_k'' e^{ik\vartheta} & \text{for} & r = R_2. \end{matrix} \tag{59.2}$$

By (35.2),

$$\Phi(z) = A \log z + \Phi^*(z),$$

where A is a real constant and $\Phi^*(z)$ is holomorphic inside the ring, so that it may be represented by a Laurent series. The function $\Psi(z)$ is holomorphic in the considered region (§ 35) and hence may likewise be expanded as a Laurent series.

Thus, inside S,

$$\Phi(z) = A \log z + \overset{+\infty}{\Sigma}\, a_k z^k, \quad \Psi(z) = \overset{+\infty}{\Sigma}\, a_k' z^k. \tag{59.3}$$

The requirement of single-valuedness of displacements is expressed by (35.7) which, since there is only one internal boundary, becomes

$$A = 0, \quad \varkappa a_{-1} + \bar{a}_{-1}' = 0. \tag{59.4}$$

However, this condition will not yet be imposed, since the more general solution has many interesting interpretations.

It will be remembered that the quantities γ_k and γ_k' of § 35 were the coefficients of terms of the form $(a \log z)$ in the expansions of the functions

$$\varphi(z) = \int^{z} \Phi(z)\,dz$$

and

$$\psi(z) = \int^{z} \Psi(z)\,dz.$$

In the present notation these terms are $a_{-1} \log z$ and $a_{-1}' \log z$ respectively.

Instead of (59.4), assume for the time being that A is *an arbitrarily given real constant*. Substituting from (59.3) in (59.2), one finds

$$2A \log r - A + \sum_{-\infty}^{+\infty} (1-k) a_k r^k e^{ik\vartheta} +$$

$$+ \overset{+\infty}{\Sigma}\, \bar{a}_k r^k e^{-ik\vartheta} - \overset{+\infty}{\Sigma}\, a_{k-2}' r^{k-2} e^{ik\vartheta} = \begin{cases} \overset{+\infty}{\Sigma}\, A_k' e^{ik\vartheta} \text{ for } r \quad R_1, \\ \Sigma\, A_k'' e^{ik\vartheta} \text{ for } r = R_2. \end{cases} \tag{59.2'}$$

Comparison of terms independent of ϑ gives

$$\begin{aligned} 2A \log R_1 - A + 2a_0 - a_{-2}' R_1^{-2} &= A_0', \\ 2A \log R_2 - A + 2a_0 - a_{-2}' R_2^{-2} &= A_0''; \end{aligned} \tag{59.5}$$

here the assumption has been made that $a_0 = \bar{a}_0$ i.e., that a_0 is real, which can always be done, since any constant imaginary part of $\Phi(z)$ has been shown not to influence the stress distribution.

Comparison of terms involving $e^{ik\vartheta}$ for $k = \pm 1, \pm 2, \ldots$ gives

$$\begin{aligned} (1-k) a_k R_1^k + \bar{a}_{-k} R_1^{-k} - a_{k-2}' R_1^{k-2} &= A_k', \\ (1-k) a_k R_2^k + \bar{a}_{-k} R_2^{-k} - a_{k-2}' R_2^{k-2} &= A_k''. \end{aligned} \tag{59.6}$$

Eliminating a'_{-2} from (59.5), one finds

$$a_0 = \frac{A''_0 R_2^2 - A'_0 R_1^2}{2(R_2^2 - R_1^2)} + \frac{A}{2} - \frac{A(R_2^2 \log R_2 - R_1^2 \log R_1)}{R_2^2 - R_1^2}. \tag{59.7}$$

Since a_0 is real, it follows that

$$\Im(A''_0 R_2^2 - A'_0 R_1^2) = 0. \tag{59.8}$$

[If a_0 had not been assumed to be real, one would have had on the left-hand side of (59.7) $\dfrac{a_0 + \bar{a}_0}{2}$ instead of a_0 and (59.8) would still have been valid.] A simple calculation shows that (59.8) expresses that the resultant moment of the external stresses must be equal to zero.

Next the remaining coefficients will be determined. Dividing the first equation of (59.6) by R_1^{k-2}, and the second by R_2^{k-2}, and subtracting, one obtains the first of the following formulae:

$$\begin{aligned} &(1-k)(R_2^2 - R_1^2)a_k + (R_2^{-2k+2} - R_1^{-2k+2})\bar{a}_{-k} = B_k, \\ &(R_2^{2k+2} - R_1^{2k+2})a_k \quad + (1+k)(R_2^2 - R_1^2)\bar{a}_{-k} = \bar{B}_{-k}, \end{aligned} \tag{59.9}$$

where

$$B_k = A''_k R_2^{-k+2} - A'_k R_1^{-k+2}; \tag{59.10}$$

the second equation (59.9) is obtained from the first by replacing k by $-k$ and by going to the conjugate complex expression. (It will now be sufficient to consider (59.9) only for $k = 1, 2, 3, \ldots$, since for $k = -1, -2, -3$ one obtains a system of equations which is conjugate to the former).

For any given value of k, the system of two equations (59.9) will determine a_k and $\bar{a}_{-k}$, provided the determinant

$$D = \begin{vmatrix} (1-k)(R_2^2 - R_1^2) & R_2^{-2k+2} - R_1^{-2k+2} \\ R_2^{2k+2} - R_1^{2k+2} & (1+k)(R_2^2 - R_1^2) \end{vmatrix}$$
$$= (1-k^2)(R_2^2 - R_1^2)^2 - (R_2^{2k+2} - R_1^{2k+2})(R_2^{-2k+2} - R_1^{-2k+2}) \tag{59.11}$$

does not vanish.

The determinant D vanishes for $k = 0, \pm 1$, and it is easily verified that for all other values of k it is different from zero. The value $k = 0$ is of no interest. For $k = +1$, (59.9) gives

$$0 = B_1, \quad (R_2^4 - R_1^4)a_1 + 2(R_2^2 - R_1^2)\bar{a}_{-1} = \bar{B}_{-1}. \tag{59.12}$$

For $k = -1$, one finds two equations, obtained from (59.12) by transition to the conjugate complex values. Thus, for the problem to be possible, one must have, in addition to (59.8),

$$B_1 = A_1''R_2 - A_1'R_1 = 0. \tag{59.13}$$

A simple calculation shows that this condition gives no new information, since it states that the resultant vector of all external forces must vanish.

To verify the earlier statement regarding the values of the determinant (59.11), consider

$$D = R_1^4 f(\xi),$$

where

$$\xi = \left(\frac{R_2}{R_1}\right)^2 > 1$$

and

$$f(\xi) = (1 - k^2)(\xi - 1)^2 + \xi^{k+1} + \xi^{-k+1} - \xi^2 + 1.$$

It is easily verified that

$$f(1) = f'(1) = f''(1) = f'''(1) = 0,$$

$$f^{IV}(\xi) = (k+1)k(k-1)[(k-2)\xi^{k-3} + (k+2)\xi^{-k-3}].$$

If $|k| \geqslant 2$, the last expression is positive for $\xi > 0$. Thus, for $\xi > 1$, one will have

$$f'''(\xi) > 0, \quad f''(\xi) > 0, \quad f'(\xi) > 0, \quad f(\xi) > 0. \quad \text{q.e.d.}$$

To show that (59.13) is the condition that the resultant vector of the external forces must vanish, consider first points of the outer circle. It is easily verified that

$$X_n + iY_n = (\widehat{rr} + i\,\widehat{r\vartheta})\,e^{i\vartheta}, \quad X_n - iY_n = (\widehat{rr} - i\,\widehat{r\vartheta})e^{-i\vartheta} \text{ on } L_2.$$

Denoting by (X'', Y'') the resultant vector of the external forces, applied to L_2, one has

$$X'' - iY'' = \int_0^{2\pi} (X_n - iY_n)R_2 d\vartheta = R_2 \int_0^{2\pi} (\widehat{rr} - i\,\widehat{r\vartheta})e^{-i\vartheta}d\vartheta = 2\pi R_2 A_1'',$$

by definition of A_1''. Similarly, one has for the inner circle

$$X' - iY' = -2\pi R_1 A_1'. \quad \text{q.e.d.}$$

When (59.13) is satisfied, the system of equations (59.12) becomes possible, although it does not permit calculation of both the coefficients a_1 and $\bar{a}_{-1}$; thus one of them may be chosen arbitrarily, *neglecting for the time being the condition of single-valuedness of displacements.*

All the other coefficients a_k ($k = \pm 2, \pm 3, \ldots$) are found by solving (59.9). For any given k, one determines simultaneously a_k and $\bar{a}_{-k}$. In fact, (59.9) gives

$$a_k = \frac{(1 + k)(R_2^2 - R_1^2) B_k - (R_2^{-2k+2} - R_1^{-2k+2})\bar{B}_{-k}}{(1 - k^2)(R_2^2 - R_1^2)^2 - (R_2^{2k+2} - R_1^{2k+2})(R_2^{-2k+2} - R_1^{-2k+2})}$$

$$(k = \pm 2, \pm 3, \ldots), \quad (59.14)$$

and $\bar{a}_{-k}$ is obtained from this formula by replacing k by $-k$ and by transition to the conjugate complex value. Thus all coefficients a_k have been determined for $k \neq 0, \pm 1$.

Finally, the coefficients a_k' may be found from one of the two formulae (59.6), with the exception of a_{-2}' which can be calculated from one of the equations (59.5). Since all a_k, with the exception of a_1, a_{-1}, have already been determined, all a_k', except for a_{-1}' and a_{-3}', can be calculated in this way.

Now *the condition of single-valuedness of displacements* will be introduced, i.e., condition (59.4). One then finds, by (59.7),

$$a_0 = \frac{A_0'' R_2^2 - A_0' R_1^2}{2(R_2^2 - R_1^2)}. \quad (59.7')$$

The coefficients a_{-1}, a_{-1}' are determined by the second equation of (59.4) and, for example, by the first equation (59.6) for $k = +1$, which gives

$$a_{-1} - a_{-1}' = \bar{A}_1' R_1. \quad (59.15)$$

Solving (59.4) and (59.15) for a_{-1} and a_{-1}', one obtains

$$a_{-1} = \frac{\bar{A}_1' R_1}{1 + \varkappa}, \quad a_{-1}' = -\frac{\varkappa A_1' R_1}{1 + \varkappa}; \quad (59.16)$$

finally, one finds from (59.12)

$$a_1 = \frac{\bar{B}_{-1}}{R_2^4 - R_1^4} - \frac{2 \bar{A}_1' R_1}{(1 + \varkappa)(R_1^2 + R_2^2)}. \quad (59.16')$$

The formulae (59.16) could have been written down immediately, using (35.9). Thus all coefficients in the expansions for Ψ and Φ have been found; in particular, a_{-3}' can now be calculated from (59.6), because a_1 and a_{-1} are known.

Note with regard to the convergence of the above series that the series for $\Phi(z)$, $\Phi'(z)$ and $\Psi(z)$ will obviously be alsolutely and uni-

formly convergent in the ring (including the boundaries), if the following series converge:

$$\begin{array}{lll} \sum_{k=1}^{\infty} |a_k| R_2^k, & \sum_{k=1}^{\infty} k |a_k| R_2^{k-1} & \sum_{k=1}^{\infty} |a_k'| R_2^k, \\ \sum_{k=1}^{\infty} |a_{-k}| R_1^{-k} & \sum_{k=1}^{\infty} k |a_{-k}| R_1^{-k-1} & \sum_{k=1}^{\infty} |a_{-k}'| R_1^{-k} \end{array} \tag{59.17}$$

Convergence of the latter series will be ensured, if it is assumed that the quantities $\widehat{rr}$ and $\widehat{r\vartheta}$, given on L_1 and L_2, have second order derivatives with respect to ϑ, which satisfy the Dirichlet condition. In fact, the coefficients A_k' and A_k'' of the series (59.1) will then satisfy inequalities of the form (§ 53)

$$|A_k'| < \frac{C}{k^3} \qquad |A_k''| < \frac{C}{k^3}, \qquad (k = \pm 1, \pm 2, \ldots).$$

Hence it is easily concluded on the basis of (59.10), (59.14) and (59.6) that the following inequalities will hold true for $k = 1, 2, 3 \ldots$:

$$|a_k| R_2^k < \frac{C}{k^3}, \quad |a_k'| R_2^k < \frac{C}{k^2}, \quad |a_{-k}| R_1^{-k} < \frac{C}{k^3}, \quad |a_{-k}'| R_1^{-k} < \frac{C}{k^2},$$

whence it follows immediately that the series (59.17) converge.

The second fundamental problem can be solved in a similar manner.

If one compares the solution deduced here with that obtained by application of Airy's function, the advantage of the introduction of functions of a complex variable becomes obvious.

With regard to the above considerations of convergence of the series, it may be shown, as in § 56, that, if one uses the boundary conditions in the form (41.5), it has to be assumed that X_n and Y_n have on L_1 and L_2 *first* order derivatives, satisfying the Dirichlet condition.

The Airy function is used in A. Almansi [2], J. H. Michell [1] and A. Timpe [1] for the solution, among others, of the first fundamental problem. Timpe has the same expression for the boundary conditions which is represented by (59.2′) above, and he deals with it at great length. The coefficients are determined, eight at a time, from systems of eight linear equations with eight unknowns.

§ 59a. Examples.

1°. Tube, subject to uniform external and internal pressures.

Let the internal and external circles be subjected to uniformly distributed normal pressures p_1 and p_2, so that $\widehat{rr} = -p_1$ on L_1, $\widehat{rr} = -p_2$

on L_2, $\widehat{r\vartheta} = 0$ on L_1, L_2. In this case

$$A_0' = -p_1, \quad A_0'' = -p_2.$$

All other coefficients A_k', A_k'' vanish. The condition for the existence of a solution is obviously satisfied. Formulae (59.7′) and (59.5) give

$$a_0 = -\frac{p_2R_2^2 - p_1R_1^2}{2(R_2^2 - R_1^2)}, \quad a'_{-2} = \frac{(p_1 - p_2)R_1^2R_2^2}{R_2^2 - R_1^2}. \tag{59.1a}$$

All the other coefficients a_k, a_k' are zero. Thus

$$\Phi(z) = -\frac{p_2R_2^2 - p_1R_1^2}{2(R_2^2 - R_1^2)}, \quad \Psi(z) = -\frac{(p_2 - p_1)R_1^2R_2^2}{R_2^2 - R_1^2}\,\frac{1}{z^2}. \tag{59.2a}$$

The polar components of stress are

$$\begin{aligned}\widehat{rr} &= -\frac{p_2R_2^2 - p_1R_1^2}{R_2^2 - R_1^2} + \frac{(p_2 - p_1)R_1^2R_2^2}{R_2^2 - R_1^2}\,\frac{1}{r^2},\\ \widehat{\vartheta\vartheta} &= -\frac{p_2R_2^2 - p_1R_1^2}{R_2^2 - R_1^2} - \frac{(p_2 - p_1)R_1^2R_2^2}{R_2^2 - R_1^2}\,\frac{1}{r^2},\\ \widehat{r\vartheta} &= 0.\end{aligned} \tag{59.3a}$$

This problem was also solved by G. Lamé.

2°. Stress distribution in a ring, rotating about its centre.

Let the ring rotate in its plane about O with a constant angular velocity ω and let no other external forces act on it. Let the system of axes Oxy rotate together with the body and hence be fixed relative to it. Then the problem reduces to a static one, the applied forces being centrifugal in origin.

One of the particular solutions of the equations of equilibrium is given by the formulae of § 28. The stresses, given by (28.6), are easily found to have the following polar components:

$$\widehat{rr} = -\frac{2\lambda + 3\mu}{4(\lambda + 2\mu)}\,\rho\omega^2r^2, \quad \widehat{\vartheta\vartheta} = -\frac{2\lambda + \mu}{4(\lambda + 2\mu)}\,\rho\omega^2r^2, \quad \widehat{r\vartheta} = 0. \tag{59.4a}$$

If one wants to apply this solution to *thin* plates (§ 26), one has to replace λ by λ^*, so that

$$\frac{2\lambda^* + 3\mu}{4(\lambda^* + 2\mu)} = \frac{3 + \sigma}{8}, \quad \frac{2\lambda^* + \mu}{4(\lambda^* + 2\mu)} = \frac{1 + 3\sigma}{8} \tag{59.5a}$$

The stresses (59.4*a*) do not satisfy the boundary conditions on the edges of the plate. In fact, one has there $\widehat{r\vartheta} = 0$, but $\widehat{rr}$ takes constant values which will be denoted by p_1 and p_2.

The solution of the present problem is obtained by superposition of the stresses (59.4a) and (59.3a) for

$$p_1 = \frac{2\lambda + 3\mu}{4(\lambda + 2\mu)} \rho\omega^2 R_1^2, \quad p_2 \qquad \frac{2\lambda + 3\mu}{4(\lambda + 2\mu)} \rho\omega^2 R_2^2. \tag{59.6}$$

The problem is thus solved. In the case of a thin plate, λ^* replaces λ. The solution, obtained in that manner, gives only mean values of the stresses. For not very thin plates this will not be sufficient. (For more complete solutions, cf. A. E. H. Love [1] § 102). When $R_2 = 0$, one finds the solution for the case of a solid rotating disc.

§ 60. Multi-valued displacements in the case of a circular ring.

Consider now the general case and study the results, given by the formulae of § 59 when one omits the conditions of single-valuedness of displacements which are expressed by (59.4), viz.,

$$A = 0, \quad \varkappa a_{-1} + \bar{a}'_{-1} = 0 \tag{60.1}$$

If these conditions no longer hold, the boundary conditions (59.2) are not sufficient for the complete determination of $\Phi(z)$ and $\Psi(z)$; some of the coefficients in the expansions for these functions remain indeterminate and a known number of arbitrary constants is retained about which more will be said below. By fixing these constants in an arbitrary manner, one obtains definite expressions for Φ and Ψ which satisfy all the conditions of the problem, except the condition of single-valuedness of displacements. In fact, if one describes a closed path L', starting from some point z, passing around the inner circle in an anti-clockwise direction and reaching again z, one finds that the increase in $u + iv$ for one circuit of this path is given by

$$[u + iv]_{L'} = \frac{\pi i}{\mu} \{(\varkappa + 1)Ai + \varkappa a_{-1} + \bar{a}'_{-1}\}; \tag{60.2}$$

this follows from (35.6), using the notation of § 59.

It has been seen in § 45 that, in spite of the multi-valuedness of the displacements, such a solution may be given a definite and very simple physical interpretation.

Firstly, this solution makes sense in the ordinary way, if one does

not apply it to the complete ring, but to a part of it obtained by removing from the ring a strip, bounded by two lines $a'b'$ and $a''b''$ connecting the inner and outer circles (in Fig. 29 the removed part has been shaded). One then has a simply connected body, i.e., a "curved beam", bounded by two circular arcs and the lines $a'b'$ and $a''b''$. In this simply connected part the functions u, v are single-valued. The functions Φ and Ψ correspond to some definite state of elastic equilibrium of the beam for which the external stresses, applied to the circular boundary, have known values, i.e., those which appear in the boundary conditions of § 59 for the solution of the problem of the continuous ring. As regards the external stresses, applied to the ends $a'b'$ and $a''b''$, they may be calculated from the functions Φ and Ψ by the help of the previously deduced formulae. The problem of equilibrium of a curved beam will be considered in § 61.

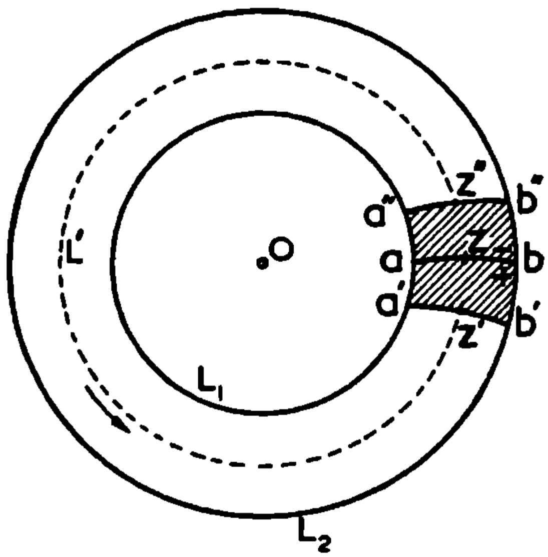


Fig. 29

Turn now to the case of the continuous ring. It has been seen in § 45 that the solution considered here, admitting multi-valued displacements, corresponds to a particular mode of deformation, which is called dislocation. This type of deformation will now be described in as far as it applies to the present problem, and, in parts, this will involve repetition of what has been said in § 45.

Produce a cut ab, joining the inner and the outer circle, and denote the sides of this cut by (+) and (—), as indicated in Fig. 29. Then, for a circuit along a closed contour L' starting from some point (x, y), considered to lie on the side (—), and ending at the same point, but now conceived to lie on the side (+), the components of displacement undergo, by (60.2), the increases

$$u^+ \quad u^- = -\varepsilon y + \alpha, \qquad v^+ - v^- = \varepsilon x + \beta, \tag{60.3}$$

where

$$\varepsilon = \frac{\pi A(1 + \varkappa)}{\mu}, \qquad \alpha + i\beta = \frac{\pi i}{\mu} (\varkappa a_{-1} + \bar{a}'_{-1}); \tag{60.4}$$

(u^+, v^+) and (u^-, v^-) denote the displacements of the point (x, y), considered to lie on the sides $(+)$ and $(-)$ respectively.

In accord with the statements of § 45, the multi-valuedness of the displacements in the present solution may be interpreted by means of the hypothesis that before deformation a (small) transverse strip with sides $a'b'$, $a''b''$ (see Fig. 29) had been removed from the ring and that the free edges had been joined. For this purpose it was assumed that before deformation the ends $a'b'$ and $a''b''$ were congruent and disposed in such a way that $a''b''$ is obtained from $a'b'$ by a rigid displacement, consisting of a rotation ε about the origin and a translation (α, β). When joining the free edges together, those points which would correspond to each other, but for the above-mentioned rigid displacement, must be combined.

Note that, as indicated in § 45, the quantities ε, α, β do not depend on the shape of the cut ab nor on its location in the ring; in the present case this follows immediately from (60.4). Thus, the transverse strip which must be cut from the ring before deformation may be taken from any part of it; one of its sides, for example $a'b'$, may be given any shape and location, and the position of the other side will be determined by the quantities ε, α, β.

It should be remembered here that it is only for the sake of convenience that reference has been made to "removal" of strips, since in practice, one has often to "add" material in one part and remove it in another.

The quantities ε, α, β represent, in the terminology of § 45, the characteristics of the dislocation. According to § 45, knowledge of these quantities and of the external stresses, applied to L_1 and L_2, will completely determine the deformation of the body under consideration. In the present case this fact is verified directly, because it is easily seen that, if those quantities are known, all coefficients in the expansions for $\Phi(z)$ and $\Psi(z)$ are determined (except for the imaginary part of a_0 which is unimportant); in fact, these coefficients may be calculated as in § 59, the only difference being that (59.4) is to be replaced by the more general condition (60.4) for given values of ε, α, β.

The coefficients in the expansions of the functions $\Phi(z)$ and $\Psi(z)$ will now be determined for the particular case, when *no external forces* are present (i.e., $N = T = 0$ on L_1 and L_2). Then all A_k' and A_k'' vanish and (60.4) gives

$$A = \frac{\mu\varepsilon}{\pi(1+\varkappa)}, \quad \varkappa a_{-1} + \bar{a}'_{-1} = \frac{\mu}{\pi i}(\alpha + i\beta). \qquad (60.4')$$

Together with the equation $a_{-1} - \bar{a}'_{-1} = 0$, obtained from (59.15), the second equation of (60.4′) leads to

$$-\frac{\mu(a+ib)i}{\pi(1+\varkappa)}, \qquad a_{-1} = \frac{\mu(\alpha - i\beta)i}{\pi(1+\varkappa)} \tag{60.5}$$

Further, (59.7) gives

$$a_0 = \frac{\mu\varepsilon}{2\pi(1+\varkappa)} - \frac{\mu\varepsilon(R_2^2 \log R_2 - R_1^2 \log R_1)}{\pi(1+\varkappa)\,(R_2^2 - R_1^2)}, \tag{60.6}$$

and, finally, by the second equation of (59.12), by (59.5) and (59.6) (for $k = 1$), one obtains

$$\begin{gathered} a_1 = -\frac{2\mu(\alpha - i\beta)i}{\pi(1+\varkappa)\,(R_1^2 + R_2^2)}, \quad a_{-2} = \frac{2\mu\varepsilon R_1^2 R_2^2}{\pi(1+\varkappa)(R_2^2 - R_1^2)} \log \frac{R_2}{R_1}, \\ a_{-3} = -\frac{2\mu(\alpha + i\beta)i}{\pi(1+\varkappa)} \frac{R_1^2 R_2^2}{R_1^2 + R_2^2}. \end{gathered} \tag{60.7}$$

All remaining coefficients are zero, and hence

$$\Phi(z) = A \log z + a_0 + a_1 z + \frac{a_{-1}}{z}, \quad \Psi(z) = \frac{a'_{-1}}{z} + \frac{a'_{-2}}{z^2} + \frac{a'_{-3}}{z^3}, \tag{60.8}$$

where the coefficients have the values stated above.

In the particular case, when $\varepsilon = \alpha = \beta = 0$ (i.e., when the displacements are single-valued), one has $\Phi(z) = \Psi(z) = 0$, as was to be expected, because it is known (§ 40) that, if the displacements are single-valued, no stresses occur in the absence of external forces.

If external forces are present, the corresponding solution may be obtained by superimposing the solution just found on that of § 59 which was deduced under the hypothesis of single-valuedness of displacements.

As has been stated in § 45, the interpretation of multi-valued displacements in the case of the circular ring was first given by A. Timpe [1], who also found formulae equivalent to those deduced here.

Turning now to the dislocations, corresponding to (60.8), it is noted that one may distinguish between the following three simple cases:

1°. $\varepsilon \neq 0$, $\alpha = \beta = 0$. This dislocation is obtained, for example, by cutting from the ring a radial sector with straight edges, forming the angle ε, and by joining the ends.

2°. $\varepsilon = 0$, $\alpha \neq 0$, $\beta = 0$. This dislocation is obtained, for example,

if one cuts the ring along the positive Ox axis, slides the lower against the upper edge by a distance α and again joins the contacting parts. The same dislocation is obtained, if one removes along the positive Oy axis a strip of thickness α and rejoins the ring by displacing the free edges parallel to the Ox axis. In the latter case, when $\alpha > 0$, a strip must be added.

3°. $\varepsilon = 0$, $\alpha = 0$, $\beta \neq 0$. This case follows from the preceding one by interchanging the parts played by the axes Ox and Oy.

Thus, it will be sufficient to state the formulae referring, for example, to the cases 1° and 3°. The expressions for the functions Φ and Ψ and for the polar components of stress will be stated here. They agree with those, obtained in a different manner by A. Timpe [1]; the method used here was taken from the Author's paper [1]. Note, however, that in the following formulae $\varkappa$ has been replaced by $\dfrac{\lambda + 3\mu}{\lambda + \mu}$.

1°. $(\varepsilon \neq 0,\ \alpha = \beta = 0)$:

$$\Phi(z) = \frac{\varepsilon\mu(\lambda + \mu)}{2\pi(\lambda + 2\mu)} \left\{\frac{1}{2} - \frac{R_2^2 \log R_2 - R_1^2 \log R_1}{R_2^2 - R_1^2}\right\} + \frac{\varepsilon\mu(\lambda + \mu)}{2\pi(\lambda + 2\mu)} \log z,$$

$$\Psi(z) = -\frac{\varepsilon\mu(\lambda + \mu)R_1^2R_2^2}{\pi(\lambda + 2\mu)(R_2^2 - R_1^2)} \log \frac{R_2}{R_1} \cdot \frac{1}{z^2}, \tag{60.9}$$

$$\widehat{rr} = \frac{\varepsilon\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} \left\{\log r + \frac{1}{r^2} \cdot \frac{R_1^2R_2^2}{R_2^2 - R_1^2} \log \frac{R_2}{R_1} - \frac{R_2^2 \log R_2 - R_1^2 \log R_1}{R_2^2 - R_1^2}\right\},$$

$$\widehat{\vartheta\vartheta} = \frac{\varepsilon\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} \left\{\log r - \frac{1}{r^2} \cdot \frac{R_1^2R_2^2}{R_2^2 - R_1^2} \log \frac{R_2}{R_1} - \frac{R_2^2 \log R_2 - R_1^2 \log R_1}{R_2^2 - R_1^2} + 1\right\}, \tag{60.10}$$

$$\widehat{r\vartheta} = 0.$$

3°. $(\varepsilon = 0,\ \alpha = 0,\ \beta \neq 0)$:

$$\Phi(z) = -\frac{\beta\mu(\lambda + \mu)}{2\pi(\lambda + 2\mu)} \left\{\frac{2z}{R_1^2 + R_2^2} - \frac{1}{z}\right\},$$

$$\Psi(z) = \frac{\beta\mu}{2\pi(\lambda + 2\mu)} \left\{\frac{1}{z} + \frac{R_1^2R_2^2}{R_1^2 + R_2^2} \cdot \frac{1}{z^3}\right\}, \tag{60.11}$$

$$\widehat{rr} = \frac{\beta\mu(\lambda+\mu)}{\pi(\lambda+2\mu)}\left\{-\frac{r}{R_1^2+R_2^2}+\frac{1}{r}-\frac{R_1^2R_2^2}{R_1^2+R_2^2}\cdot\frac{1}{r^3}\right\}\cos\vartheta,$$

$$\widehat{\vartheta\vartheta} = \frac{\beta\mu(\lambda+\mu)}{\pi(\lambda+2\mu)}\left\{-\frac{3r}{R_1^2+R_2^2}+\frac{1}{r}+\frac{R_1^2R_2^2}{R_1^2+R_2^2}\cdot\frac{1}{r^3}\right\}\cos\vartheta, \quad (60.12)$$

$$\widehat{r\vartheta} = \frac{\beta\mu(\lambda+\mu)}{\pi(\lambda+2\mu)}\left\{-\frac{r}{R_1^2+R_2^2}+\frac{1}{r}-\frac{R_1^2R_2^2}{R_1^2+R_2^2}\cdot\frac{1}{r^3}\right\}\sin\vartheta.$$

In the case of a thin plate (§ 26) λ must be replaced by λ^*.

Hitherto it has been assumed that no external loading is present. If arbitrary external stresses occur, these solutions have to be combined with those of § 59 for single-valued displacements.

§ 61. Supplement. *Bending of a curved beam by forces, applied to the ends, for arbitrary distribution of external stresses on the curved boundaries.*

Let it be assumed that one is dealing with a part of a ring, bounded by two radii. First, let the curved boundaries be free from external loading. The solutions, obtained in § 60, satisfy, of course, all equilibrium conditions and give zero external stresses on the circular boundaries. The displacements will be single-valued in the region considered here. However, the stresses, applied to the straight edges (ends) of the beam, will be different from zero and will depend on the three constants α, β, ε. Generally speaking, it is impossible to choose these constants, so that one obtains at the ends a given external stress distribution. But, as will now be shown, one may always arrange that the stresses, applied to one of the ends, will be statically equivalent to a given force and couple, i.e., that they have a known resultant vector and moment. The forces, applied to the other end, will then be statically equivalent to a force and couple, opposite to the former.

If the length of the beam is large compared with its width, the given resultant vector and moment of the forces, applied to an end, may be replaced by a fictitious distribution of forces, using Saint-Venant's Principle (§ 23). In the sequel, when speaking of the force and couple applied to an end of the beam, this will refer to application of external forces which are statically equivalent to those given in the problem. For example, it may be assumed that one of the ends is clamped; then, due

to the clamping, reactions will occur which statically balance the force and couple applied to the other end.

Let the part of the ring, to be considered here, correspond to values of ϑ in the interval $\vartheta_1 \leqslant \vartheta \leqslant \vartheta_2$. Consider first the solution 1° of § 60. The resultant vector of the forces applied to either of the ends will be zero. In fact, it is easily seen that, if $\widehat{\vartheta\vartheta}$ is determined from (60.10),

$$\int_{R_1}^{R_2} \widehat{\vartheta\vartheta}\, dr = 0.$$

The resultant moment of the forces, acting on the end $\vartheta = \vartheta_2$, is given by

$$M = 2h\int_{R_1}^{R_2} \widehat{\vartheta\vartheta}\, r\, dr = h \frac{(R_2^2 - R_1^2)^2 - 4R_1^2R_2^2\left(\log \dfrac{R_2}{R_1}\right)^2}{2(R_2^2 - R_1^2)} \frac{\varepsilon\mu(\lambda+\mu)}{\pi(\lambda+2\mu)}, \tag{61.1}$$

where $2h$ is the thickness of the beam in the direction perpendicular to the Oxy plane.

The solution of the problem of flexure of a curved circular beam by couples, applied to its ends, is thus obtained by substituting in the formulae (60.10)

$$\frac{\varepsilon\mu(\lambda+\mu)}{\pi(\lambda+2\mu)} = \frac{1}{h} \frac{2M(R_2^2 - R_1^2)}{(R_2^2 - R_1^2)^2 - 4R_1^2R_2^2\left(\log \dfrac{R_2}{R_1}\right)^2} \tag{61.2}$$

It is easily seen that the denominator on the right-hand side of (61.2) is always positive; in fact, let

$$(R_2^2 - R_1^2)^2 - 4R_1^2R_2^2\left(\log \frac{R_2}{R_1}\right)^2 = R_1^4 f(x),$$

where

$$x = \frac{R_2^2}{R_1^2} > 1, \qquad f(x) = (x-1)^2 - x(\log x)^2.$$

But

$$f(1) = f'(1) = f''(1) = f'''(1) = 0,$$

and

$$f^{\mathrm{IV}}(x) = \frac{2\log x}{x^2}.$$

Hence, $f'''(x) > 0$ for $x > 1$, whence follows that $f''(x) > 0$, $f'(x) > 0$, $f(x) > 0$ for $x > 1$, q.e.d.

Next consider the solution 3° of § 60 and assume that the direction of the coordinate axis is such that $\vartheta_2 = \pi/2$. On the end $\vartheta = \vartheta_2$ one has $\widehat{\vartheta\vartheta} = 0$, as is shown by (60.12). Thus the resultant vector of the external forces, applied to this end, passes through O, is parallel to the axis Oy and its magnitude (i.e., its projection on the Oy axis) is

$$P = 2h \int_{R_1}^{R_2} \widehat{r\vartheta}\, dr = 2h \frac{(R_1^2 + R_2^2)\log \frac{R_2}{R_1} - R_2^2 + R_1^2}{R_1^2 + R_2^2} \cdot \frac{\beta\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)}. \qquad (61.3)$$

Hence one can solve the problem of bending of a beam by a transverse force, applied to the end $\vartheta = \vartheta_2$, by substituting in (60.12)

$$\frac{\beta\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} = \frac{1}{2h} \frac{P(R_1^2 + R_2^2)}{(R_1^2 + R_2^2)\log \frac{R_2}{R_1} - R_2^2 + R_1^2} \qquad (61.4)$$

It is easily verified that the denominator on the right-hand side of (61.4) is always positive [cf. remarks following (61.2)].

The problem for the case of forces normal to the end of the beam may be solved in the same manner. The solution can either be found directly, as in the preceding case, or by adapting this solution. In fact, consider the part of the ring included between the radii $\vartheta = 0$ and $\vartheta = \pi/2$. The preceding solution gives on the end $\vartheta = \pi/2$ a system of forces, statically equivalent to a single force, parallel to Oy and passing through O. Consequently, the forces applied to the section $\vartheta = 0$ will be equal in magnitude and opposite in direction to that force, i.e., they will be equal to a force, normal to the straight end $\vartheta = 0$; the line of action of this force passes through O. By adding a suitable couple (using the solution already found for flexure by a couple), one can always obtain a force the line of action of which passes through an arbitrary point.

The preceding solutions of the problem of bending of curved beams by forces and couples applied to the ends (and likewise for other types of loading) were found by Kh. Golovin [1]; Golovin's paper remained unknown outside Russia and his solutions were rediscovered independently by several other authors.

Thus the complete solution has been found for the case when the curved sides of the beam are free from external stresses. Now suppose that these sides are likewise loaded in an arbitrary manner. The solution may then the obtained by the following method.

Imagine that the beam be extended into a complete ring and impose arbitrary loads on the curved boundaries of the added part, in such a way, however, that these loads, together with those given for the curved sides of the original beam, are statically equivalent to zero; then solve the problem for the complete ring by the method of § 59.

Such a solution will satisfy on that part of the ring, which corresponds to the original beam, the known conditions on the curved boundaries. There only remains to select the solutions of the present section in such a way that one obtains, by their superposition, at the straight ends forces which give the known forces and couples (where, of course, the latter must be such that they statically balance the forces given on the curved boundaries).

Note that by varying the loads on the curved boundaries of the complementary part of the ring, different solutions may be obtained. This does not contradict the uniqueness theorem, because only the resultant vector and moment, and not the stress distribution at the straight ends have been taken into consideration. All the different solutions, mentioned above, will correspond to different distributions of the external stresses at the ends (which, however, give the same resultant vectors and moments). All these solutions, by Saint-Venant's Principle, will differ little from each other in parts of the beam which are not too close to the ends, provided the width of the beam is small compared with its length.

§ 62. Thermal stresses in a hollow circular cylinder. Since the problem of dislocations in a circular ring has been solved (§ 60), the problem of deformation of a hollow cylinder, the transverse sections of which are circular rings and which is placed in a two-dimensional axially symmetrical flow of heat, can likewise be solved on the basis of the results of § 46. Consideration will be limited here to one simple application.

The notation of § 46 will be retained. In the present case the region S is bounded by two concentric circles with radii R_1, R_2 ($R_1 < R_2$) and the origin as centre.

Suppose that the hollow cylinder under consideration is heated by being placed in a heat flow and that $T = T_1$ for $r = R_1$ and $T = T_2$ for $r = R_2$, where T_1 and T_2 are constant and r is the distance of a point (x, y) from the origin. Then, as is easily verified (cf. Note at end of this section),

$$T = \frac{T_2 - T_1}{\log R_2 - \log R_1} \log r + \frac{T_1 \log R_2 - T_2 \log R_1}{\log R_2 - \log R_1} \tag{62.1}$$

Hence, denoting by $F(z)$ the same quantity as in § 46 and omitting the imaginary part of an arbitrary constant, one obtains

$$F(z) = \frac{T_2 - T_1}{\log R_2 - \log R_1} \log z + \frac{T_1 \log R_2 - T_2 \log R_1}{\log R_2 - \log R_1}. \tag{62.2}$$

Thus, in the present case, one has by (46.6), omitting again a constant,

$$u^* + iv^* = \frac{T_2 - T_1}{\log R_2 - \log R_1} z \log z + \\ + \frac{T_1(\log R_2 + 1) - T_2(\log R_1 + 1)}{\log R_2 - \log R_1} z, \tag{62.3}$$

and it is seen that, since there is only one internal contour L_1, (cf. § 46)

$$B_1 = \frac{T_2 - T_1}{\log R_2 - \log R_1}, \quad \alpha_1^* = \beta_1^* = 0. \tag{62.4}$$

This means that the solution of the "auxiliary" problem of § 46 is obtained by substituting in (60.9) and (60.10) [cf. (46.16)] the value

$$\varepsilon = -\frac{\pi\nu}{\lambda + \mu} \cdot \frac{T_2 - T_1}{\log R_2 - \log R_1}. \tag{62.5}$$

Since the stresses X_x, Y_y, X_y in the auxiliary problem are the same as in the original one, these stresses are obtained directly from (60.10) by substituting for ε the above value. In this way a well known formula has been obtained (cf. for example A. Föppl [1]). In order to calculate the displacements, one has to find the displacements u' and v' of the auxiliary problem. Then u and v will be given by (46.8) and (62.3).

NOTE. Several additional remarks will now be made with regard to the present problem, i.e., to the case when the cross-section is a circular ring.

If the temperature distribution T is not given, but only its values on the circles L_1 and L_2, it may be calculated in the following manner.

The problem of finding T is a particular case of the so-called first fundamental problem of the theory of the logarithmic potential (Dirichlet problem); it consists of the determination of a function (i.e., T), harmonic in a region, for given values on the boundary. It may be shown that this problem has always a unique solution (under very general conditions). The general solution of this problem for the case when the region is a circular ring is given below.

For the determination of $F(z)$ one has

$$2T = F(z) + F(z) \tag{62.6}$$

and, by (46.12), putting $z_1 = 0$,

$$F(z) = A \log z + \sum^{+\infty} a_k z^k, \tag{62.7}$$

where A is a real constant.

The function $F(z)$ must be determined for the boundary conditions

$$F(z) + \overline{F(z)} = 2f_1(\vartheta) \text{ for } r = R_1,$$

$$F(z) + F(z) = 2f_2(\vartheta) \text{ for } r = R_2,$$

where $f_1(\vartheta)$, $f_2(\vartheta)$ are the known values of T on L_1 and L_2. Let these functions be represented by their complex Fourier series

$$f_1(\vartheta) = \sum^{+\infty} A_k' e^{ik\vartheta}, \quad f_2(\vartheta) = \sum^{+\infty} A_k'' e^{ik\vartheta}, \tag{62.8}$$

where by § 52, since the functions f_1 and f_2 are real,

$$A_k' = \bar{A}'_{-k}, \quad A_k'' = \bar{A}''_{-k},$$

and, in particular,

$$A_0' = \bar{A}_0', \quad A_0'' = \bar{A}_0'',$$

i.e., A_0' and A_0'' are real.

The boundary conditions may now be written

$$2A \log r + \sum^{+\infty} a_k r^k e^{ik\vartheta} + \sum^{+\infty} \bar{a}_k r^k e^{-ik\vartheta} = \begin{cases} 2 \sum\limits_{-\infty} A_k' e^{ik\vartheta} & \text{for } r = R_1, \\ 2 \sum\limits_{-\infty}^{+\infty} A_k'' e^{ik\vartheta} & \text{for } r = R_2. \end{cases}$$

Hence

$$2A \log R_1 + a_0 + \bar{a}_0 = 2A_0', \qquad 2A \log R_2 + a_0 + \bar{a}_0 = 2A_0'', \tag{62.9}$$

$$a_k R_1^k + \bar{a}_{-k} R_1^{-k} = 2A_k', \qquad a_k R_2^k + \bar{a}_{-k} R_2^{-k} = 2A_k'' \quad (k \neq 0); \tag{62.10}$$

the equations (62.9) determine A and $a_0 + \bar{a}_0$, while each pair of equations (62.10) gives a_k and $\bar{a}_{-k}$, where it is sufficient to give k only positive values, in order to find all the coefficients. The imaginary part of a_0 remains indeterminate, as was to be expected, and may be fixed arbitrarily.

For example, if $T = T_1$ for $r = R_1$ and $T = T_2$ for $r = R_2$, where T_1 and T_2 are constants, one has

$$A_0' = T_1, \quad A_0'' = T_2, \quad A_k' = A_k'' = 0 \quad (k \neq 0),$$

and $F(z)$ is given by (62.2).

Note the important fact that multi-valued terms in the function $\int F(z)dz$ can only originate from the term $A \log z$ and the term $a_{-1}z^{-1}$ in the expansion (62.7). However, as shown by (62.9) and (62.10), the constants A and a_{-1} are determined solely by A_0', A_0'', A_1' and A_1''. Consequently, the characteristics of the dislocations for the auxiliary problem, and therefore also the stresses X_x, Y_y, X_y in the original problem, depend solely on the quantities

$$A_0' = \frac{1}{2\pi}\int_0^{2\pi} f_1(\vartheta)d\vartheta, \quad A_0'' = \frac{1}{2\pi}\int_0^{2\pi} f_2(\vartheta)d\vartheta,$$

$$A_1' = \frac{1}{2\pi}\int_0^{2\pi} f_1(\vartheta)e^{-i\vartheta}d\vartheta, \quad A_1'' = \frac{1}{2\pi}\int_0^{2\pi} f_2(\vartheta)e^{-i\vartheta}d\vartheta,$$

i.e., on

$$\int_0^{2\pi} f_1(\vartheta)d\vartheta, \quad \int_0^{2\pi} f_2(\vartheta)d\vartheta, \quad \int_0^{2\pi} f_1(\vartheta)\cos\vartheta\, d\vartheta, \quad \int_0^{2\pi} f_2(\vartheta)\cos\vartheta\, d\vartheta,$$

$$\int_0^{2\pi} f_1(\vartheta)\sin\vartheta\, d\vartheta, \quad \int_0^{2\pi} f_2(\vartheta)\sin\vartheta\, d\vartheta.$$

APPLICATION OF CONFORMAL MAPPING

It has been seen in the earlier chapters of this Part that the use of power series for the unknown functions leads to effective results in the case of regions bounded by one or two concentric circles. By mapping given simply or doubly connected regions on a circle or circular ring, such expansions of the unknown functions will likewise secure effective solutions. The present chapter deals briefly with this problem, while a more satisfactory application of conformal mapping by other means will be described in Parts V and VI.

§ 63. Case of simply connected regions. Consider first the case of a finite region S bounded by a simple contour L which may be mapped on the circle $|\zeta| < 1$ by the function $z = \omega(\zeta)$; denote by γ the circumference $|\zeta| = 1$ of that circle.

Since, in the notation of § 50, the functions $\varphi_1(z)$ and $\psi_1(z)$ are holomorphic in S, the functions $\varphi(\zeta)$ and $\psi(\zeta)$ will be holomorphic in γ. Hence one will have inside γ the expansions

$$\varphi(\zeta) = \sum_{0} a_k \zeta^k, \quad \psi(\zeta) = \sum_{0}^{+\infty} a_k' \zeta^k, \quad \varphi'(\zeta) = \sum_{0}^{+\infty} k a_k \zeta^{k-1} \tag{63.1}$$

One can thus try to solve the fundamental boundary problems by substituting these series (assuming them to converge on γ, i.e., for $\zeta = \sigma = e^{i\vartheta}$) in (51.1) or (41.5) which give certain systems of equations for the determination of the coefficients a_k and a_k'.

This procedure will be explained for the case of the first fundamental problem. The boundary condition (51.1) will now be written

$$\sigma) + \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = f_1 + if_2, \tag{63.2}$$

omitting an arbitrary constant on the right-hand side. It can always be assumed that the point $z = 0$ corresponds to the point $\zeta = 0$, i.e., that $\omega(0) = 0$. It is also known that $\varphi_1(0)$ and the imaginary part of

$\varphi_1'(0)$, or, in terms of $\varphi(\zeta)$, that $\varphi(0) = a_0$ and the imaginary part of $\dfrac{\varphi'(0)}{\omega'(0)}$, i.e., $\Im[a_1/\omega'(0)]$, may be fixed arbitrarily. Hence, in the following, a_0 will be put equal to zero, while the imaginary part of $a_1/\omega'(0)$ will be left indeterminate for the time being. Further, suppose that $\omega(\sigma)/\overline{\omega'(\sigma)}$ may be expanded (for $\sigma = e^{i\vartheta}$) in a series of the form

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \sum_{-\infty}^{+\infty} b_k e^{ik\vartheta} = \sum_{-\infty}^{+\infty} b_k \sigma^k \tag{63.3}$$

which will be assumed to be absolutely convergent. It is easily shown that this condition will be met, if the contour L satisfies the conditions of § 47.

In fact, this follows from the well known theorem of S. N. Bernstein [1] which states that, if a function $f(\vartheta)$ satisfies the Hölder condition for the index $\alpha > \frac{1}{2}$ (cf. § 65 for a definition of this term), the series of Fourier coefficients of this function is absolutely convergent. In the present case, $\omega(\sigma)/\overline{\omega'(\sigma)}$ has a continuous first order derivative and hence satisfies the Hölder condition for $\alpha = 1$.

Developing $f_1 + if_2$ in a complex Fourier series (assuming this to be possible)

$$f_1 + if_2 = \sum_{-\infty}^{+\infty} A_k e^{ik\vartheta} = \sum_{-\infty}^{+\infty} A_k \sigma^k \tag{63.4}$$

and substituting (63.1), (63.3) and (63.4) in (63.2), one obtains

$$\sum_{k=1}^{\infty} a_k \sigma^k + \sum_{l=-\infty}^{+\infty} b_l \sigma^l \sum_{k=1}^{\infty} k\bar{a}_k \sigma^{-k+1} + \sum_{k=1}^{\infty} \bar{a}_k' \sigma^{-k} = \sum_{-\infty}^{+\infty} A_k \sigma^k. \tag{63.5}$$

Multiplying out the series of the middle term on the left-hand side — the operation being known to be permissible, if it is assumed, for example, that the series for $\varphi'(\sigma)$ as well as the series (63.3) converge absolutely — and comparing coefficients of σ^m $(m = 1, 2, \ldots)$, one finds

$$a_m + \sum_{k=1}^{\infty} k\bar{a}_k b_{m+k-1} = A_m \quad (m = 1, 2, \ldots); \tag{63.6}$$

similarly, one obtains from σ^{-m} $(m = 0, 1, 2, \ldots)$

$$\bar{a}_m' + \sum_{k=1}^{\infty} k\bar{a}_k b_{-m+k-1} = A_{-m} \quad (m = 0, 1, 2, \ldots). \tag{63.7}$$

The equations (63.6) form an infinite system of equations for the infinitely many unknowns a_k. Each of these equations provides two

real equations for the quantities α_k, β_k, where

$$\alpha_k + i\beta_k = a_k, \quad \alpha_k - i\beta_k = \bar{a}_k.$$

If one succeeds in solving this system by one or the other method, the function $\varphi(\zeta)$ will be determined. The coefficients a'_m in the expansion for $\psi(\zeta)$ can then be found from (63.7). Thus, the basic problem consists of the solution of the system (63.6), i.e., of the determination of the function $\varphi(\zeta)$. Further, if the series for $\varphi(\zeta)$, $\psi(\zeta)$, $\varphi'(\zeta)$, determined in this manner, are found to be uniformly convergent for $|\zeta| = 1$ and if the series for $\varphi'(\zeta)$ is, in addition, absolutely convergent, one can be sure that the conditions of the problem will be satisfied.

Obviously, uniform convergence for $|\zeta| = 1$ entails uniform convergence for $|\zeta| \leq 1$ and, hence, continuity of φ, φ' and ψ up to the contour, i.e., the regularity of the solutions (§ 42).

Note that, having found $\varphi(\zeta)$, the function $\psi(\zeta)$ may be determined directly without recourse to (63.7). In fact, if $\varphi(\zeta)$ is known, the boundary value of $\psi(\zeta)$ on $|\zeta| = 1$ is given by

$$\psi(\sigma) = f_1 - if_2 - \overline{\varphi(\sigma)} - \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\varphi'(\sigma)$$

[cf. (63.2)]. Hence, the function $\psi(\zeta)$ may be calculated directly by means of Cauchy's formula.

In many cases, the actual solution of the system of equations (63.6) will present no difficulties. An analogous system, obtained for one particular case by D. M. Volkov and A. A. Nazarov [1, 2], was solved by the method of successive approximation. Still earlier, P. Sokolov [1] gave the solutions of a number of particular problems, which are of practical importance, by an analogous method. The present treament will be restricted to the following remarks of a general character with respect to the system (63.6), and a start will be made with the simple case when $\omega(\zeta)$ is a polynomial

$$\omega(\zeta) = c_1\zeta + c_2\zeta^2 + \ldots + c_n\zeta^n \quad (c_1 \neq 0,\ c_n \neq 0). \tag{63.8}$$

In this work use will be made of the following notation which is a particular case of a somewhat more general notation explained in § 76 and which will be widely used. If

$$f(\zeta) = a_0 + a_1\zeta + \ldots + a_n\zeta^n$$

is some polynomial, then $\bar{f}(\zeta)$, where the bar extends only over f, will be understood to be the polynomial obtained from $f(\zeta)$ by replacing the

coefficients by their conjugate complex values, so that, by definition,

$$\bar{f}(\zeta) = \bar{a}_0 + \bar{a}_1\zeta + \ldots + \bar{a}_n\zeta^n.$$

In this notation one will have (remembering that $\sigma = e^{i\vartheta}$, $\bar{\sigma} = e^{-i\vartheta} = \sigma^{-1}$)

$$\overline{f(\sigma)} = \bar{a}_0 + \bar{a}_1\bar{\sigma} + \ldots + \bar{a}_n\bar{\sigma}^n = \bar{a}_0 + \bar{a}_1\sigma^{-1} + \ldots + \bar{a}_n\sigma^{-n} = \bar{f}\left(\frac{1}{\sigma}\right).$$

Applying this notation to the expression $\omega(\sigma)/\overline{\omega'(\sigma)}$, figuring in (63.2), one finds

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \frac{\omega(\sigma)}{\bar{\omega}'\left(\frac{1}{\sigma}\right)} = \frac{c_1\sigma + c_2\sigma^2 + \ldots + c_n\sigma^n}{\bar{c}_1 + 2\bar{c}_2\sigma^{-1} + \ldots + n\bar{c}_n\sigma^{-n}} =$$

$$= \sigma^n \cdot \frac{c_1 + c_2\sigma + \ldots + c_n\sigma^{n-1}}{\bar{c}_1\sigma^{n-1} + 2\bar{c}_2\sigma^{n-2} + \ldots + n\bar{c}_n}.$$

The right-hand side, considered as a function of the complex variable σ in the entire plane (and it is, obviously, a rational function), has no poles outside the circle γ, including its boundary, with the exception of the point $\sigma = \infty$, because $\omega'(\zeta)$ does not vanish inside and on γ (§ 47) and hence $\bar{\omega}'(1/\zeta)$ does not vanish on or outside γ. In fact, suppose that at some point ζ_0, for $|\zeta_0| \geqslant 1$, one has $\bar{\omega}'(1/\zeta_0) = 0$; taking the conjugate complex value, one has $\omega'(1/\bar{\zeta}_0) = 0$ or $\omega'(\zeta_1) = 0$, where $\zeta_1 = 1/\bar{\zeta}_0$ and $|\zeta_1| \leqslant 1$, which is impossible. Thus, one has an expansion of the form

$$\frac{\omega(\sigma)}{\bar{\omega}'\left(\frac{1}{\sigma}\right)} = b_n\sigma^n + b_{n-1}\sigma^{n-1} + \ldots + b_1\sigma + b_0 + \sum_{k=1}^{\infty} b_{-k}\sigma^{-k},$$

which holds for $|\sigma| \geqslant 1$ and, in particular, for $\sigma = e^{i\vartheta}$. Hence in the present case the series (63.3) contains only a finite number of terms with positive powers of σ, and, in fact, one finds

$$b_k = 0 \quad (k \geqslant n + 1). \tag{63.9}$$

The equations (63.6) now reduce to the following:

$$a_m = A_m \quad (m \geqslant n + 1) \tag{63.6'}$$

and

$$\begin{aligned} a_1 + \bar{a}_1b_1 + 2\bar{a}_2b_2 + \ldots + n\bar{a}_nb_n &= A_1, \\ a_2 + \bar{a}_1b_2 + 2\bar{a}_2b_3 + \ldots + (n-1)\bar{a}_{n-1}b_n &= A_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_n + \bar{a}_1b_n &= A_n; \end{aligned} \tag{63.6''}$$

one also finds from (63.7)

$$+\sum_{k=1}^{m+n+1} k\bar{a}_k b_{-m+k-1} = A_{-n} \quad (m = 0, 1, \ldots). \tag{63.7'}$$

Thus, one has for the determination of the coefficients $a_1, \ldots, a_n$ the n equations (63.6″) which correspond to $2n$ real equations for the determination of the $2n$ real coefficients α_k, β_k $(k = 1, \ldots, n)$, where $\alpha_k + i\beta_k = a_k$. If the equations (63.6″) have a solution, the remaining coefficients are determined by (63.6′) and (63.7′) and it is easily proved directly that the series for $\varphi(\zeta)$ and $\psi(\zeta)$, obtained in this manner, will satisfy the conditions of the problem, provided the given functions f_1 and f_2 are sufficiently regular, i.e., for example, they have second order derivatives with respect to ϑ, satisfying the Dirichlet condition.

In fact, if this last condition is satisfied, one will have inequalities of the form

$$|A_m| < \frac{C}{|m|^3} \quad (m = \pm 1, \pm 2, \ldots), \tag{a}$$

whence it follows by (63.6′) that the series for $\varphi(\zeta)$ and $\varphi'(\zeta)$ will be absolutely and uniformly convergent for $|\zeta| \leq 1$. Further, (63.7′) shows that $a'_m = -c_m + A_{-m}$, where

$$c_m = \sum_{k=1}^{m+n+1} k\bar{a}_k b_{-m+k-1}.$$

But the series ΣA_{-m} converges absolutely by (a); the series Σc_m is likewise absolutely convergent, because its terms are found in a number of terms of an absolutely convergent series, obtained by multiplying out the absolutely convergent series $\Sigma k\bar{a}_k$ and Σb_k. It follows directly from this that the series for $\psi(\zeta)$ is absolutely and uniformly convergent for $|\zeta| \leq 1$.

Thus, the solution of the problem will be obtained, provided the system (63.6″) can be solved. However, it is clear that the system (63.6″) cannot give definite values for all the coefficients $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$; in fact, it is known beforehand that the imaginary part of $a_1/\omega'(0) = (\alpha_1 + i\beta_1)/\omega'(0)$ remains always arbitrary. This means that the determinant of the system (63.6″) must vanish, and from this it is known to follow that, for the existence of a solution, the quantities $A_1, \ldots, A_n$ must satisfy a certain additional condition which will be deduced by excluding unknowns from (63.6″). This condition will obviously express the demand that the resultant moment of the external forces must vanish (and this condition had been allowed for by assuming $f_1 + if_2$ to be continuous on the contour), because the present problem has a solution for this

(and only this) sufficient condition (cf. the existence proof in Part V.). It follows from the theorem of uniqueness of solutions that all coefficients $a_1, \ldots, a_n$ are completely determined, with the exception of the first only the real part of which [or, better, the real part of $a_1/\omega'(0)$] will be fixed, while the imaginary part may be given an arbitrary value.

As an example, consider the case when L is Pascal's Limaçon. By § 48, 2° (writing a instead of m)

$$z = \omega(\zeta) = R(\zeta + a\zeta^2), \quad R > 0, \quad 0 \leqslant a < \tfrac{1}{2}.$$

One has

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \frac{\sigma + a\sigma^2}{1 + 2a\bar{\sigma}} = a\sigma^2 + (1 - 2a^2)\sigma - \frac{2a(1-2a^2)}{1 + \dfrac{2a}{\sigma}}$$

$$= a\sigma^2 + (1 - 2a^2)\sigma - 2a(1-2a^2)\sum_{k=0}^{\infty}(-1)^k\left(\frac{2a}{\sigma}\right)^k,$$

and hence in the present case $n = 2$,

$$b_2 = a, \quad b_1 = 1 - 2a^2, \quad b_{-k} = (-1)^{k+1}(2a)^{k+1}(1-2a^2),$$
$$(k = 0, 1, 2, \ldots).$$

The system (63.6″) will now reduce to the following:

$$a_1 + \bar{a}_1(1 - 2a^2) + 2\bar{a}_2 a = A_1, \quad a_2 + \bar{a}_1 a = A_2.$$

Substituting $\bar{a}_2 = -aa_1 + \bar{A}_2$, obtained from the second of these equations, into the first equation, one finds

$$a_1 + \bar{a}_1 = \frac{A_1 - 2a\bar{A}_2}{1 - 2a^2}$$

which determines the real part of a_1. Hence, in order that the problem have a solution, one requires that the imaginary part of $A_1 - 2a\bar{A}_2$ be zero. It is easily verified directly that this is the condition for the vanishing of the resultant moment of the external forces.

Putting for definiteness $\mathfrak{J}(a_1) = 0$, one finds

$$a_1 = \frac{A_1 - 2a\bar{A}_2}{2(1-2a^2)}, \quad a_2 = A_2 - \bar{a}_1 a = A_2 - a_1 a,$$

$$A_m (m \geqslant 3), \quad \bar{a}'_m = -\sum_{k=1}^{m+3} k\bar{a}_k b_{-m+k-1} + A_{-m}, \qquad (m > 0)$$

and the problem is solved. The series obtained for $\varphi(\zeta)$ and $\psi(\zeta)$ can be

summed and expressed by means of Cauchy type integrals, but this will not be done here, since the corresponding formulae are more easily obtained by another method (cf. § 84).

Now consider the general case when $\omega(\zeta)$ is not a polynomial. By omitting from the expansion

$$\omega(\zeta) = c_1\zeta + c_2\zeta^2 + \qquad + c^n\zeta^n + c_{n+1}\zeta^{n+1} +$$

all terms, beginning with $c_{n+1}\zeta^{n+1}$,one obtains instead of $\omega(\zeta)$ a polynomial $\omega_n(\zeta)$ which does not map the region S, but a region S_n on to the unit circle, where S_n represents an approximation to S which improves with increasing n. The solution of the problem for the region S_n has been seen to present essentially no difficulties. The simplification brought about by replacement of $\omega(\zeta)$ by the polynomial $\omega_n(\zeta)$ is equivalent to omitting in (63.6) and (63.7) all terms involving b_k for $k \geqslant n + 1$.

It has been seen that in this case the problem reduces to the solution of a finite number of linear equations with a finite number of unknown coefficients, namely, to the solution of (63.6″) and to the calculation of the remaining coefficients by means of (63.6′) and (63.7′). This is one of the methods of approximate solution of the infinite systems (63.6) and (63.7), i.e., of approximate solution of the original problem. Letting now n increase beyond all bounds, the regions S_n will tend to S and the approximate solution will tend to the exact one, i.e., the functions φ and ψ, found for the regions S_n, will tend to definite functions giving the exact solution for the region S. Under known general assumptions with respect to the contour of the region S and to the functions f_1 and f_2 given on the contour, this statement may be proved rigorously (cf. § 89 for a more detailed discussion).

Quite similar remarks refer to the method of solution of the second fundamental problem. This problem is even simpler, since, when the displacements on the contour are known, the coefficient a_1 is completely determined and the boundary problem will not be subject to any additional conditions. Thus, the system analogous to (63.6″) will always have a unique solution.

In the case of infinite regions, mapped on to the circle $|\zeta| < 1$ by a function of the form

$$\omega(\zeta) = \frac{c}{\zeta} + c_1\zeta + c_2\zeta^2 + \ldots + c_n\zeta^n, \tag{63.8'}$$

results are obtained which are as simple as for finite regions. No further detail will be given on this, since in the above (and likewise in more

general) cases effective solutions may be deduced by the method of Part IV.

The method of solution of the fundamental problems, studied in this section, is given in more detail and with certain additional interesting extensions in Chapter VI of the book by L. V. Kantorovich and V. I. Krylov [1]; the presentation given here has been reproduced without essential changes from the earlier editions of the present book.

§ 64. Example of application of mapping on to a circular ring. Solution of the fundamental problems for a continuous ellipse.

It is natural to try to generalize the method of § 63 to the case of doubly-connected regions by using transformations on to the circular ring. However, even for regions of a very simple form, direct application of this method does not lead to simple results. Mapping on to the circular ring will be used here to solve the fundamental problems for *the continuous ellipse*.

The first fundamental problem for a region, bounded by two eccentric circles, has been solved by G. B. Jeffery [1], using a method which is essentially close to that of § 63. It is of interest to deduce by the same method the solution for regions, bounded by two confocal ellipses. The solution, given for this case by A. Timpe [2], has been found to be incorrect, as will be shown below. In fact, Timpe tried to obtain the solution of the problem by expanding the corresponding Airy function as a series in terms of a certain system of particular solutions of the biharmonic equation. But it is not difficult to verify that his system of particular solutions is incomplete. The complete system is easily constructed, starting from the complex representation of the biharmonic functions and utilizing conformal transformation on the circular ring.

The fundamental problems for the continuous ellipse were solved by O. Tedone [1] and T. Boggio [3] by other, more complex, means. The solution, given in this section, was first presented in the Author's paper [16] and it was also contained in the earlier editions of this book. Recently D. I. Sherman [18] gave a solution, based on the integral equations of G. Lauricella. The final formulae of Sherman agree with the Author's earlier formulae, if the latter are somewhat transformed. In fact, the Author originally gave only the equations (64.21) and (64.21′) for the calculation of the coefficients c_k, appearing in (64.23). However, by taking into consideration the coefficients of the expansion (64.24), formula (64.27) follows directly from those mentioned earlier. By substituting the expression for c_k on the right-hand side of (64.19), one obtains a formula which agrees with that of D. I. Sherman.

It is a fact that the finite region, bounded by an ellipse, may, like every region bounded by a single contour, be mapped on to a circle. But the corresponding transforming function is complicated and in-

convenient. That is the reason why another transformation will be used here.

Imagine that the ellipse has been cut along the segment connecting its foci. This cut may likewise be conceived as an ellipse which is confocal with the original one and whose minor axis is zero. Thus, one has the limiting case of regions, lying between two confocal ellipses. This region may be mapped on the ring between two circles γ_1 and γ_2 in the ζ plane by putting (§ 48, 5°)

$$z = \omega(\zeta) \quad R\left(\zeta + \frac{1}{\zeta}\right), \quad R > 0. \tag{64.1}$$

Circles of radius ρ in the ζ plane will correspond to ellipses in the z plane, where the parametric representation of the latter is

$$x = R\left(\rho + \frac{1}{\rho}\right)\cos\vartheta, \quad y = R\left(\rho - \frac{1}{\rho}\right)\sin\vartheta. \tag{64.2}$$

The circle $\rho = 1$ of the ζ plane will correspond to the segment AB of the Ox axis (of the z plane), enclosed between the points

$$x = -2R \qquad \text{and} \qquad x = +2R.$$

When the point ζ describes the circle $\rho = 1$, the corresponding point z moves along the segment AB in accordance with the law

$$= x = 2R\cos\vartheta = R\left(\sigma + \frac{1}{\sigma}\right) \quad (\sigma = e^{i\vartheta}), \tag{64.3}$$

so that the points $\sigma = e^{i\vartheta}$ and $\bar{\sigma} = e^{-i\vartheta}$ of the ζ plane correspond to one and the same point of the segment AB.

Thus, one can take for γ_1 the unit circle; the radius of γ_2 will be denoted by ρ_0 ($\rho_0 > 1$). The magnitude of ρ_0 will be determined by the given eccentricity $2R$ and the major semi-axis of the ellipse $a = R(\rho_0 + 1/\rho_0)$, whence it follows that

$$\rho_0 = \frac{a + \sqrt{a^2 - 4R^2}}{2R} \tag{64.4}$$

(since $-\sqrt{a^2 - 4R^2}$ would give $\rho_0 < 1$).

In the notation of § 50, the functions $\varphi_1(z)$ and $\psi_1(z)$ must be holomorphic inside the uncut ellipse. Furthermore, they must be holomorphic in the ellipse, cut along AB. Hence $\varphi(\zeta)$ and $\psi(\zeta)$ must be holomorphic

in the ring between γ_1 and γ_2 and they will have expansions of the form

$$\varphi(\zeta) = \sum_{-\infty}^{+\infty} a_k \zeta^k, \quad \psi(\zeta) = \sum_{-\infty}^{+\infty} a_k' \zeta^k, \tag{64.5}$$

which are convergent for $1 < |\zeta| < \rho_0$.

Actually, the series (64.5) will even converge for $1/\rho_0 < |\zeta| < \rho_0$, since it can easily be shown that $\varphi(\zeta)$ and $\psi(\zeta)$ may be analytically continued into the region between γ_1 and the circle of radius $\rho' = 1/\rho_0$. For this purpose it is sufficient to consider the Riemann surface of two sheets, superimposed on the z plane, with branch points at A and B. The relation (64.1) gives the transformation of this surface on the ring $1/\rho_0 < |\zeta| < \rho_0$, and hence the above statements are justified.

For $\rho = \rho_0$, the functions (64.5) must satisfy the boundary condition

$$\overline{\varphi(\zeta)} + \frac{\overline{\omega(\zeta)}}{\omega'(\zeta)} \varphi'(\zeta) + \psi(\zeta) = f_1 - if_2, \tag{64.6}$$

where $f_1 - if_2$ is a known function of ϑ [cf. (63.2); the conjugate complex expression has been given here solely for the sake of convenience].

Further, one must have on γ_1

$$\varphi(\sigma) = \varphi(\bar{\sigma}), \quad \psi(\sigma) = \psi(\bar{\sigma}), \tag{64.7}$$

because the points σ and $\bar{\sigma}$ correspond to one and the same point of the segment AB in the z plane. Conversely, if this condition is satisfied, the functions $\varphi_1(z)$ and $\psi_1(z)$ will take one and the same value, when the point z approaches the segment AB from either side, and hence they will be analytic functions in the uncut ellipse.

It follows from (64.5) and (64.7) that

$$a_k = a_{-k}, \quad a_k' = a_{-k}'. \tag{64.8}$$

Introducing the series (64.5) into (64.6), noting that for $\rho = \rho_0$

$$\omega'(\zeta) = R\left(1 - \frac{1}{\zeta^2}\right) = R\left(1 - \frac{\bar{\zeta}^2}{\rho_0^4}\right),$$

$$\overline{\omega(\zeta)} = R\left(\bar{\zeta} + \frac{1}{\bar{\zeta}}\right) = R\left(\frac{\rho_0^2}{\zeta} + \frac{\zeta}{\rho_0^2}\right),$$

and multiplying both sides of (64.6) by $1 - \dfrac{1}{\zeta^2}$, one obtains

$$\left(1 - \frac{\bar{\zeta}^2}{\rho_0^4}\right) \sum_{-\infty}^{+\infty} \bar{a}_k \bar{\zeta}^k + \left(\frac{\rho_0^2}{\zeta} + \frac{\zeta}{\rho_0^2}\right) \sum_{-\infty}^{+\infty} k a_k \zeta^{k-1} +$$

$$+ \sum_{-\infty}^{+\infty} b_k \zeta^k = (f_1 - if_2)\left(1 - \frac{1}{\zeta^2}\right) \text{ for } \rho = \rho_0, \tag{64.9}$$

where use has been made of the notation

$$\left(1-\frac{1}{\zeta^2}\right)\psi(\zeta)=\left(1-\frac{1}{\zeta^2}\right)\sum_{-\infty}^{+\infty}a'_k\zeta^k=\sum_{-\infty}^{+\infty}(a'_k-a'_{k+2})\zeta^k=\sum_{-\infty}^{+\infty}b_k\zeta^k, \tag{64.10}$$

so that

$$b_k = a'_k \qquad {}_{k+2}. \tag{64.11}$$

Expanding the right-hand side of (64.9) in the complex Fourier series

$$(f_1-if_2)\,(1-\rho_0^{-2}e^{-2i\vartheta})=\sum_{-\infty}^{+\infty}A_ke^{ik\vartheta}, \tag{64.12}$$

putting $\zeta=\rho_0e^{i\vartheta}$ and comparing coefficients of $e^{ik\vartheta}$, one finds

$$\rho_0^{-k}\bar{a}_{-k}-\rho_0^{-k-4}\bar{a}_{-k-2}+(k+2)\rho_0^{k+2}a_{k+2}+k\rho_0^{k-2}a_k+b_k\rho_0^k=A_k,$$

or, noting that by (64.8) $\bar{a}_{-k}=\bar{a}_k$, $\bar{a}_{-k-2}=\bar{a}_{k+2}$,

$$(k+2)\rho_0^{k+2}a_{k+2}-\rho_0^{-k-4}\bar{a}_{k+2}+k\rho_0^{k-2}a_k+\rho_0^{-k}\bar{a}_k+b_k\rho_0^k=A_k. \tag{64.13}$$

Replacing k by $-k-2$ and noting that by (64.8) and (64.11)

$$b_{-k-2}=a'_{-k-2}-a'_{-k}=a'_{k+2}-a'_k=-b_k, \tag{64.8'}$$

one finds

$$-(k+2)\rho_0^{-k-4}a_{k+2}+\rho_0^{k+2}\bar{a}_{k+2}-k\rho_0^{-k}a_k-\rho_0^{k-2}\bar{a}_k-b_k\rho_0^{-k-2}=A_{-k-2}. \tag{64.13'}$$

Elimination of b_k from (64.13) and (64.13') finally leads to

$$\begin{aligned}(k+2)\,(\rho_0^2-\rho_0^{-2})a_{k+2}+(\rho_0^{2k+4}-\rho_0^{-2k-4})\bar{a}_{k+2}-{}\\-k(\rho_0^2-\rho_0^{-2})a_k-(\rho_0^{2k}-\rho_0^{-2k})\bar{a}_k=B_k,\end{aligned} \tag{64.14}$$

where

$$B_k=A_k\rho_0^{-k}+A_{-k-2}\rho_0^{k+2} \tag{64.15}$$

The coefficients a_k can be determined from the recurrence formula (64.14), provided a_0 and a_1 are known. [In actual fact, (64.14) gives for each k two equations obtained by separating real and imaginary parts; instead of this, one may deduce a second equation by going to conjugate complex quantities (see later).] The coefficient a_0 may be fixed arbitrarily, since one can always add an arbitrary constant to $\varphi(\zeta)$. The formulae (64.14) show that, as was to be expected, the coefficient a_2 (and hence also a_4, a_6 ...) does not depend on a_0; in fact, for $k=0$, the terms involving a_0 cancel out. In order to calculate $a_1=a_{-1}$, put $k=-1$ in (64.14) which gives

$$a_1+\bar{a}_1=\frac{B_{-1}}{2(\rho_0^2-\rho_0^{-2})}=\frac{A_{-1}\rho_0}{\rho_0^2-\rho_0^{-2}}. \tag{64.16}$$

This relation permits calculation of the real part of a_1 and at the same time shows that one must have

$$A_{-1} = \text{a real quantity}, \tag{64.17}$$

so that the problem can be solved. It is easily verified directly that (64.17) expresses the condition for the vanishing of the resultant moment of the external forces. (The vanishing of the resultant force vector has already been taken care of by assuming f_1 and f_2 to be continuous on the boundary.)

The imaginary part of a_1 remains arbitrary, as was to be expected, since one can always add to $\varphi_1(z)$ a term of the form Ciz, where C is an arbitrary real constant; hence one can add to $\varphi(\zeta)$ a term of the form $Ciz = CiR(\zeta + 1/\zeta)$. It is easily seen that this imaginary part does not affect a_3, a_5, etc. Thus, by giving arbitrary values to a_0 and the imaginary part of a_1 and by determining successively all remaining coefficients by means of (64.14), one obtains an expression for $\varphi(\zeta)$.

After this the coefficients b_k can be found from one of the formulae (64.13) or (64.13′). In this way one finds an expression for

$$\left(1 - \frac{1}{\zeta^2}\right) \psi(\zeta) = \sum_{-\infty}^{+\infty} b_k \zeta^k = \sum_{k=0}^{\infty} b_k \zeta_k + \sum_{k=1} b_{-k} \zeta^{-k}$$

or, remembering that $b_{-k} = -b_{k-2}$ (i.e., in particular, $b_{-1} = -b_{-1} = 0$),

$$\left(1 - \frac{1}{\zeta^2}\right) \psi(\zeta) = \sum_{k=0}^{\infty} b_k \left(\zeta^k - \frac{1}{\zeta^{k+2}}\right). \tag{64.18}$$

It will be shown below that for definite conditions the series for $\psi(\zeta)$ and $\varphi(\zeta)$ converge in the relevant region. The right-hand side of (64.18) vanishes for $\zeta = \pm 1$ and, consequently, the function $\psi(\zeta)$, obtained by dividing the right-hand side by $1 - 1/\zeta^2$, will not be singular for $\zeta = \pm 1$ (see remarks following (63.7) with regard to direct determination of $\psi(\zeta)$ by use of $\varphi(\zeta)$ and the boundary condition.)

Thus the problem is solved. The second fundamental problem can be solved in a similar manner.

Before turning to the question of the convergence of the above series, it may be noted that calculation of the coefficients a_k $(k = 2, 3, \ldots)$ can be simplified as follows. For convenience put

$$k(\rho_0^2 - \rho_0^{-2}) a_k + (\rho_0^{2k} - \rho_0^{-2k}) \bar{a}_k = c_k, \tag{64.19}$$

in which case (64.14) becomes

$$c_{k+2} - c_k = B_k. \tag{64.20}$$

Substituting in this formula successively $k = 0, 2, \ldots 2n - 2$, adding the results thus obtained and noting that $c_0 = 0$, one obtains

$$c_{2n} = \sum_{k=0}^{n-1} B_{2k} = \sum_{k=0}^{n-1} (A_{2k}\rho_0^{-2k} + A_{-2k-2}\rho_0^{2k+2}). \tag{64.21}$$

Similarly, substituting in (64.20) successively $k = 1, 3, \ldots, 2n - 1$ and adding, one finds

$$c_{2n+1} = c_1 + \sum_{k=1} B_{2k-1} = c_1 + \sum_{k=1} (A_{2k-1}\rho_0^{-2k+1} + A_{-2k-1}\rho_0^{2k+1}), \tag{64.21'}$$

where, by (64.19) and (64.16),

$$c_1 = (\rho_0^2 - \rho_0^{-2})(a_1 + \bar{a}_1) = A_{-1}\rho_0. \tag{64.22}$$

Thus, one has found closed expressions for the quantities c_k. The coefficients a_k, however, can be expressed very simply in terms of the c_k; in fact, writing down the conjugate complex equation of (64.19) and solving for a_k, one finds.

$$a_k = \frac{k(\rho_0^2 - \rho_0^{-2})c_k - (\rho_0^{2k} - \rho_0^{-2k})\bar{c}_k}{k^2(\rho_0^2 - \rho_0^{-2})^2 - (\rho_0^{2k} - \rho_0^{-2k})^2} \qquad (k = 2, 3, \ldots). \tag{64.23}$$

The expressions (64.21) and (64.21') for c_k may be further simplified, if one introduces instead of the coefficients A_k of (64.12) the coefficients C_k of the expansion of the function $f_1 - if_2$ as a complex Fourier series

$$f_1 - if_2 = \sum C_k e^{ik\vartheta} \tag{64.24}$$

Comparing (64.24) with (64.12), one observes that

$$A_k = C_k - \rho_0^{-2} C_{k+2}. \tag{64.25}$$

The expression (64.22) for c_1 takes then the form

$$c_1 = C_{-1}\rho_0 - C_1\rho_0^{-1}. \tag{64.26}$$

Substituting (64.25) and (64.26) on the right-hand side of (64.21) and (64.21'), one finally finds the simple formula

$$c_k = C_{-k}\rho_0^k - C_k\rho_0^{-k} \quad (k = 1, 2, \ldots). \tag{64.27}$$

Finally consider the question of the convergence of the series obtained above and suppose that the functions f_1 and f_2 have second order derivatives, satisfying the Dirichlet condition (or more generally, being of

bounded variation). Then one has for the coefficients C_k of (64.24) inequalities of the form

$$|C_k| < \frac{C}{|k|^3} \quad (k = \pm 1, \pm 2, \ldots).$$

On the basis of (64.27), (64.23) and (64.13) or (64.13′) one easily finds the inequalities

$$|a_k|\rho_0^k < \frac{C}{|k|^3}, \quad |b_k|\rho_0^k < \frac{C}{k^2} \quad (k = \pm 1, \pm 2, \ldots), \qquad (64.28)$$

from which follows immediately the absolute and uniform convergence of the series for $\varphi(\zeta)$, $\varphi'(\zeta)$, $(1 - 1/\zeta^2)\,\psi(\zeta)$ in the interval

$$\frac{1}{\rho_0} \leqslant |\zeta| \leqslant \rho_0,$$

and hence the suitability of the solutions.

PART IV

ON CAUCHY INTEGRALS

In the subsequent chapters wide use will be made of so-called Cauchy integrals. A systematic study of the properties of these integrals may be found in the Author's book [25], but for the convenience of the reader, who wants to limit himself to the information actually required for the understanding of what follows, the essentials will be given in the present Part. Some deductions will be stated without proofs; these may be found in the book mentioned above, or in I. I. Privalov's book [1]. On the other hand, a number of elementary formulae and results will be given here which are of practical value and which are not contained in those books.

CHAPTER 12

FUNDAMENTAL PROPERTIES OF CAUCHY INTEGRALS

§ 65. Notation and terminology.

1°. In the sequel, unless stated otherwise, L will be a simple smooth contour, a simple finite smooth arc or the union of a finite number of such disconnected arcs and contours in the plane Oxy (Fig. 30). In this case L will be called *a simple smooth line*, where the words "simple" and "smooth" will often be omitted.

Thus the line L may consist of parts placed separately from each other. If L contains arcs, their ends will be called *ends of the line L*.

The line L will always be given a definite positive direction; in the case when it consists of disconnected parts a positive direction must be chosen on each of these parts.

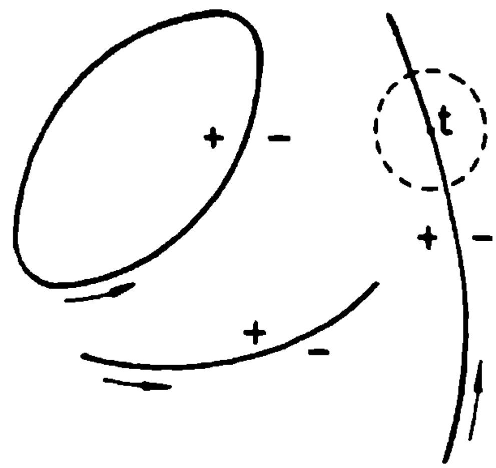


Fig. 30.

If one draws about any point of the (simple, smooth) line L, which does not coincide with one of its ends, a circle of sufficiently small radius, this circle will be divided by L into two parts one of which will lie on the left and the other on the right of the line (looking in the positive direction of L; cf. Fig. 30). In accordance with this, a distinction may be made between "left" and "right" neighbourhoods of the point t on L, other than one of its ends. For example, the left neighbourhood of t consists of points, not on L and in the left section of a circle drawn with sufficiently small radius about t.

In a similar manner one may distinguish between left and right neighbourhoods of any part of a line L, the ends of which are not ends of L. As before, a *part* of the line L will *always* be a part consisting of (a finite number of) arcs or contours belonging to L. The left and right neighbourhoods will be denoted by $(+)$ and $(-)$ respectively.

2°. The definitions of § 37 will now be recalled and partly extended.

Let $F(z)$ be some function, given in the neighbourhood of L, but not on L, and assume that $F(z)$ is continuous there. (As in § 37, the function $F(z)$ will not be assumed to be analytic). Also let t be some point of L other than its ends (if the last exist).

The function $F(z)$ will be said to be *continuous at t from the left* (*or right*), if $F(z)$ tends to a limit as z tends to t along any path remaining, however, on the left (or right) of L [i.e., z may take any position in the left (right) neighbourhood of t]. The limiting values of $F(z)$ as $z \to t$ from the left or from the right will be denoted by

$$F^+(t) \text{ or } F^-(t)$$

respectively and they will be called the *boundary values* of $F(z)$.

This notation, and the term "boundary value" will *only be used in such cases* when the corresponding limiting values exist as z tends to t *along any path* on the left or on the right of L, i.e., when $F(z)$ is continuous at t from the left or from the right.

Let L' be some part of L the ends of which do not coincide with those of L (if such exist). The function $F(z)$ will be said to be continuous at L' from the left [or right], if the limiting value $F^+(t)$ [or $F^-(t)$] exists for all points t of L'.

As mentioned in § 37, if $F(z)$ is continuous on L' from the left [or right], the function $F^+(t)$ [or $F^-(t)$] will be continuous on L'. Hence it follows that, if the line L' be added to the left [or right] neighbourhoods of L' and if the function $F(z)$ be given its values $F^+(t)$ [or $F^-(t)$] on L', then $F(z)$ will be continuous in the left [or right] neighbourhood, including the line L'.

3°. Let $f(t)$ be some, in general complex, function of the point t of L; this means that

$$f(t) = f_1(t) + i f_2(t), \tag{65.1}$$

where $f_1(t)$ and $f_2(t)$ are real functions of t on L.

In future, t will denote a point as well as its coordinates, i.e., $t = x + iy$.

The function $f(t)$ will be said to satisfy on L *the Hölder condition*, or just the H condition, if for every pair of points t_1, t_2 of L the following inequality holds true:

$$| f(t_2) - f(t_1) | < A \, | t_2 - t_1 |^{\mu}, \tag{65.2}$$

where A and μ are positive constants and $0 < \mu \leq 1$; A is called the *Hölder constant* and μ the *Hölder index*.

The condition (65.2) is easily seen to be equivalent to

$$| f(t_2) - f(t_1) | \leqslant B\sigma_{12}^{\mu}, \tag{65.3}$$

where B is a positive constant and σ_{12} is the length of arc of L between t_1 and t_2; if t_1 and t_2 lie on a contour forming part of L, the shorter of the two arcs between t_1 and t_2 must be taken for σ_{12}. If L consists of several different parts, the condition (65.3) must be understood to be fulfilled for any pair of points lying on one and the same part.

The equivalence of the conditions (65.2) and (65.3) follows from these propositions which are easily proved:

1°. If (65.2) is satisfied for any pair of points whose distance does not exceed some fixed number δ, it will be satisfied for the whole of L, provided, if this should be necessary, A is replaced by a larger value.

2°. For any pair of points t_1, t_2 whose distance does not exceed δ

$$k \leqslant \frac{|t_1 - t_2|}{\sigma_{12}} \leqslant 1,$$

where k is a positive constant. Proofs of these propositions may be found, for example, in the Author's book [25].

If $\mu > 1$ in (65.2) or (65.3), it is easily seen that the derivative of $f(t)$ with respect to the arc s of L will be zero; hence, in this case, $f(t) = \text{const.}$ on L or, if L consists of different parts, on each of these parts. This case is of no interest, and for this reason consideration will be restricted to $\mu \leqslant 1$.

NOTE. If for a *given* point t_0 of L one has the inequality

$$| f(t) - f(t_0) | \leqslant A \, | t - t_0 |^{\mu}$$

for all t of L, sufficiently close to t_0, the function $f(t)$ will be said to satisfy the H condition on L *at the given point* t_0; however, this does not mean that $f(t)$ satisfies the H condition *in the neighbourhood* of t_0, i.e., that (65.2) holds true for any pair of points in the neighbourhood of t_0 on L.

4°. In the sequel the following well known notation will sometimes be used. Let ζ be a variable quantity which runs through some set of values and tends to 0 [or to ∞]. Then $O(\xi)$ will denote a quantity such that $O(\xi)/\xi$ remains bounded for sufficiently small [or sufficiently large] values of $|\xi|$. In other words, for those values of ξ

$$O(\xi) \leqslant C . |\xi|,$$

where C is a finite constant. Further, $o(\xi)$ will denote a quantity such

that (the modulus of) $o(\xi)/\xi$ will be as small as desired, when $|\xi|$ is sufficiently small [or sufficiently large], i.e.,

$$|o(\xi)| \leqslant c \cdot |\xi|,$$

where c is a positive quantity which only depends on $|\xi|$ and which tends to zero for $\xi \to 0$ [$\xi \to \infty$].

For example, if $f(t)$ satisfies the H condition in the neighbourhood of the point t_0, this condition may be written

$$|f(t_2) - f(t_1)| = O(|t_2 - t_1|^\mu)$$

for all points t_1, t_2 which are sufficiently near to t_0.

One particular case of this notation should be noted. Consider the expression $O(|\xi|^\alpha)$, where α is a real number. By definition, $O(|\xi|^\alpha)/|\xi|^\alpha$ remains bounded when $|\xi| \to 0$ [$|\xi| \to \infty$]. In particular, for $\alpha = 0$, the expression $O(|\xi|^\alpha)$ becomes $O(1)$. Thus $O(1)$ denotes a quantity which remains bounded for sufficiently small [or sufficiently large] values of $|\xi|$. Similarly, $o(1)$ denotes a quantity which tends to zero for $|\xi| \to 0$ [or $|\xi| \to \infty$], i.e., $|o(1)| < \varepsilon$, where ε only depends on $|\xi|$ and $\lim \varepsilon = 0$ for $|\xi| \to 0$ [or $|\xi| \to \infty$].

For example, the condition that $f(t)$ is continuous on L may be written

$$|f(t_2) - f(t_1)| = o(1)$$

for $|t_2 - t_1| \to 0$.

§ 66. Cauchy integrals. Let L be the same as in the preceding section and let $f(t) = f_1(t) + if_2(t)$ be some, in general, complex function given on L. Unless stated otherwise, it will always be assumed that $f(t)$ is finite and integrable in an ordinary (Riemann) sense.

The integral of the form

$$\frac{1}{2\pi i} \int \frac{f(t)dt}{t - z} \qquad (a)$$

taken over L with z some point in the plane of L, will be called an *integral of the Cauchy type* or *Cauchy integral*; the factor $1/2\pi i$ is not essential and has only been introduced for the sake of convenience.

For the present it will be assumed that the point z does not lie on L. In that case the integral (a) has a definite meaning and represents a function of the complex variable z throughout the entire plane with the

exception of the points of L. This function will be denoted by $F(z)$, so that

$$F(z) = \frac{1}{2\pi i} \int_L \frac{f(t)dt}{t - z}. \tag{66.1}$$

It is easily seen that $F(z)$ is holomorphic in the whole plane, excluding the points of L. If L consists of contours, as shown in Fig. 30, the preceding statement must be understood in the sense that $F(z)$ is holomorphic inside all parts into which the plane is divided by L. (It must not be thought that $F(z)$ is analytically continued when z passes from one part to another; this will become clear from the following work.)

Further, it is easily seen that for $z \to \infty$ $F(z)$ tends to zero, i.e.,

$$F(\infty) = 0. \tag{66.2}$$

§ 67. Values of Cauchy integrals on the path of integration. Principal value. Hitherto it has been assumed that the point z in (66.1) does not lie on the line of integration L. Now let z coincide with some point t_0 of L. For the present write formally

$$\frac{1}{2\pi i} \int_L \frac{f(t)dt}{t - t_0}. \tag{67.1}$$

If $f(t_0) \neq 0$, the integrand becomes infinite like $| t - t_0 |^{-1}$ as $t \to t_0$. Hence the integral has no meaning in the ordinary sense. However, for certain conditions referring to $f(t)$, the integral (67.1) may be given a definite interpretation. In fact, assume that t_0 is not an end of L (if such exist) and separate from L a sufficiently small arc $t_1 t_2$ which contains t_0 in such a way that

$$|t_1 - t_0| = |t_2 - t_0| \tag{67.2}$$

Denote the arc $t_1 t_2$ by l and the remaining part of L by $L - l$ and consider the integral

$$\frac{1}{2\pi i} \int_{L-l} \frac{f(t)dt}{t - t_0}. \tag{67.3}$$

This integral is completely defined in the ordinary sense, since, as t travels along the path of integration $L - l$, $| t - t_0 | \geqslant \delta$ where δ is some positive constant.

Next suppose that t_1 and t_2 tend to t_0 in such a way that (67.2) remains satisfied. If under these conditions the integral (67.3) tends to a definite

limit, this limit is called the ***principal value of the Cauchy integral*** (67.1).

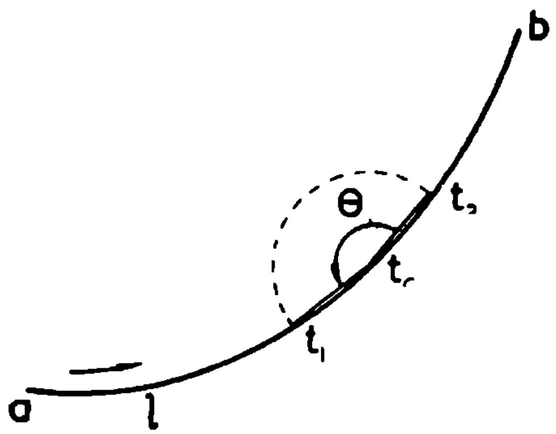


Fig. 31.

Clearly, if (67.1) has a meaning in the ordinary (Riemann) sense, its principal value will exist, but the converse proposition is, generally speaking, not true. (In this connection the term ordinary means that t_1 and t_2 tend to t_0 in an arbitrary manner so that (67.2) is no longer fulfilled).

The principal value of an integral, if it exists, will be denoted by the same symbol as the ordinary integral, i.e., by (67.1), where it must be understood that, if the integral has no meaning in the ordinary sense, its principal value must be taken (provided that it exists). Some authors use a special symbol for the principal value; for example, the integral sign is accented (′) or the letters VP (Valeur Principale) are put in front of it.

No consideration will be given to the problem of finding the most general conditions for the existence of a principal value, but instead one very important case (which is completely sufficient for the purpose of this book) will be stated when this existence is definitely ensured. In fact, *the principal value of the integral* (67.1) *exists, if the function* $f(t)$ *satisfies the H condition in the neighbourhood of the point* t_0, i.e., the condition (cf. § 65)

$$| f(t_2) - f(t_1) | \leqslant A \, | t_2 - t_1 |^{\mu}, \quad 0 < \mu \leqslant 1. \tag{67.4}$$

This proposition will now be proved by actually expressing the principal value of the integral by an ordinary integral. For this purpose investigate the integral (67.3) and consider first the case (cf. Fig. 31) when L consists of a simple closed arc ab, i.e., consider the integral

$$\frac{1}{2\pi i} \int\limits_{ab-l} \frac{f(t)dt}{t - t_0}, \tag{67.3′}$$

where the positive direction is from a to b.

This integral may be written as follows:

$$\frac{1}{2\pi i} \int\limits_{ab-l} \frac{f(t)dt}{t - t_0} = \frac{1}{2\pi i} \int\limits_{ab-l} \frac{f(t) - f(t_0)}{t - t_0} dt + \frac{f(t_0)}{2\pi i} \int\limits_{ab-l} \frac{dt}{t - t_0}. \tag{67.5}$$

The first integral on the right-hand side tends to the limit

$$\frac{1}{2\pi i}\int_{ab}\frac{f(t)-f(t_0)}{t-t_0}\,dt,$$

as $t_1 \to t_0$, $t_2 \to t_0$, because it converges in the ordinary (Riemann) sense; in fact, by (67.4),

$$\left|\frac{f(t)-f(t_0)}{t-t_0}\right| \leqslant \frac{A}{|t-t_0|^{1-\mu}},$$

and since $1-\mu < 1$, this inequality ensures convergence of the integral by a well-known elementary convergence theorem.

Next consider the second integral on the right-hand side of (67.5) which is easily represented in the form

$$\frac{1}{2\pi i}\int_{ab-l}\frac{dt}{t-t_0} = [\log(t-t_0)]_a^{t_1} + [\log(t-t_0)]_{t_2}^{b},$$

where by $\log(t-t_0)$ on the parts at_1 and t_2b of the line ab must be understood any branches of this function which change continuously with t on each of the parts at_1, t_2b separately. These branches may be chosen arbitrarily on each of these parts, but for the sake of definiteness they will be related by the following condition: the value $\log(t-t_0)$ for $t = t_2$ is to be obtained from the value $\log(t-t_0)$ for $t = t_1$ by means of a continuous change of $\log(t-t_0)$, as the point t moves from t_1 to t_2 along a (infinitely small) semi-circle, lying to the left of L (cf. Fig. 31). Under this condition the branch $\log(t-t_0)$ on at_1 completely determines the choice of the branch on t_2b, and, provided this choice has been made, one may write

$$\frac{1}{2\pi i}\int_{ab-l}\frac{dt}{t-t_0} = \frac{1}{2\pi i}\log\frac{b-t_0}{a-t_0} + \frac{1}{2\pi i}\log\frac{t_1-t_0}{t_2-t_0} \tag{67.6}$$

where

$$\log\frac{b-t_0}{a-t_0} = \log(b-t_0) - \log(a-t_0),$$

$$\log\frac{t_1-t_0}{t_2-t_0} = \log(t_1-t_0) - \log(t_2-t_0),$$

for the choice of logarithms, stated above. Further, since by supposition

$$|t_1 - t_0|/|t_2 - t_0| \to 1,$$

$$\log \frac{t_1 - t_0}{t_2 - t_0} \to i\theta,$$

where θ is the angle, shown in Fig. 31. Obviously, as $t_1 \to t_0$, $t_2 \to t_0$, one has: $\lim \theta = \pi$. Hence, proceeding in (67.6) to the limit, one obtains

$$\lim \frac{1}{2\pi i} \int\limits_{ab-l} \frac{dt}{t - t_0} = \tfrac{1}{2} + \frac{1}{2\pi i} \log \frac{b - t_0}{a - t_0}.$$

Thus, the integral (67.3′) has a definite limiting value and this limit is, by definition, the principal value of the integral

$$\frac{1}{2\pi i} \int\limits_{ab} \frac{f(t)dt}{t - t_0};$$

it is given by the formula

$$\frac{1}{2\pi i} \int\limits_{ab} \frac{f(t)dt}{t - t_0} = \tfrac{1}{2} f(t_0) + \frac{1}{2\pi i} f(t_0) \log \frac{b - t_0}{a - t_0} +$$
$$+ \frac{1}{2\pi i} \int\limits_{ab} \frac{f(t) - f(t_0)}{t - t_0}\, dt, \qquad (67.7)$$

where the integral on the right-hand side has an ordinary (Riemann) value.

The preceding statement implies that the condition (67.2) is satisfied. If this condition were not fulfilled, one would have instead of (a)

$$\log \frac{t_1 - t_0}{t_2 - t_0} = \log \frac{r_1}{r_2} + i\theta, \qquad (b)$$

where $r_1 = |t_1 - t_0|$, $r_2 = |t_2 - t_0|$; hence, if (67.2) is omitted, the expression (b) would not tend to a limit.

Now let L be an arbitrary line of the form discussed in § 65, 1°. Then, selecting on L some arc ab containing t_0 (in such a way that a or b do not coincide with t_0), one may rewrite the integral (67.3)

$$\frac{1}{2\pi i} \int\limits_{L} \frac{f(t)dt}{t - t_0} = \frac{1}{2\pi i} \int\limits_{ab} \frac{f(t)dt}{t - t_0} + \frac{1}{2\pi i} \int\limits_{L-ab} \frac{f(t)dt}{t - t_0}.$$

Provided (67.2) is satisfied, the first integral on the right-hand side

has just been shown to tend to a definite limit, as $t_1 \to t_0$, $t_2 \to t_0$; the second integral obviously does not depend on t_1 and t_2. Hence (67.3) tends to a definite limit which is, by definition, the principal value of (67.1); on the basis of the preceding work, this principal value is given by

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-t_0} = \tfrac{1}{2}f(t_0) + \frac{1}{2\pi i} f(t_0)\log\frac{b-t_0}{a-t_0} + \\ + \frac{1}{2\pi i}\int_{ab}\frac{f(t)-f(t_0)}{t-t_0}\,dt + \frac{1}{2\pi i}\int_{L-ab}\frac{f(t)dt}{t-t_0} \tag{67.8}$$

This formula shows little symmetry and it will not be used below; it has only been introduced here to show that the Cauchy principal value exists under the conditions referring to $f(t)$, stated earlier, and that it may be expressed by means of ordinary integrals.

NOTE. 1. The formula (67.8) is considerably simplified, if L is a simple contour. The relevant formula for this case may be deduced from (67.7) by letting the end b of the arc ab tend to the end a, so that one obtains in the limit the contour L. Assuming, for definiteness, that the positive direction on this contour is chosen in such a way that the finite part of the plane bounded by L lies to the left when looking along L in the positive direction, one will have (for $b = a$)

$$\log\frac{b-t_0}{a-t_0} = 0,$$

and (67.7) takes the form

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-t_0} = \tfrac{1}{2}f(t_0) + \frac{1}{2\pi i}\int_L \frac{f(t)-f(t_0)}{t-t_0}\,dt. \tag{67.7'}$$

NOTE. 2. It is unnecessary to introduce the condition (67.2), in order to define the principal value of the Cauchy integral.

In fact, it is sufficient to assume that

$$\lim\frac{|t_2-t_0|}{|t_1-t_0|} = 1,$$

as $t_2 \to t_0$, $t_1 \to t_0$, i.e., that $r_1 = |t_1-t_0|$, $r_2 = |t_2-t_0|$ are infinitesimal quantities of equal order of magnitude. Obviously one will again have

under this condition that

$$\lim \frac{t_1 - t_0}{t_2 - t_0} \qquad i\pi$$

(cf. remarks following 67.7), and hence all the preceding conclusions and formulae will remain valid.

In particular, the condition (67.2) may be replaced by

$$\sigma_1 = \sigma_2, \tag{67.2'}$$

where σ_1 and σ_2 denote the lengths of the arcs t_1t_0 and t_2t_0, so that the point t_0 divides the arc t_1t_2 into two equally long parts.

NOTE. 3. Obviously the formulae and conclusions of this section will remain in force, if $f(t)$ satisfies the H condition only at a point t_0 (cf. § 65, 3°), i.e., if

$$| f(t) - f(t_0) | \leqslant A \, | t - t_0 |^{\mu}$$

(for t sufficiently close to t_0); thus there is no need for the H condition to be satisfied for any pair of points in the neighbourhood of t_0. But in that case the above work will only hold true for the given value t_0.

§ 68. Boundary values of Cauchy integrals. The Plemelj formulae. As regards the values of the Cauchy integral

$$F(z) = \frac{1}{2\pi i} \int \frac{f(t)dt}{t - z} \tag{68.1}$$

on the line of integration, considered in § 67, a distinction must be made between its boundary values, i.e., between the limits of $F(z)$ as z tends to t_0 on L from the left or from the right. The following important proposition holds true with regard to these boundary values.

If the function $f(t)$, given on L, satisfies the H condition in the neighbourhood of a point t_0 of L, other than one of its ends, the integral $F(z)$ is continuous at L from the left and from the right, i.e., the boundary values $F^+(t_0)$ and $F^-(t_0)$ exist. (Naturally, the H condition is only satisfied for points of L near t_0, since $f(t)$ is not given for other points of the plane. See also § 65, 2°).

These boundary values are given by the formulae

$$F^+(t_0) = \tfrac{1}{2}f(t_0) + \frac{1}{2\pi i} \int_L \frac{f(t)dt}{t - t_0}, \tag{68.2}$$

$$F^-(t_0) = -\tfrac{1}{2}f(t_0) + \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t - t_0}, \tag{68.3}$$

where the principal values of the integrals must be taken on the right-hand sides.

The formulae (68.2) and (68.3) may be replaced by the equivalent expressions

$$F^+(t_0) - F^-(t_0) = f(t_0), \tag{68.4}$$

$$F^+(t_0) + F^-(t_0) = \frac{1}{\pi i}\int \frac{f(t)dt}{t - t_0}. \tag{68.5}$$

Further, the following proposition holds true:

If the function $f(t)$ satisfies the H condition on some part L' of L, the boundary values $F^+(t_0)$ and $F^-(t_0)$ also satisfy the H condition on L', except possibly in an arbitrarily small neighbourhood of the ends of L' (if the latter exist).

The above theorems were first given by J. Plemelj [1] and refined by I. I. Privalov. The formulae (68.2) and (68.3) were likewise given by J. Plemelj (in the same paper) and for this reason they will be called Plemelj formulae. The proof of the formulae and theorems of the present section may be found in I. I. Privalov's book [1] or in the Author's book [25].

NOTE. 1. The expression (68.4) follows from (68.2) and (68.3) which have been obtained under the supposition that $f(t)$ satisfies the H condition in the neighbourhood of the point t_0. But it may also be extended (in a conventional way) to the case when $f(t)$ is *only continuous* in the neighbourhood of t_0. Draw through t_0 some straight line Δ which does not coincide with the tangent at t_0 and select on this line two points t' and t'' in such a way that the segment $t't''$ is bisected by t_0. Then, if $f(t)$ is continuous (on L) near t_0, the difference

$$F(t'') - F(t')$$

tends to the limit $f(t_0)$ as $t' \to t_0$, $t'' \to t_0$ (provided t' and t'' are all the time equidistant from t_0). Hence, denoting this limit by $F^+(t_0) - F^-(t_0)$, one finds (68.4) to be still valid under the new conditions for $f(t)$.

This circumstance was likewise noted by J. Plemelj [1]; the proof may be found in the Author's book [25]. It can also be shown that $F(t'') - F(t')$ tends uniformly to the limit $f(t_0)$ (with regard to the position of t_0 on

some sufficiently small part of L), if the non-obtuse angle between the straight line Δ and the tangent to L at t_0 is not less than some fixed acute angle (the proof of this proposition is given in the Author's book [25]).

NOTE. 2. The following result follows immediately from the statements of the preceding Note: If the function $f(t)$ is continuous on L in the neighbourhood of t_0 and if the boundary value $F^+(t_0)$ [or $F^-(t_0)$] exists, the boundary value $F^-(t_0)$ [or $F^+(t_0)$] also exists and these boundary values are related to each other by (68.4).

NOTE. 3. In contrast to what has been said in Note. 3 of § 67, it is not sufficient for the existence of the boundary values $F^+(t_0)$, $F^-(t_0)$ to assume that $f(t)$ satisfies the H condition only at a given point t_0 (cf. § 65, 3°.) and not in some (arbitrarily small) neighbourhood of t_0 (on L). However, under this last supposition, there will exist limits of the function $F(z)$ as $z \to t_0$ from the left or from the right, if it is assumed that this transition takes place along a definite path not tangential to L.

NOTE. 4. Let L be a simple closed arc the ends of which will be denoted by a and b and the positive direction of which is from a to b. The behaviour of the function $F(z)$ near the ends is easily determined. In fact, let it first be assumed that $f(a) = 0$. Then, extending the line L beyond the end a, for example by a segment of the tangent there, and putting on the additional part $f(t) = 0$, one arrives at the case where a is not an end. Hence, applying the earlier results, it is easily concluded that $F(z)$ tends to a definite limit as z tends to a along any path. [By (68.2) and (68.3), one will have $F^+(a) = F^-(a)$, because in the present case $f(t_0) = f(a) = 0$.] If $f(a) \neq 0$, the formula (68.1) can be rewritten in the form

$$F(z) = \frac{1}{2\pi i}\int_{ab} \frac{f(a)dt}{t-z} + \frac{1}{2\pi i}\int_{ab} \frac{f(t)-f(a)}{t-z}\, dt =$$

$$= \frac{f(a)}{2\pi i}\log\frac{b-z}{a-z} + \frac{1}{2\pi i}\int_{ab} \frac{f(t)-f(a)}{t-z}\, dt,$$

and it is easily seen, on the basis of the preceding remarks, that near a

$$F(z) = \frac{f(a)}{2\pi i}\log\frac{1}{z-a} + F^*(z), \tag{68.6}$$

where $F^*(z)$ tends to a definite limit as $z \to a$. Similarly, one has for the

end b

$$F(z) = -\frac{f(b)}{2\pi i}\log\frac{1}{z-b} + F^{**}(z), \tag{68.7}$$

where $F^{**}(z)$ tends to a definite limit as $z \to b$.

These results can be immediately extended to the case when L contains an arbitrary number of closed arcs $a_k b_k$.

§ 69. The derivatives of Cauchy integrals. As before, let

$$F(z) = \frac{1}{2\pi i}\int \frac{f(t)dt}{t-z}, \tag{69.1}$$

where $f(t)$ and L are the same as at the beginning of § 66 and z is a point not on L. Derivatives of any order of $F(z)$ may be obtained by differentiating the integral on the right-hand side with respect to z, so that

$$F'(z) = \frac{1}{2\pi i}\int_L \frac{f(t)dt}{(t-z)^2}, \tag{69.2}$$

and, in general,

$$F^{(k)}(z) = \frac{k!}{2\pi i}\int_L \frac{f(t)dt}{(t-z)^{k+1}}. \tag{69.3}$$

Now the question arises with regard to the behaviour of these derivatives when z approaches L from one or the other side. This question is easily answered, if it is assumed that $f(t)$, given on L, satisfies certain conditions.

For example, suppose that $f(t)$ has on some arc ab of L a first derivative with respect to t which satisfies on L the H condition. By the derivative of $f(t)$ with respect to t will, of course, be understood the limit

$$\lim \frac{f(t')-f(t)}{t'-t}$$

as $t' \to t$ in an arbitrary manner, remaining all the time on the arc ab; this derivative will be denoted, as usually, by $f'(t)$ or $df(t)/dt$.

Subdivide the integral on the right-hand side of (69.2) into two integrals one of which is taken over the arc ab and the other over the remainder of L. Obviously, the second integral represents a function of z which is holomorphic near the points of the arc ab, other than its ends.

The first integral can be transformed by an integration by parts:

$$\frac{1}{2\pi i}\int_{ab}\frac{f(t)dt}{(t-z)^2} = -\frac{1}{2\pi i}\int_{ab} f(t)d\,\frac{1}{t-z} = -\left[\frac{f(t)}{t-z}\right]_{t=a}^{t=b} + \frac{1}{2\pi i}\int_{ab}\frac{f'(t)dt}{t-z}.$$

Since, by supposition, $f'(t)$ satisfies the H condition on ab, it is clear that, using the results of § 68, the right-hand side of the preceding formula is continuous at ab from the left and from the right, if one excludes the ends a and b; hence the same will be true with regard to $F'(z)$.

Proceeding progressively to the higher order derivatives, it is easily shown that the function $F^{(n)}(z)$ is continuous at ab from the left and from the right, excluding the ends a and b, provided the function $f(t)$ has an n-th order derivative with respect to t which satisfies on the same arc ab of L the H condition.

Using the results of § 68, it may be shown that under the stated conditions the boundary values $[F^{(n)}(t)]^+$ and $[F^{(n)}(t)]^-$ satisfy the H condition on ab, if (arbitrarily small) neighbourhoods of the ends a and b are excluded.

NOTE. If one determines the position of the point t on ab by means of the arc s, measured from some fixed point (say a) of L in the positive direction, one obviously has

$$\frac{df(t)}{ds} = f'(t)\cdot\frac{dt}{ds} = e^{i\alpha}f'(t), \tag{69.4}$$

where α is the angle between the positive tangent to L at t and the Ox axis. Hence

$$f'(t) = \frac{df(t)}{ds}\,\frac{ds}{dt} = e^{-i\alpha}\frac{df(t)}{ds} \tag{69.5}$$

By supposition, the line L was to be smooth, i.e., such that the angle α changes continuously with t (or with s). It does not follow from this that α satisfies the H condition. Therefore, if $f'(t)$ satisfies the H condition, this is not necessarily true for $df(t)/ds$.

If, in addition, it is assumed that α satisfies the H condition, then it follows from the fact that $f'(t)$ satisfies the H condition that also $df(t)/ds$ satisfies that condition, and vice versa.

Further, it does not follow from the existence of the second derivative $f''(t)$ with respect to t

$$f''(t) = \frac{df'(t)}{dt}$$

that $d^2f(t)/ds^2$ exists, even if it is assumed that α satisfies the H condition. But, if it is assumed that the derivative $d\alpha/ds$ (which is known to represent the curvature of the line L at t) exists, then the derivative $d^2f(t)/ds^2$ exists and it may be expressed by the formula

$$\frac{d^2f(t)}{ds^2} = f''(t)\left(\frac{dt}{ds}\right)^2 + f'(t)\,\frac{d^2t}{ds^2} = e^{2i\alpha}f''(t) + ie^{i\alpha}\frac{d\alpha}{ds}\,f'(t), \quad (69.6)$$

which follows from (69.4); this derivative will satisfy the H condition, if that condition is satisfied by $f''(t)$ and by $d\alpha/ds$.

Similar reasoning may be applied to derivatives of higher order.

§ 70. Some elementary formulae, facilitating the calculation of Cauchy integrals. A number of simple formulae will now be deduced which facilitate calculations in many cases.

Let L be a *simple smooth contour*. Denote by S^+ the finite part of the plane bounded by L, and by S^- the infinite part of the plane consisting of the points lying outside L. The contour L will not be included with S^+ or S^-. The region, consisting of S^+ and of the points of L, will, by an obvious notation, be denoted by $S^+ + L$, and the region, consisting of S^- and of the points of L, by $S^- + L$. The *positive direction* of L will be chosen so that *the region S^+ lies on the left.*

Now the following well-known formulae will be recalled.

1°. Let $f(z)$ be a function, holomorphic in S^+ and continuous in $S^+ + L$. Then

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = f(z) \quad \text{for } z \text{ in } S^+, \quad (70.1)$$

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = 0 \quad \text{for } z \text{ in } S^-; \quad (70.2)$$

(70.1) is Cauchy's formula and (70.2) is a direct consequence of Cauchy's theorem, because in this case the integrand $f(t)/(t-z)$, considered as a function of t, is holomorphic in S^+ and continuous in $S^+ + L$.

2°. Let $f(z)$ be a function, holomorphic in S^- including the point at infinity and continuous in $S^- + L$. (It will be remembered that this means that for sufficiently large $|z|$

$$f(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots,$$

so that $c_0 = f(\infty)$.) Then

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = -f(z) + f(\infty) \qquad \text{for } z \text{ in } S^-, \tag{70.1'}$$

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = f(\infty) \qquad \text{for } z \text{ in } S^+; \tag{70.2'}$$

(70.1′) will be called *Cauchy's formula for the infinite region* S^-. The signs on the right-hand sides of (70.1′) and (70.2′) must be inverted, if the positive direction on L is chosen in such a way that S^- (and not S^+) lies to the left.

Note how the formulae (70.1′) and (70.2′) may be deduced from Cauchy's formula and theorem for finite regions. Assume for the time being that $f(\infty) = 0$. Let Γ be a circle with centre at the origin and with so large a radius that the contour L and the point z lie inside Γ. Then, assuming z to lie in the region between L and Γ, one has by Cauchy's formula

$$f(z) = -\frac{1}{2\pi i}\int_{\Gamma+L} \frac{f(t)dt}{t-z} = -\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} - \frac{1}{2\pi i}\int_\Gamma \frac{f(t)dt}{t-z},$$

where $\Gamma = L$ denotes the union of the contours Γ and L and the positive direction on Γ is assumed to be clockwise; the (—) sign on the right-hand side follows from the fact that the region between Γ and L lies to the right for motion along L and Γ in the positive direction.

It will now be shown that the integral

$$I = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)dt}{t-z}$$

is zero. In fact, the value of I does not change, if the radius R of the circle Γ is arbitrarily increased, since the function $f(t)$ is holomorphic outside L. On the other hand, as $f(\infty) = 0$, one will have for sufficiently large $|t|$

$$|f(t)| < \frac{C}{|t|},$$

where C is a positive constant. Hence, putting

$$t = Re^{i\vartheta}, \quad \text{whence } dt = iRe^{i\vartheta}d\vartheta, \quad |dt| = R\,|d\vartheta|,$$

one has

$$|I| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(t)|\, R d\vartheta}{|t-z|} \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{CRd\vartheta}{R\,|t-z|} \leqslant \frac{C}{2\pi}\int_0^{2\pi} \frac{d\vartheta}{R-r} = \frac{C}{R-r},$$

where $r = |z|$. Thus, when $R \to \infty$, $I \to 0$. But since I does not depend on R, $I = 0$. Thus (70.1′) has been proved under the supposition $f(\infty) = 0$.

In order to prove (70.2′) under the same condition, it will be assumed that z lies in S^+. Then

$$\frac{f(t)}{t-z},$$

considered as a function of t, is holomorphic in the region between L and Γ. Therefore, by Cauchy's theorem,

$$0 = \frac{1}{2\pi i}\int_{\Gamma+L} \frac{f(t)dt}{t-z} = \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} + \frac{1}{2\pi i}\int_\Gamma \frac{f(t)dt}{t-z}.$$

But the last integral, denoted earlier by I, is zero, and hence the required formula (70.2′) follows for $f(\infty) = 0$.

If now $c_0 = f(\infty) \neq 0$, on applying the formulae just deduced to the function $f(z) - c_0$, which vanishes at infinity, and noting that

$$\frac{1}{2\pi i}\int_L \frac{c_0 dt}{t-z} = \begin{cases} 0 & \text{for } z \text{ in } S^- \\ c_0 & \text{for } z \text{ in } S^+ \end{cases},$$

one obtains (70.1′) and (70.2′) for the general case.

With a view to the generalization of the preceding formulae, the following terminology will now be introduced. Let a be some finite point of the z plane and let the function $f(z)$ have the form

$$f(z) = G(z) + f_0(z) \tag{a}$$

in the neighbourhood of this point, where

$$G(z) = \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \ldots + \frac{A_l}{(z-a)^l} \tag{b}$$

($A_1, A_2, \ldots, A_l$ being constants). Then it will be said that $f(z)$ has at that point *a pole of order l with the principal part* $G(z)$.

Similarly, if in the neighbourhood of the point $z = \infty$, i.e., for sufficiently large $|z|$, the formula (*a*) holds true, where now $f_0(z)$ is holomorphic near $z = \infty$ *and vanishes at that point*, and where

$$G(z) = A_0 + A_1 z + \ldots + A_l z^l \tag{c}$$

(A_0, A_1, ..., A_l being constants), then it will be said that $f(z)$ has at $z = \infty$ *a pole of order l with the principal part $G(z)$.*

It will be noted that *in the case of the point at infinity the constant A_0 has been added to the principal part.*

A function $f_0(z)$, holomorphic in the neighbourhood of the point a, may be expanded in a series of the form

$$f_0(z) = c_0 + c_1(z - a) + c_2(z - a)^2 + \dots .$$

Even in the case when $f(z)$ is holomorphic near the point at infinity, i.e., when for sufficiently large $|z|$

$$f(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots ,$$

the function $f(z)$ will be said to have at $z = \infty$ a pole of zero order with the principal part c_0.

The following simple formulae will now be proved.

3°. Let $f(z)$ be holomorphic in S^+ and continuous in $S^+ + L$ with the possible exception of the points $a_1, a_2, \dots, a_n$ of S^+, where it may have poles with the principal parts $G_1(z), G_2(z), \dots, G_n(z)$. Then

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t - z} = f(z) - G_1(z) - G_2(z) - \dots - G_n(z) \quad \text{for } z \text{ in } S^+ \tag{70.3}$$

and

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t - z} = - G_1(z) - G_2(z) - \dots - G_n(z) \quad \text{for } z \text{ in } S^-. \tag{70.4}$$

4°. Let $f(z)$ be holomorphic in S^- and continuous in $S^- + L$ with the possible exclusion of the finite points $a_1, a_2, \dots, a_n$ of S^- and also the point $z = \infty$, where it may have poles with the principal parts $G_1(z), G_2(z), \dots, G_n(z), G_\infty(z)$. Then

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t - z} = - f(z) + G_1(z) + \dots + G_n(z) + G_\infty(z) \text{ for } z \text{ in } S^- \tag{70.3'}$$

and

$$\frac{1}{2\pi i}\int \frac{f(t)dt}{t - z} = G_1(z) + \dots + G_n(z) + G_\infty(z) \text{ for } z \text{ in } S^+. \tag{70.4'}$$

These formulae are easily established. Because of the similarity in the proof, attention will be restricted to (70.3) and (70.3′); the reader may verify the other two formulae in a similar manner.

First consider (70.3). Applying Cauchy's formula to the function

$$f_0(z) = f(z) - G_1(z) - \ldots - G_n(z)$$

which is holomorphic in S^+, one finds (assuming z to lie in S^+)

$$f_0(z) = \frac{1}{2\pi i}\int_L \frac{f_0(t)dt}{t-z} = \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} - \frac{1}{2\pi i}\int_L \frac{G_1(t)dt}{t-z} - \ldots - \frac{1}{2\pi i}\int_L \frac{G_n(t)dt}{t-z}.$$

But each of the functions $G_k(z)$, $k = 1, \ldots, n$, is holomorphic in S^- and vanishes at infinity, because these functions are of the form (*b*). Hence, by (70.2′),

$$\frac{1}{2\pi i}\int_L \frac{G_k(t)dt}{t-z} = 0, \qquad k = 1, 2, \ldots, n.$$

Thus

$$f_0(z) = \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z},$$

and so (70.3) follows.

To prove (70.3′), let Γ be a circle with centre at the origin and with radius so large that L and the points $z, a_1, a_2, \ldots, a_n$ lie inside Γ. Applying Cauchy's formula to the function

$$f_0(z) = f(z) - G_1(z) - G_2(z) - \ldots - G_n(z) - G_\infty(z)$$

which is holomorphic in the region between L and Γ, one has (with the former convention regarding the positive direction on Γ)

$$f_0(z) = \frac{1}{2\pi i}\int_{L+\Gamma} \frac{f_0(t)dt}{t-z} = -\frac{1}{2\pi i}\int_L \frac{f_0(t)dt}{t-z} - \frac{1}{2\pi i}\int_\Gamma \frac{f_0(t)dt}{t-z}$$

(assuming, of course, that z lies in S^-). But, by (70.2′), the last integral vanishes, since $f_0(z)$ is holomorphic outside Γ and vanishes at infinity. Hence

$$f_0(z) = -\frac{1}{2\pi i}\int_L \frac{f_0(t)dt}{t-z} = -\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} +$$

$$+ \frac{1}{2\pi i}\int_L \frac{G_1(t)dt}{t-z} + \ldots + \frac{1}{2\pi i}\int_L \frac{G_n(t)dt}{t-z} + \frac{1}{2\pi i}\int_L \frac{G_\infty(t)}{t-z}dt.$$

But all the integrals on the right-hand side containing $G_1(t), G_2(t), \ldots, G_n(t), G_\infty(t)$ vanish, since these functions are holomorphic in S^+ and the point z lies in S^-. Hence

$$f_0(z) = -\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z},$$

and so (70.3′) follows.

§ 71. On Cauchy integrals, taken along infinite straight lines

Hitherto consideration has been restricted to integrals taken along finite lines. There is no difficulty in extending the definition of Cauchy integrals to the case where the line of integration goes to infinity; it is only necessary to study the question of convergence of such integrals with infinite integration limits.

In the sequel only those cases of infinite lines of integration will be considered which are straight lines. Without affecting generality it can be assumed that the line of integration is the real axis. This case will be considered below in detail.

Thus let L be the real axis and consider the Cauchy type integral

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} \equiv \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)dt}{t-z}; \tag{71.1}$$

in the present case t is a real variable, which assumes all real values, and $f(t)$ is a function (in general complex) of the real variable t:

$$f(t) = f_1(t) + if_2(t),$$

where $f_1(t)$ and $f_2(t)$ are real functions. Unless stated otherwise, it will always be assumed that $f(t)$ is finite and integrable in the ordinary sense on every finite segment of the straight line L.

For the present let it be assumed that z does not lie on L. The integral (71.1) will converge uniformly, if, for sufficiently large $|t|$, the inequality

$$|f(t)| < \frac{B}{|t|^\mu} \tag{71.2}$$

holds, where B and μ are positive constants. (This condition is, of course, sufficient, but not necessary.) In fact, in this case the integrand is of order $|t|^{-1-\mu}$ for large $|t|$ and the above statement follows from a known convergence criterion for integrals with infinite limits.

However, in the sequel the more general case will occur where $f(t) \to c$ as $|t| \to \infty$, the limit c being the same for $t \to +\infty$ and for $t \to -\infty$. This limit will be denoted by $f(\infty)$. It will now be assumed that for sufficiently large $|t|$

$$f(t) = c + O\left(\frac{1}{|t|^{\mu}}\right) = f(\infty) + O\left(\frac{1}{|t|^{\mu}}\right), \quad \mu > 0. \tag{71.3}$$

Then (71.1) will diverge, i.e.,

$$\int_{N'}^{N''} \frac{f(t)dt}{t}$$

will not tend to a limit as N' and N'' tend independently of one another to $-\infty$ and $+\infty$ respectively. In fact,

$$\int_{N'}^{N''} \frac{f(t)dt}{t-z} = \int_{N'}^{N''} \frac{f(t)-c}{t-z}\,dt + c\int_{N'}^{N''} \frac{dt}{t-z}.$$

Elementary reasoning shows that

$$\int_{N'}^{N''} \frac{dt}{t-z} = \pm \alpha i + \log \frac{r''}{r'}, \tag{a}$$

where α $(0 < \alpha < \pi)$ denotes the angle between the straight lines connecting z with N' and N'' (Fig. 32) and r', r'' the distances of z from N', N''. The $(+)$ sign refers to the case when z lies in the upper half-plane and the $(-)$ sign to the case when z lies in the lower half-plane.

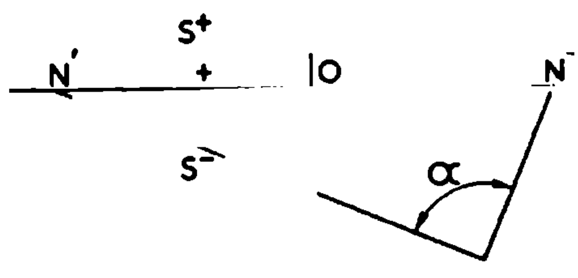


Fig. 32.

If N' and N'' tend (independently of one another) to $-\infty$ and $+\infty$ respectively, α tends to π, but $\log r''/r'$ does not have a limit. Hence the preceding integral does not tend to a limit and the same may be said

with respect to the left-hand side of (a), because the first integral on the right-hand side converges on the basis of (71.3). However, if N' and N'' do not increase independently of each other, but if it is assumed that at all times $N' = -N''$, then $\log r''/r'$ tends to 0 and

$$\lim_{N\to\infty}\int_{-N}^{+N}\frac{f(t)dt}{t-z}=\int^{+\infty}\frac{f(t)-c}{t-z}\,dt\pm\pi ic. \tag{71.4}$$

The expression on the left-hand side is called the Cauchy principal value of the integral

$$\int_{-\infty}^{+\infty}\frac{f(t)dt}{t-z}\quad\text{or}\quad\int_{L}\frac{f(t)dt}{t-z},$$

taken between infinite integration limits. *In future, when using integrals with infinite limits, their principal values will be understood* whenever these integrals do not exist in the ordinary sense.

It has been seen that, provided (71.3) is satisfied, the principal value exists and

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{f(t)dt}{t-z}=\frac{1}{2\pi i}\int_{\infty}^{+\infty}\frac{f(t)-f(\infty)}{t-z}\,dt\pm\tfrac{1}{2}f(\infty), \tag{71.5}$$

where on the left-hand side the principal value must be taken, while the integral on the right-hand side exists in the ordinary sense; the signs (+) or (—) must be chosen according to whether z is in the upper or in the lower half-plane. (Note that for the definition of the principal value it is not necessary to assume $N' = -N''$, but it will be sufficient if $\lim N'/N'' = -1$.)

Thus the term "principal value" will be used in two different, but analogous senses: when the integrand becomes infinite at some point (as in the preceding sections) or when the integration limits are infinite.

Next suppose that the point $z = t_0$ lies on the path of integration, i.e., on the real axis L. Then the integral

$$\int_{L}\frac{f(t)dt}{t-t_0}=\int_{-\infty}^{+\infty}\frac{f(t)dt}{t-t_0}$$

must be taken as principal value in both the senses stated above, i.e.,

its value will be defined as

$$\int_{-\infty}^{+\infty} \frac{f(t)dt}{t-t_0} = \lim_{\substack{N\to\infty\\ \varepsilon\to 0}} \left\{ \int_{-N}^{t_0-\varepsilon} \frac{f(t)dt}{t-t_0} + \int_{t_0+\varepsilon}^{N} \frac{f(t)dt}{t-t_0} \right\}, \tag{71.6}$$

if that limit exists.

It is easily seen that

$$\int \frac{dt}{t-t_0} = 0. \tag{b}$$

The principal value (71.6) will clearly exist, if(71.3) is fulfilled and if $f(t)$ satisfies the H condition near t_0. It follows from (b) that the principal value of (71.6) may then be expressed by either of the following formulae:

$$\int \frac{f(t)dt}{t-t_0} = \int \frac{f(t)-f(\infty)}{t-t_0}\, dt \tag{71.6'}$$

or

$$\int \frac{f(t)dt}{t-t_0} = \int \frac{f(t)-f(t_0)}{t-t_0}\, dt, \tag{71.6''}$$

where the principal values on the right-hand sides may be understood only in one of the senses indicated above; in the first case, it will be the limit

$$\lim_{\varepsilon\to 0} \left\{ \int_{-\infty}^{\varepsilon} \frac{f(t)-f(\infty)}{t-t_0}\, dt + \int_{\varepsilon}^{\infty} \frac{f(t)-f(\infty)}{t-t_0}\, dt \right\},$$

since both integrals in the curly brackets converge; in the second, it will be the limit of the ordinary integral

$$\lim_{N\to\infty} \int_{-N}^{+N} \frac{f(t)-f(t_0)}{t-t_n}\, dt,$$

since the integrand is now integrable in the ordinary sense.

Let $f(t)$ satisfy (71.3) and, of course, the conditions of integrability

and finiteness imposed at the beginning of this section. Then $F(z)$, defined by

$$F(z) = \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)dt}{t-z}, \tag{71.7}$$

will, obviously, be holomorphic in the upper as well as in the lower half-plane (but, generally speaking, not on L). Denote these half-planes by S^+ and S^- respectively; the boundary L will not be included with either of these regions. The Plemelj formulae and the theorems on the boundary values stated in § 68 are extended without difficulty to the present case.

In fact, if t_0 is a point on L (lying at a finite distance from the origin) and if $f(t)$ satisfies the H condition near this point,

$$F^+(t_0) = \quad \tfrac{1}{2}f(t_0) + \frac{1}{2\pi i}\int \frac{f(t)dt}{t-t_0}, \tag{71.8}$$

$$F^-(t_0) = -\tfrac{1}{2}f(t_0) + \frac{1}{2\pi i}\int \frac{f(t)dt}{t-t_0}; \tag{71.9}$$

$F^+(t_0)$ and $F^-(t_0)$ denote here the limiting values of $F(z)$ as $z \to t_0$ along any path on the left and right of L respectively, i.e., in S^+ or S^-. Further, if $f(t)$ satisfies the H condition on some segment of L, $F^+(t_0)$ and $F^-(t_0)$ satisfy the H condition there, except possibly near the ends of the segment. The statements in the Notes at the end of § 68 will also remain true in the present case.

In order to verify the correctness of (71.8) and (71.9) and of the subsequent statements, it is sufficient, for example, to represent the integral (71.7) in the form (71.5) and to divide the integral on the right-hand side into two integrals: the one to be taken over a finite segment, containing t_0, the other over the remaining part of the straight line.

Hitherto, when speaking of the behaviour of the function $F(z)$ near a point of the boundary L and of its boundary values, points in the finite part of the plane have always been implied. In order to study the behaviour and the boundary values of $F(z)$ near the point at infinity (which in the present case lies on L), one may, for example, proceed in the following manner.

Introduce the coordinate transformation

$$z = -\frac{1}{\zeta} \tag{71.10}$$

then the point $\zeta = 0$ of the ζ plane corresponds to the point $z = \infty$ of the z plane and vice versa; the real axis of the z plane becomes the real axis of the ζ plane and upper and lower half-planes correspond to one another; when the point $z = t$ travels along the real axis in the positive direction from $t = -\infty$ to $t = +\infty$, the corresponding point

$$\sigma = -\frac{1}{t} \tag{71.10'}$$

of the ζ plane also travels along the real axis in the positive direction as follows: from $\sigma = 0$ to $\sigma = +\infty$, from $\sigma = -\infty$ to $\sigma = 0$ (since the points $\sigma = -\infty$ and $\sigma = +\infty$ represent the same point $\zeta = \infty$ of the ζ plane).

Introducing the transformation (71.10) in (71.7), changing the integration variable in accordance with (71.10′) and introducing the notation

$$F(z) = F\left(-\frac{1}{\zeta}\right) = F^*(\zeta), \quad f(t) = f\left(-\frac{1}{\sigma}\right) = f^*(\sigma), \tag{71.11}$$

one finds

$$F(z) = F^*(\zeta) = \frac{\zeta}{2\pi i}\int \frac{f^*(\sigma)d\sigma}{\sigma(\sigma-\zeta)} \tag{71.12}$$

or

$$F(z) = F^*(\zeta) = \frac{1}{2\pi i}\int \frac{f^*(\sigma)d\sigma}{\sigma-\zeta} - \frac{1}{2\pi i}\int \frac{f^*(\sigma)d\sigma}{\sigma} \tag{71.13}$$

all these integrals must be taken as Cauchy principal values. Obviously these will exist, if, as it has been assumed, $f(t)$ satisfies the H condition for all finite t and the condition (71.3) for large $|t|$. The second integral on the right hand side of (71.13) is constant, and hence the study of the function $F(z)$ near $z = \infty$ is reduced to that of the integral

$$\frac{1}{2\pi i}\int^{+\infty} \frac{f^*(\sigma)d\sigma}{\sigma-\zeta} \tag{71.13'}$$

near $\zeta = 0$, i.e., to a problem discussed earlier.

In general, the study of the integral (71.1) may be reduced to that of an integral of the same form taken over a finite closed line, e.g., a circle. For this purpose it is sufficient, for example, to introduce the transformation

$$z = -i - \frac{1}{\zeta + i}. \tag{A}$$

Then the real axis L of the z plane becomes the circle l of the ζ plane which is tangential to the real axis and passes through the point $\zeta = -i$, and the integral (71.1) takes the form

$$\frac{1}{2\pi i}\int_l \frac{f^*(\sigma)d\sigma}{\sigma - \zeta} - \frac{1}{2\pi i}\int_l \frac{f^*(\sigma)d\sigma}{\sigma + i}$$

where

$$f^*(\sigma) = f\left(\frac{-i\sigma}{\sigma + i}\right).$$

Note that (A) transforms the half-plane S^+ on to the region bounded by the circle l.

In order to utilize immediately the earlier results, impose on $f(t)$ the condition that $f^*(\sigma)$ is to satisfy the H condition near $\sigma = 0$, i.e., that

$$| f^*(\sigma_2) - f^*(\sigma_1) | \leqslant B \mid \sigma_2 - \sigma_1 |^{\mu}, \quad 0 < \mu \leqslant 1.$$

This leads for $f(t)$ to the condition

$$| f(t_2) - f(t_1) | \leqslant A \left| \frac{1}{t_2} - \frac{1}{t_1} \right|^{\mu} \quad 0 < \mu \leqslant 1 \tag{71.14}$$

for sufficiently large $| t_1 |$, $| t_2 |$; (71.14) will be called *the H condition for the neighbourhood of the point at infinity.* Note that (71.3) is obviously a consequence of (71.14) for the neighbourhood of the point at infinity, but that the converse statement is not true. (71.3) may be called the H condition for the point $z = \infty$ (but not for its neighbourhood).

Assuming that $f(t)$ satisfies the H condition in the neighbourhood of the point at infinity, i.e., the condition (71.14), it will now be shown that the boundary values of $F(z)$ exist when z tends to infinity along any path which remains in either the upper or lower half-plane. These boundary values will be denoted by $F^+(\infty)$ and $F^-(\infty)$ respectively and (71.13) will be used to prove their existence and to calculate their values.

If $z \to \infty$, remaining in the upper or lower half-plane, then $\zeta \to 0$, also remaining in the upper or lower half-plane. Hence, applying (71.8)

to the first integral on the right-hand side of (71.13), one obtains

$$F^+(\infty) = F^{*+}(0) = \tfrac{1}{2}f^*(0) + \frac{1}{2\pi i}\int^{+\infty} \frac{f^*(\sigma)d\sigma}{\sigma} - \frac{1}{2\pi i}\int^{+\infty} \frac{f^*(\sigma)a}{\sigma}$$

and the first of the following formulae is deduced:

$$F^+(\infty) = \tfrac{1}{2}f(\infty), \quad F^-(\infty) = -\tfrac{1}{2}f(\infty); \tag{71.15}$$

the second formula may be proved in an analogous manner.

The following property of the integral $F(z)$, as defined by (71.7), will now be noted. Suppose that not only $f(t)$ but also the product $tf(t)$ satisfies the H condition near the point at infinity.

It is easily seen that, if $tf(t)$ satisfies the H condition near the point at infinity, then also $f(t)$ satisfies that condition; in addition, obviously $f(\infty) = 0$, so that, by (71.15), $F^+(\infty) = F^-(\infty) = 0$.

Under these conditions the product $zF(z)$ tends to a definite limit as $z \to \infty$ along any path remaining in the upper or lower half-plane. In fact, putting

$$tf(t) = f_1(t), \tag{71.16}$$

one has

$$zF(z) = \frac{1}{2\pi i}\int^{+\infty} \frac{zf_1(t)dt}{t(t-z)} = \frac{1}{2\pi i}\int^{+\infty} \frac{f_1(t)dt}{t-z} - \frac{1}{2\pi i}\int^{+\infty} \frac{f_1(t)dt}{t},$$

whence, by (71.15),

$$\lim_{z\to\infty} [zF(z)] = \pm \tfrac{1}{2}f_1(\infty) - \frac{1}{2\pi i}\int_{-\infty} \frac{f_1(t)dt}{t} = \pm \tfrac{1}{2}f_1(\infty) - \frac{1}{2\pi i}\int_{-\infty} f(t)dt, \tag{71.17}$$

where the upper or lower sign must be chosen according to whether z remains in the upper or lower half-plane.

This formula may also be written

$$F(z) = \frac{A}{z} + o\left(\frac{1}{z}\right) \text{ in each half-plane,} \tag{71.18}$$

where A is a constant (which may have different values in the different half-planes) and $o(1/z)$ indicates, as always, that $z.o(1/z)$ tends to zero as $| z |$ grows beyond all bounds.

Similarly, it may be shown that, if in addition to $tf(t)$ also the product

$$t^2 f'(t) = f_2(t) \tag{71.19}$$

satisfies the H condition in the neighbourhood of the point at infinity, then

$$F'(z) = -\frac{A}{z^2} + o\left(\frac{1}{z^2}\right) \quad \text{in each half-plane,} \tag{71.20}$$

where A is the same constant as in (71.18). In fact, integrating by parts, one obtains

$$F'(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)dt}{(t-z)^2} = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f'(t)dt}{t-z},$$

whence, noting that

$$\frac{1}{t-z} = \frac{t^2}{z^2(t-z)} - \frac{1}{z} - \frac{t}{z^2},$$

one easily deduces

$$z^2F'(z) = \frac{1}{2\pi i}\int \frac{f_2(t)dt}{t-z} - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} tf'(t)dt.$$

Letting $z \to \infty$, one finds by (71.15)

$$\lim_{z\to\infty} [z^2F'(z)] = \pm \tfrac{1}{2}f_2(\infty) - \frac{1}{2\pi i}\int^{+\infty} tf'(t)dt,$$

and it is easily verified that the right-hand side agrees with the right-hand side of (71.17), taken with the opposite signs. [Using the substitution $t = -1/\sigma$, it is seen that $f_2(\infty) = -f_1(\infty)$.]

It is just as easily shown that, if in addition to (71.16) and (71.19) the following relation also holds true:

$$t^3 f''(t) = f_3(t), \tag{71.21}$$

where $f_3(t)$ satisfies the H condition in the neighbourhood of the point at infinity, then

$$F''(z) = \frac{2A}{z^3} + o\left(\frac{1}{z^3}\right), \tag{71.22}$$

where A is the same constant as before.

The generalization of the above work to any order derivatives is obvious; however, only derivatives up to and including the second order will be encountered in subsequent chapters.

§ 72. On Cauchy integrals, taken over infinite straight lines (continued). A number of formulae, analogous to those of § 70, may be deduced, in order to simplify the calculation of Cauchy integrals, taken over an infinite straight line L. Consideration will be limited here to the simplest of these formulae which may easily be generalized by the reader.

1°. Let $f(z)$ be a function, holomorphic in S^+ and continuous in $S^+ + L$ including the point at infinity, and let $f(\infty) = a$. Then

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = f(z) - \tfrac{1}{2}a \quad \text{for } z \text{ in } S^+, \tag{72.1}$$

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = -\tfrac{1}{2}a \quad \text{for } z \text{ in } S^-. \tag{72.2}$$

2°. Let $f(z)$ be a function, holomorphic in S^- and continuous in $S^- + L$ including the point at infinity, and let $f(\infty) = a$. Then

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = \tfrac{1}{2}a \quad \text{for } z \text{ in } S^+, \tag{72.1'}$$

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = -f(z) + \tfrac{1}{2}a \quad \text{for } z \text{ in } S^-. \tag{72.2'}$$

The condition that $f(z)$ is continuous in $S^+ + L$ [or in $S^- + L$] and at $z = \infty$ may be expressed as follows:

$$f(z)=f(\infty)+o(1)=a+o(1) \text{ for } z\to\infty \text{ in } S^+ + L \text{ [or in } S^- + L]. \tag{72.3}$$

(The notation $o(1)$ denotes a quantity which tends uniformly to zero as $|z| \to \infty$; cf. § 65, 4°).

The formulae (72.1) and (72.2') may be called *Cauchy formulae* for the regions S^+ and S^- respectively.

Formula (72.1) will now be proved. Draw about the origin as centre a circle with sufficiently large radius R, so that the point z lies inside. Consider the contour Γ, consisting of the segment AB of the real axis

contained in the circle and of the semi-circle, lying in S^+; select the positive direction on Γ in such a way that AB is in the direction Ox. Since, by supposition, the point z is inside Γ, one has by Cauchy's formula

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(t)dt}{t-z} = \frac{1}{2\pi i}\int_{AB} \frac{f(t)dt}{t-z} + \frac{1}{2\pi i}\int_{\gamma} \frac{f(t)dt}{t-z},$$

where γ is the semi-circle, forming part of the path of integration.

The second integral on the right-hand side tends, thanks to (72.3), to the limit

$$a.\frac{\pi i}{2\pi i} = \tfrac{1}{2}a \text{ as } R \to \infty;$$

the first term then tends to a definite limit, as $R \to \infty$, and this limit is given by $f(z) - \frac{1}{2}a$. But

$$\lim_{R\to\infty} \frac{1}{2\pi i}\int_{AB} \frac{f(t)dt}{t-z} = \lim_{R\to\infty} \frac{1}{2\pi i}\int_{-R}^{R} \frac{f(t)dt}{t-z}$$

is also, by definition, the principal value of the integral

$$\frac{1}{2\pi i}\int \frac{f(t)dt}{t-z},$$

and so (72.1) is proved. Note that also the existence of the principal value of the preceding integral has been proved by this argument; this was not obvious beforehand, since in the present case $f(t)$ is subject to the condition: $f(t) = a + o(1)$, and not to the condition: $f(t) = a + O(|t|^{-\mu})$ under which the existence of the principal value had been proved earlier.

The other formulae of this section can be proved in an analogous manner.

CHAPTER 13

BOUNDARY VALUES OF HOLOMORPHIC FUNCTIONS

§ 73. Some general propositions. Let L be a simple contour, S^+ and S^- the finite and infinite parts of the plane, bounded by L; let the positive direction on L be such that S^+ remains on the left. The contour L will not be included in S^+ or S^-. Further, let

$$f(t) = f_1(t) + if_2(t)$$

be a continuous function given on L.

Consider the question as to whether $f(t)$ can be the boundary value of some function $F(z) = U(x, y) + iV(x, y)$, holomorphic in S^+, where reference here is, of course, to boundary values as $z \to t$ from S^+.

It is easily seen that, in general, this cannot be the case, if the continuous function $f(t)$ is otherwise arbitrary. In fact, it is known that it is sufficient to give the boundary value $f_1(t)$ on L of a function $U(x, y)$, harmonic in S^+, in order to completely determine this function; but then also its conjugate function $V(x, y)$ will be completely determined, neglecting an arbitrary constant term, and hence also the boundary value $f_2(t)$ of this function, if indeed it exists. Clearly the roles played by $f_1(t)$ and $f_2(t)$ may be interchanged.

The problem of determining a harmonic function from its boundary values represents the well-known Dirichlet problem. Also note that it does not follow from the existence of the boundary value of $U(x, y)$ that those of its conjugate function $V(x, y)$ exist.

It follows from the above that only one of the two real functions $f_1(t)$, $f_2(t)$ may be given arbitrarily, if the function $f(t) = f_1(t) + if_2(t)$ is required to be the boundary value of some function, holomorphic in S^+. Hence it will be of great interest to find the necessary and sufficient condition that a continuous function $f(t)$, given on L, represents the boundary value of some function $F(z)$, holomorphic in S^+; an analogous question will arise with regard to the region S^-. The following theorems answer these questions:

I. *A necessary and sufficient condition for a continuous function $f(t)$, given on L, to be the boundary value of some function, holomorphic in S^+, is*

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = 0 \text{ for all } z \text{ in } S^-. \tag{73.1}$$

II. *A necessary and sufficient condition for a continuous function $f(t)$, given on L, to be the boundary value of some function, holomorphic in S^- (including the point at infinity), is*

$$\frac{1}{2\pi i}\int \frac{f(t)dt}{t-z} = a \text{ for all } z \text{ in } S^+, \tag{73.2}$$

where a is some constant which is equal to the value of the above-mentioned holomorphic function at infinity.

These propositions are almost obvious on the basis of the results of the preceding sections. In fact, if $f(t)$ is the boundary value of some function holomorphic in S^+, condition (73.1) holds true by (70.2); hence (73.1) is necessary. It is also sufficient, for, assuming it to be fulfilled, one may write

$$F(z) = \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z}. \tag{73.3}$$

Taking into consideration that $F(z) = 0$ for z in S^-, and hence $F^-(t_0) = 0$ on L, one obtains from (68.4) and Note 2 of § 68 that

$$F^+(t_0) = f(t_0),$$

i.e., if (73.1) is satisfied, $f(t)$ represents the boundary value $F^+(t)$ of the function $F(z)$, defined by (73.3).

The second theorem may be proved in an analogous manner. If $f(t)$ is the boundary value of a function, holomorphic in S^-, (73.2) is necessary by (70.2′); it is also sufficient, since, if it is satisfied, the function

$$F(z) = -\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} + a \tag{73.4}$$

is holomorphic in S^- and takes the boundary value $F^-(t_0) = f(t_0)$; the last conclusion follows from (73.2), (68.4) and from Note 2 of § 68.

Hitherto it has been assumed that the function $f(t)$ is only continuous. If, in addition, it is assumed that it satisfies on L the H condition (§ 65),

then (73.1) and (73.2) may be given a new form which is in many respects very convenient. In fact, denoting by t_0 some point of L and performing in (73.1) and (73.2) the limiting process $z \to t_0$ from S^+ and S^- respectively, one obtains, on the basis of the Plemelj formulae (§ 68),

$$-\tfrac{1}{2}f(t_0) + \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t - t_0} = 0 \tag{73.1'}$$

and

$$\tfrac{1}{2}f(t_0) + \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t - t_0} = a \tag{73.2'}$$

respectively (for all t_0 on L). These conditions are equivalent to the conditions (73.1) and (73.2). In fact, (73.1') expresses that the boundary value of the function

$$F(z) = \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t - z},$$

holomorphic in S^-, is zero along the entire boundary L of S^-; hence, applying Cauchy's formula to S^- or from § 37, 2°, $F(z) = 0$ throughout S^-, which is the condition (73.1). Similar reasoning applies to (73.2) and (73.2'). The conditions (73.1') and (73.2') were stated by J. Plemelj [1].

So far it has been assumed that L is a simple contour. Consider now the case when L is an infinite straight line and let the real axis represent this line. As in § 71, let S^+ and S^- represent the upper and the lower half-planes respectively. The following theorems are easily proved in a manner analogous to that used in the preceding proofs.

Let $f(t)$ be a function, continuous on L, for which for large $|t|$

$$f(t) = a + O(|t|^{-\mu}) = f(\infty) + O(|t|^{-\mu}), \tag{73.5}$$

where a and μ are constants and $\mu > 0$. Then

III. *A necessary and sufficient condition for the function $f(t)$ to be the boundary value of a function, holomorphic in S^+ and continuous in $S^+ + L$ (including the point $z = \infty$), is*

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t - z} = -\tfrac{1}{2}a \text{ for all } z \text{ in } S^-. \tag{73.6}$$

IV. *A necessary and sufficient condition for the function $f(t)$ to be the boundary value of a function, holomorphic in S^- and continuous in*

$S^- + L$ *(including the point* $z = \infty$*), is*

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = \tfrac{1}{2}a \text{ for all } z \text{ in } S^+. \tag{73.7}$$

If, in addition, $f(t)$ satisfies on L, including the point at infinity [cf. (71.14)], the H condition, (73.6) and (73.7) may be replaced by

$$-\tfrac{1}{2}f(t_0) + \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-t_0} = -\tfrac{1}{2}a \tag{73.6'}$$

and

$$\tfrac{1}{2}f(t_0) + \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-t_0} = \tfrac{1}{2}a \tag{73.7'}$$

respectively, where t_0 can be any point of L.

The proofs of these theorems will be left to the reader.

§ 74. Generalization. The formulae and theorems of the preceding section, referring to the case of regions bounded by one simple contour, can immediately be extended to the case when the boundary consists of several such contours.

It is easily seen that the conditions (73.1), (73.2), (73.1′) and (73.2′) remain valid, if S^+ is a connected finite region bounded by simple contours $L_1, L_2 \ldots, L_m, L_{m+1}$ which do not intersect each other and the last of which surrounds all the others, if L is the union of these contours and, finally, if S^- is the part of the plane which is the complement of the region $S^+ + L$ with regard to the entire plane. Thus the region S^- consists of the finite regions $S_1^-, S_2^-, \ldots, S_m^-$, bounded by $L_1, L_2, \ldots, L_m$ respectively, and of the infinite region S_{m+1}^-, consisting of the points outside L_{m+1}. The function $F(z)$, holomorphic in S^-, must then be conceived as the union of the functions, holomorphic in $S_1^-, S_2^-, \ldots, S_{m+1}^-$.

§ 75. Harnack's theorem. A theorem which is frequently used and which is due to *A*. Harnack [1] follows almost immediately from the results of the preceding sections.

Let L be a simple contour and let S^+, S^- *be the finite and infinite parts into which the plane is divided by* L (which does not itself belong to S^+ or S^-). *Let* $f(t)$ *be a real and continuous function* on L. Then, if

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = 0 \text{ for all } z \text{ in } S^+, \tag{75.1}$$

$f(t) = 0$ *everywhere on* L. *Also, if*

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = 0 \text{ for all } z \text{ in } S^-, \tag{75.2}$$

$f(t) =$ *const. on* L.

In fact, it follows from (75.1), on the basis of the results of § 73, that $f(t)$ is the boundary value of some function $F(z) = U(x, y) + iV(x, y)$, holomorphic in S^-, i.e., $f(t) = U^- + iV^-$. But since $f(t)$ is a real function, the boundary value V^- of the function $V(x, y)$, harmonic in S^-, is zero everywhere on L. Hence $V(x, y) = 0$ everywhere in S^-. Therefore $U = C =$ $=$ const. in S^-, and hence $f(t) = U^- = C$ on L. Substituting this value in (75.1) and noting that

$$\frac{1}{2\pi i}\int_L \frac{C dt}{t-z} = C,$$

it is verified that $C = 0$.

It may be shown in the same manner that it follows from (75.2) that $f(t) = C =$ const.; however, in this case it is impossible to conclude that $C = 0$, since, substituting $f(t) = C$ in (75.2), one obtains the identity $0 = 0$.

Thus the theorem is proved. It will be left to the reader to generalize it to the case of the regions considered in § 74. In that case it follows from (75.1) that $f(t) = C_k$ on L_k $(k = 1, 2, \ldots, m)$, $f(t) = 0$ on L_{m+1}, and from (75.2) that $f(t) = C$ on L, where $C, C_1, C_2, \ldots, C_m$ are constants.

It is also easy to formulate a theorem, analogous to the preceding one, for the case when L is an infinite straight line.

NOTE. 1. The following conclusion follows directly from Harnack's theorem (having in mind the case when L is a simple contour).

Let $f_1(t)$, $f_2(t)$ be two *real* continuous functions, given on L. Then, if

$$\frac{1}{2\pi i}\int_L \frac{f_1(t)dt}{t-z} = \frac{1}{2\pi i}\int_L \frac{f_2(t)dt}{t-z} \text{ for all } z \text{ in } S^+, \tag{75.3}$$

$f_1(t) = f_2(t)$ on L; also, if

$$\frac{1}{2\pi i}\int_L \frac{f_1(t)dt}{t-z} = \frac{1}{2\pi i}\int_L \frac{f_2(t)dt}{t-z} \text{ for all } z \text{ in } S^-, \tag{75.4}$$

$f_2(t) = f_1(t) + \text{const.}$ on L. This result is verified by applying Harnack's theorem to the function $f_2(t) - f_1(t)$.

NOTE. 2. It is not difficult to show that the preceding theorem remains true, if it is not assumed that $f(t)$ is continuous, but if it is allowed to have a finite number of first order discontinuities. This case will not be considered further here and it will only be noted that the theorem, if properly formulated, will hold for much more general conditions.

§ 76. Some special formulae for the circle and the half-plane.

When L is a circle or a straight line, the formulae of § 75 may be given a form which is convenient for future applications.

1°. First some special notation will be introduced. Let

$$F(z) = U(x, y) + iV(x, y) \tag{76.1}$$

be a function of the complex variable z, defined in some region of the plane z. Then $\bar{F}(z)$ [*where the bar only extends over* F] is to denote the function, having the conjugate complex value of $F(z)$ at the point $\bar{z}$, which results from a reflection of the point z in the real axis, i.e., which is simply the conjugate complex value of z (Fig. 33).

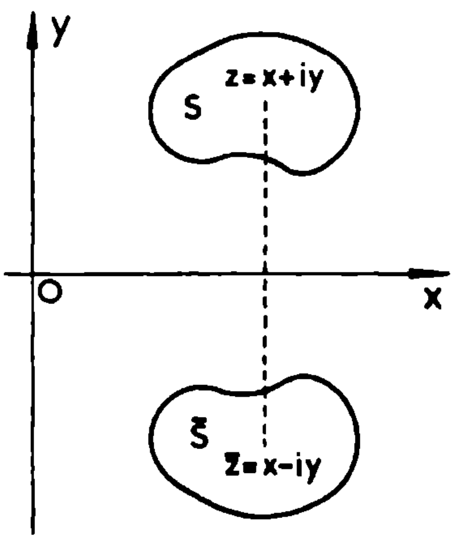


Fig. 33.

Thus, by definition,

$$\bar{F}(z) = \overline{F(\bar{z})} \tag{76.2}$$

or

$$\bar{F}(z) = U(x, -y) - iV(x, -y). \tag{76.2'}$$

For example, if $F(z)$ is a polynomial

$$F(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n, \tag{76.3}$$

then obviously by (76.2)

$$\bar{F}(z) = \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \ldots + \bar{a}_n, \tag{76.3'}$$

i.e., $\bar{F}(z)$ is obtained from $F(z)$ by replacing the coefficients by their conjugate complex values. Similarly, if $F(z)$ is a rational function

$$F(z) \quad \frac{a_0 z^n + a_1 z^{n-1} + \ldots + a_n}{b_0 z^n + b_1 z^{n-1} + \ldots + b_n} \tag{76.4}$$

then

$$\bar{F}(z) = \frac{\bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \ldots + \bar{a}_n}{\bar{b}_0 z^n + \bar{b}_1 z^{n-1} + \ldots + \bar{b}_n}. \tag{76.4'}$$

It is easily seen that, if $F(z)$ is holomorphic in some region S, $\bar{F}(z)$ is holomorphic in the region $\bar{S}$, obtained from S by reflection in the real axis (cf. Fig. 33).

In fact, putting

$$\bar{F}(z) = U_1(x, y) + iV_1(x, y),$$

one has by (76.2')

$$U_1(x, y) = U(x, -y), \quad V_1(x, y) = -V(x, -y).$$

Hence, if $U(x, y)$, $V(x, y)$ satisfy the Cauchy—Riemann conditions

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

in S, then $U_1(x, y)$, $V_1(x, y)$ will satisfy the Cauchy—Riemann conditions

$$\frac{\partial U_1}{\partial x} = \frac{\partial V_1}{\partial y}, \quad \frac{\partial U_1}{\partial y} = -\frac{\partial V_1}{\partial x}$$

in the region $\bar{S}$.

If the function $F(z)$ is holomorphic in S, except at certain points where it has poles, the function $\bar{F}(z)$ will have the same properties in $\bar{S}$ and its poles will be at points obtained from the poles of the function $F(z)$ by reflection in the real axis.

Note also that the function $\overline{F(z)}$, conjugate complex to $F(z)$, may be represented as

$$\overline{F(z)} = \bar{F}(\bar{z}); \tag{76.5}$$

this follows from (76.2) by replacing z by $\bar{z}$.

Now suppose that the function $F(z)$ is defined in one of the half-planes S^+, S^- into which the z plane is divided by the real axis, say, in the region S^+. Then the function $\bar{F}(z)$ will be defined in the region S^-. Further, if the boundary value $F^+(t)$ exists, where t is some point of the real axis, it follows immediately from (76.2) that also the boundary value $\bar{F}^-(t)$ exists and that

$$\bar{F}^-(t) = \overline{F^+(t)} \tag{76.6}$$

(since, if in (76.2) $z \to t$ from S^-, $\bar{z} \to t$ from S^+).

Obviously the roles played by S^+ and S^- may be interchanged; in that case one will have

$$\bar{F}^+(t) = \overline{F^-(t)}. \tag{76.6'}$$

2°. Let γ be the unit circle with centre at the origin of the plane of the complex variable ζ; the points of γ will be denoted by σ so that

$$\sigma = e^{i\vartheta}, \quad 0 \leqslant \vartheta < 2\pi. \tag{76.7}$$

Denote by Σ^+ and Σ^- the regions $|\zeta| < 1$ and $|\zeta| > 1$ respectively and choose the positive direction on γ so that the region Σ^+ remains on the left.

Let $F(\zeta)$ be a function, defined in Σ^+[or Σ^-]. Consider the function $F_*(\zeta)$, defined in Σ^-[or Σ^+] in the following manner:

$$F_*(\zeta) = \bar{F}\left(\frac{1}{\zeta}\right) \qquad (76.$$

or, remembering the meaning of the symbol $\bar{F}$,

$$F_*(\zeta) = \overline{F\left(\frac{1}{\bar{\zeta}}\right)}. \qquad (76.8')$$

The last formula shows that $F_*(\zeta)$ may be defined as follows: the function $F_*(\zeta)$ takes values, conjugate complex to those of $F(\zeta)$ at points which are reflections of the point ζ in the circle γ (Fig. 34). [It will be remembered (§ 48, 1°) that the reflection of the point ζ in the circle γ is the point $\zeta' = 1/\bar{\zeta}$, because in the present case the radius of the circle is unity.]

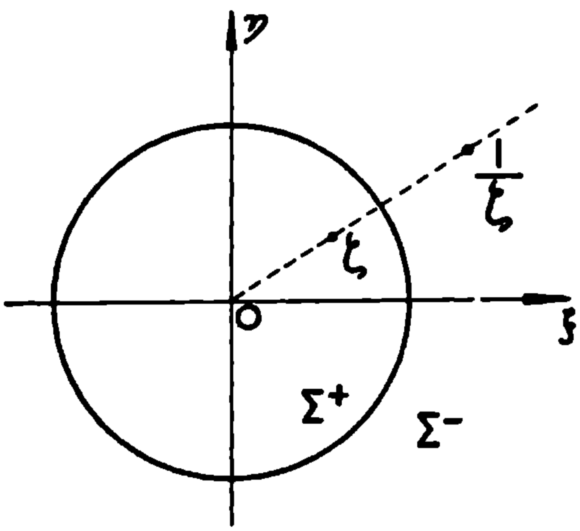


Fig. 34.

It is easily seen that, if $F(\zeta)$ is holomorphic in Σ^+[or Σ^-], the function $F_*(\zeta)$ is holomorphic in Σ^-[or Σ^+], and vice versa. For example, if $F(\zeta)$ is holomorphic in Σ^+, it may be represented by the series

$$F(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + \ldots \tag{76.9}$$

which is absolutely convergent in Σ^+, i.e., for the function $F_*(\zeta)$ will then be represented by the series

$$F_*(\zeta) = \bar{F}\left(\frac{1}{\zeta}\right) = \bar{a}_0 + \frac{\bar{a}_1}{\zeta} + \qquad (76.9')$$

absolutely convergent in Σ^-, i.e., for $|\zeta| > 1$.

Now suppose that $F(\zeta)$, defined in Σ^+, has the boundary value $F^+(\sigma)$ for $\zeta \to \sigma$, where σ is a point on γ. Then it is easily seen from (76.8′) that also the function $F_*(\zeta)$ has the boundary value $F_*^-(t)$, defined in Σ^-, and

$$F_*^-(\sigma) = F^+(\sigma), \tag{76.10}$$

because, if in (76.8′) $\zeta \to \sigma$ on γ remaining in Σ^-, then $\zeta' = 1/\bar{\zeta}$ tends to $1/\bar{\sigma} = \sigma$ remaining in Σ^+. Clearly the roles of Σ^+ and Σ^- may be interchanged; instead of (76.10) one will then have

$$F_*^+(\sigma) = F^-(\sigma). \tag{76.11}$$

3°. Using the fact that every function, holomorphic in Σ^+ [or Σ^-], corresponds to a function $\bar{F}(1/\zeta)$, holomorphic in Σ^- [or Σ^+], one may, in the case of circular boundaries, modify the formulation of the propositions I and II of § 73 which hold in the general case. In fact, the following theorems are easily proved:

I. *A necessary and sufficient condition for the function $f(\sigma)$, continuous on the circle γ, to be the boundary value of some function, holomorphic inside γ, is*

$$\frac{1}{2\pi i}\int \frac{\overline{f(\sigma)}d\sigma}{\sigma - \zeta} = \bar{a} \text{ for all } \zeta \textit{ inside } \gamma, \tag{76.12}$$

where $\bar{a}$ is a constant which is equal to the value of the above-mentioned function at $\zeta = 0$.

II. *A necessary and sufficient condition for the function $f(\sigma)$, continuous on the circle γ, to be the boundary value of a function, holomorphic outside γ, is*

$$\frac{1}{2\pi i}\int \frac{\overline{f(\sigma)}d\sigma}{\sigma - \zeta} = 0 \text{ for all } \zeta \textit{ outside } \gamma. \tag{76.13}$$

The conditions (76.12) and (76.13) follow directly from the conditions (73.2) and (73.1) and from the statements of the present section. For example, if $f(\sigma)$ is to be the boundary value $F^+(\sigma)$ of some function $F(\zeta)$, holomorphic inside γ, the function $\overline{f(\sigma)}$ must be the boundary value $F_*^-(\sigma)$ of the function $F_*(\zeta) = \bar{F}(1/\zeta)$, holomorphic outside γ; this follows directly from (76.10). Hence, applying (73.2), one obtains immediately (76.12), where $\bar{a} = F_*(\infty) = \bar{F}(0) = \overline{F(0)}$.

The condition (76.13) may be proved in an analogous manner. However, one special point must be noted: let (76.13) be fulfilled and let it be

required to find the function $F(\zeta)$, holomorphic outside γ and taking the boundary value $f(\sigma)$ on γ; if one wants to use for this purpose Cauchy's formula for the infinite region Σ^- (§ 70, 2°), viz,

$$F(\zeta) = -\frac{1}{2\pi i}\int \frac{f(\sigma)d\sigma}{\sigma - \zeta} + F(\infty), \tag{76.14}$$

one has to know $F(\infty)$. As is easily seen, this quantity is given by

$$F(\infty) = \frac{1}{2\pi i}\int \frac{f(\sigma)}{\sigma} d\sigma. \tag{76.15}$$

Introducing (76.15) in (76.14), one may write

$$F(\zeta) = -\frac{\zeta}{2\pi i}\int_\gamma \frac{f(\sigma)d\sigma}{\sigma(\sigma - \zeta)}. \tag{76.14'}$$

By (70.2'), for ζ inside γ,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\sigma)d\sigma}{\sigma - \zeta} = F(\infty),$$

whence, for $\zeta = 0$, one obtains (76.15). Thus (76.15) may, obviously, be replaced by

$$F(\infty) = \frac{1}{2\pi i}\int_\gamma \frac{f(\sigma)d\sigma}{\sigma - \zeta_0},$$

where ζ_0 is any point inside γ.

Note the following formulae which will be used in the sequel. Let

$$\varphi(\zeta) = a_0 + a_1\zeta + \ldots = \varphi(0) + \zeta\varphi'(0) + \frac{\zeta^2}{1.2}\varphi''(0) + \ldots \tag{76.16}$$

be a function, holomorphic inside and continuous up to γ. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{\sigma^k\overline{\varphi(\sigma)}d\sigma}{\sigma - \zeta} = \bar{a}_0\zeta^k + \bar{a}_1\zeta^{k-1} + \ldots + \bar{a}_k \quad (k=0, 1, 2, \ldots) \tag{76.17}$$

for all ζ inside γ. In fact, $\sigma^k\overline{\varphi(\sigma)}$ is the boundary value of $\zeta^k\bar{\varphi}(1/\zeta)$, holomorphic outside γ, except at the point $\zeta = \infty$ near which it has the form

$$\zeta^k\bar{\varphi}\left(\frac{1}{\zeta}\right) = \zeta^k\left(\bar{a}_0 + \frac{\bar{a}_1}{\zeta} + \frac{\bar{a}_2}{\zeta^2} + \ldots\right) =$$

$$\bar{a}_0\zeta^k + \bar{a}_1\zeta^{k-1} + \ldots + \bar{a}_k + O\left(\frac{1}{\zeta}\right),$$

and (76.17) follows immediately from (70.4′). In particular, for $k = 0$, one has for all ζ inside γ

$$\frac{1}{2\pi i}\int \frac{\overline{\varphi(\sigma)}d\sigma}{\sigma-\zeta} = \overline{\varphi(0)}. \tag{76.18}$$

This formula is the same as (76.12), but written somewhat differently.

4°. As before, using the fact that to every function $F(z)$, holomorphic in the upper [or lower] half-plane S^+ [or S^-], corresponds the function $\bar{F}(z)$, holomorphic in the lower [or upper] half-plane S^- [or S^+], one deduces the following propositions from the conditions (73.7) and (73.6).

As in § 73, let $f(t)$ denote a function given on the real axis L, where it is continuous and such that for large $|t|$

$$f(t) = a + O(|t|^{-\mu}) = f(\infty) + O(|t|^{-\mu}), \quad \mu = \text{const.} > 0. \tag{76.19}$$

Then

III. *A necessary and sufficient condition for the function $f(t)$ to be the boundary value of a function, holomorphic in S^+, is*

$$\frac{1}{2\pi i}\int_L \frac{\overline{f(t)}dt}{t-z} = \tfrac{1}{2}\bar{a} \textit{ for all } z \textit{ in } S^+. \tag{76.20}$$

IV. *A necessary and sufficient condition for the function $f(t)$ to be the boundary value of a function, holomorphic in S^-, is*

$$\frac{1}{2\pi i}\int \frac{\overline{f(t)}dt}{t-z} \quad \tfrac{1}{2}\bar{a} \textit{ for all } z \textit{ in } S^-. \tag{76.21}$$

§ 77. Simple applications: solutions of the fundamental problems of potential theory for a circle and half-plane.

As simple applications of the preceding results the solutions of the fundamental problems of the theory of the logarithmic potential will now be given for the cases of a circle and a half-plane.

The *first fundamental problem* (Dirichlet problem) consists of determining a function, harmonic in a region, when its boundary values are given. (The solution of this problem for the circular ring by use of infinite series was stated at the end of § 62.)

The *second fundamental problem* (Neumann problem) consists of determining a function, harmonic in a given region, when the boundary values of its normal derivative are given.

1°. First fundamental problem for a circle. For simplicity, let the radius of the circle be unity and its centre at the origin. As before, denote the circumference of the circle by γ and its points by $\sigma = e^{i\vartheta}$; other points of the plane will be denoted by ζ. Let the unknown harmonic function be P and its conjugate complex function Q. The latter function is known to be determined apart from an arbitrary constant, if the function P is known. Finally, put

$$F(\zeta) = P + iQ, \tag{77.1}$$

where $F(\zeta)$ must be holomorphic inside γ.

By the condition of the problem, the unknown function P must take the definite boundary value P^+, as ζ tends to the point σ of γ (from the inside of γ), which must be equal to the real function $f(\sigma)$ or $f(\vartheta)$, given on γ; it will be assumed that the given function $f(\vartheta)$ is continuous on γ. Hence the boundary condition of the problem may be written

$$P = f(\vartheta), \tag{77.2}$$

where, for simplicity, P has been written for P^+.

As a matter of fact, if it is assumed that P takes (definite, finite) boundary values for *all* points on γ, then the given function $f(\vartheta)$ must *necessarily* be assumed to be continuous; this follows from the statements in § 37, 1°.

The problem will now be restricted by assuming that not only the function P, but also its conjugate complex Q, and hence also the function $F(\zeta)$ take definite boundary values. (This condition is not necessary and has only been introduced to simplify the reasoning.) Denoting $F^+(\sigma)$ by $F(\sigma)$, the boundary condition (77.2) may now be written

$$F(\sigma) + F(\sigma) = 2f(\vartheta). \tag{77.3}$$

Multiplying (77.3) by $\dfrac{1}{2\pi i}\,\dfrac{d\sigma}{\sigma - \zeta}$, where ζ is a point inside γ, and integrating around γ, one finds

$$\frac{1}{2\pi i}\int \frac{F(\sigma)d\sigma}{\sigma - \zeta} + \frac{1}{2\pi i}\int \frac{\overline{F(\sigma)}d\sigma}{\sigma - \zeta} = \frac{1}{\pi i}\int \frac{f(\vartheta)d\sigma}{\sigma - \zeta} \tag{77.4}$$

On the basis of Harnack's theorem (§ 75), this condition is completely equivalent to the preceding one. (Cf. § 75, Note 1.)

By Cauchy's theorem, the first integral on the left-hand side is equal

to $F(\zeta)$; by (76.18), the second integral equals

$$F(0) = \alpha_0 - i\beta_0,$$

where α_σ and β_0 are real (for the present, unknown) constants. Thus

$$F(\zeta) = \frac{1}{\pi i}\int \frac{f(\vartheta)d\sigma}{\sigma - \zeta} - \alpha_0 + i\beta_0. \tag{77.5}$$

There remains still to determine $\alpha_0 - i\beta_0$. For this purpose put $\zeta = 0$ in (77.5) which gives

$$2\alpha_0 = \frac{1}{\pi i}\int \frac{f(\vartheta)d\sigma}{\sigma} = \frac{1}{\pi}\int^{2\pi} f(\vartheta)d\vartheta. \tag{77.6}$$

Thus α_0 may be determined from this formula; the quantity β_0, however, remains quite arbitrary, as was to be expected, because the function Q, conjugate complex to P, was determined by P apart from an arbitrary real constant, and hence $F(\zeta)$ must be determined apart from an imaginary constant.

Introducing the value of α_0 in (76.5), one finds

$$F(\zeta) = \frac{1}{\pi i}\int_\gamma \frac{f(\vartheta)d\sigma}{\sigma - \zeta} - \frac{1}{2\pi i}\int_\gamma \frac{f(\vartheta)d\sigma}{\sigma} + i\beta_0 =$$

$$= \frac{1}{2\pi i}\int f(\vartheta)\,\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} + i\beta_0. \tag{77.7}$$

This last formula is the well known *Schwarz formula*; the unknown harmonic function P is obtained from it by separating real and imaginary parts

$$P = \Re\, F(\zeta) = \Re\, \frac{1}{2\pi i}\int f(\vartheta)\,\frac{\sigma+\zeta}{\sigma-\zeta}\cdot\frac{d\sigma}{\sigma} \tag{77.7'}$$

It has only been proved that, if the solution of the problem satisfying all the imposed conditions exists, it is necessarily given by this formula. There remains to prove that this formula actually gives the solution. This will be done, assuming that $f(\vartheta)$ satisfies the H condition. In this case, on the basis of the statements of § 68, the function $F(\zeta)$, determined by (77.7), takes definite boundary values which satisfy (77.3); hence P satisfies (77.2).

The existence and uniqueness of the solution may be proved for much more general conditions (it is sufficient, if $f(\vartheta)$ is continuous), but no space has been devoted to this here, since this problem is considered in any textbook on complex function theory or potential theory.

It follows from the equivalence of the conditions (77.3) and (77.4) that the boundary values of $F(\zeta)$ satisfy (77.3). That $F(\zeta)$ satisfies (77.3) may be verified directly on the basis of the Plemelj formulae (68.2). In fact, denoting by $\sigma_0 = e^{i\vartheta_0}$ some point on γ, one has

$$F^+(\sigma_0) = f(\vartheta_0) + \frac{1}{\pi i}\int_\gamma \frac{f(\vartheta)d\sigma}{\sigma-\sigma_0} - \frac{1}{\pi i}\int_\gamma \frac{f(\vartheta)d\sigma}{\sigma} + i\beta_0 = $$
$$= f(\vartheta_0) + \frac{1}{2\pi i}\int_\gamma f(\vartheta)\,\frac{\sigma+\sigma_0}{\sigma-\sigma_0}\cdot\frac{d\sigma}{\sigma} + i\beta_0.$$

Writing under the integral sign $\sigma = e^{i\vartheta}$, $\sigma_0 = e^{i\vartheta_0}$, one finds

$$F^+(\sigma_0) = f(\vartheta_0) + \frac{1}{2\pi i}\int_0^{2\pi} f(\vartheta)\cot\frac{\vartheta-\vartheta_0}{2}\,d\vartheta + i\beta_0 = f(\vartheta_0) + \text{an imaginary quantity},$$

whence

$$\Re F^+(\sigma_0) = f(\vartheta_0).$$

Substituting in (77.7′)

$$\sigma = e^{i\vartheta}, \quad \zeta = \rho e^{i\psi}, \quad d\sigma = ie^{i\vartheta}d\vartheta,$$

one easily deduces *Poisson's formula*

$$P = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-\rho^2)\,f(\vartheta)d\vartheta}{1-2\rho\cos(\vartheta-\psi)+\rho^2} \tag{77.8}$$

which solves the above problem without the use of complex variables.

2°. The second fundamental problem for a circle. Let $F(\zeta)$ denote the same function as in 1°; it will be assumed that the derivative $F'(\zeta)$ takes the definite boundary value $F'(\sigma)$.

One deduces from the equation

$$2P = F(\zeta) + \overline{F(\zeta)} = F(\rho e^{i\vartheta}) + \overline{F(\rho e^{i\vartheta})}$$

that

$$2\frac{\partial P}{\partial \rho} = e^{i\vartheta}F'(\rho e^{i\vartheta}) + \overline{e^{i\vartheta}F'(\rho e^{i\vartheta})} = \frac{\zeta}{\rho}F'(\zeta) + \frac{\bar\zeta}{\rho}\overline{F'(\zeta)}. \tag{77.9}$$

The boundary condition of the problem under consideration has the form (denoting by n the outward normal)

$$\frac{\partial P}{\partial n} = f(\vartheta) \ \text{ or } \ \frac{\partial P}{\partial \rho} = f(\vartheta) \ \text{ on } \gamma, \tag{77.10}$$

where $f(\vartheta)$ is a given continuous function. By (77.9) this condition may be written

$$\sigma F'(\sigma) + \overline{\sigma F'(\sigma)} = 2f(\vartheta) \text{ on } \gamma. \tag{77.11}$$

In the same manner as in 1° one obtains

$$\frac{1}{2\pi i}\int \frac{\sigma F'(\sigma)d\sigma}{\zeta - \sigma} + \frac{1}{2\pi i}\int \frac{\overline{\sigma F'(\sigma)}d\sigma}{\sigma - \zeta} = \frac{1}{\pi i}\int \frac{f(\vartheta)d\sigma}{\sigma - \zeta},$$

where ζ is an arbitrary point inside γ; hence, applying Cauchy's formula and (76.18) and noting that $\sigma F'(\sigma)$ vanishes for $\sigma = 0$, one finds

$$\zeta F'(\zeta) = \frac{1}{\pi i}\int \frac{f(\vartheta)d\sigma}{\sigma - \zeta}. \tag{77.12}$$

This formula determines $F'(\zeta)$ and shows that the right-hand side must vanish for $\zeta = 0$, if the problem is to have a solution. This means that, in order for the problem to be possible, one must have

$$\int \frac{f(\vartheta)d\sigma}{\sigma} = 0 \qquad \text{or} \qquad \int_0^{2\pi} f(\vartheta)d\vartheta = 0. \tag{77.13}$$

In contrast to the Dirichlet problem, the Neumann problem does not always have a solution, but only when (77.13) is satisfied.

If the condition (77.13) is satisfied, the function $F'(\zeta)$, determined by (77.12), will be holomorphic also for $\zeta = 0$. The function $F(\zeta)$ is determined by integration

$$F(\zeta) = \frac{1}{\pi i}\int \frac{d\zeta}{\zeta}\int \frac{f(\vartheta)d\sigma}{\sigma - \zeta} + \text{const.}, \tag{77.14}$$

where const. denotes an arbitrary complex constant. The value of the unknown function $P = \Re F(\zeta)$ is thus determined apart from an arbitrary real constant. This was to be expected, because, if P is a solution of the Neumann problem, $P +$ const. will obviously be a solution of the same problem.

It is easily seen (cf. the preceding problem) that the formulae, just obtained, actually solve the stated boundary problem, if, for example, the given function $f(\vartheta)$ satisfies the H condition.

Inverting the order of integration and evaluating the inner integral on the right-hand side of (77.14), noting that

$$\int \frac{d\zeta}{\zeta(\sigma-\zeta)} = \frac{1}{\sigma} \log \zeta - \tfrac{1}{2} \log (\sigma - \zeta) + \text{const.},$$

and using (77.13), one obtains the formula which was stated by T. Boggio [4]

$$F(\zeta) = -\frac{1}{\pi i} \int_{\gamma} f(\vartheta) \log (\sigma - \zeta) \frac{d\sigma}{\sigma} + \text{const.} =$$
$$= -\frac{1}{\pi} \int_0^{2\pi} f(\vartheta) \log (\sigma - \zeta) d\vartheta + \text{const.}; \quad (77.15)$$

separating real and imaginary parts gives the formula of U. Dini [1]

$$P = -\frac{1}{\pi} \int_0^{2\pi} f(\vartheta) \log r \, d\vartheta + \text{const.}, \qquad (77.16)$$

where $r = |\sigma - \zeta|$ and const. is an arbitrary real constant. However, in most applications, it is convenient to use the formulae (77.12) and (77.14).

3°. *The first and second fundamental problems for the half-plane* may be reduced to the corresponding problems for the circle by means of conformal transformation (cf. § 71) or may be solved directly by a method, analogous to that used in the earlier problems for the circle. In view of the complete analogy with the above work, only short remarks will be made here.

Let $f(t)$ be a real continuous function, given on the real axis L, and let it be required to find $P(x, y)$, harmonic in the upper half-plane S^+ and taking the boundary value $P^+ = f(t)$ on L including the point at infinity, so that for $z \to \infty$ (in $S^+ + L$) $P \to a$, where a is the real constant

$$a = f(\infty). \qquad (77.17)$$

Introducing the function of a complex variable

$$F(z) = P + iQ, \qquad (77.18)$$

holomorphic in S^+, and assuming that this function has a definite boundary value $F^+(t)$ for all points of L, including the point at infinity, one can write down the boundary condition of the problem

$$F(t) + \overline{F(t)} = 2f(t) \text{ on } L, \qquad (77.19)$$

where $F(t)$ has been written instead of $F^+(t)$. Let

$$F(\infty) = a + ib,$$

where a is the same as in (77.17) and b is some other real constant; multiply both sides of (77.19) by $\frac{1}{2\pi i} \cdot \frac{dt}{t-z}$, where z is an arbitrary point of S^+, and integrate along L:

$$\frac{1}{2\pi i} \int_L \frac{F(t)dt}{t-z} + \frac{1}{2\pi i} \int_L \frac{\overline{F(t)}dt}{t-z} = \frac{1}{\pi i} \int_L \frac{f(t)dt}{t-z}.$$

Noting that $F(t)$ is the boundary value of $F(z)$ holomorphic in the upper half-plane, $\overline{F(t)}$ is the boundary value of $\bar{F}(z)$ holomorphic in the lower half-plane, $F(\infty) = a + ib$ and $\bar{F}(\infty) = a - ib$, and applying (72.1) and (72.1'), one concludes that the first integral on the left-hand side equals $F(z) - \frac{1}{2}(a + ib)$, while the second integral equals $\frac{1}{2}(a - ib)$. Hence

$$F(z) = \frac{1}{\pi i} \int_L \frac{f(t)dt}{t-z} + ib, \tag{77.20}$$

where, as was to be expected, the quantity b remains arbitrary.

It is easily verified that (77.20) solves the problem, if the function $f(t)$, for example, satisfies the H condition on L (including the point at infinity, cf. § 71.)

The second fundamental problem may be solved in an analogous manner.

PART V

APPLICATION OF CAUCHY INTEGRALS TO THE SOLUTION OF BOUNDARY PROBLEMS OF PLANE ELASTICITY

As mentioned earlier, the solution of the fundamental boundary problems of the theory of elasticity for regions of general form presents great practical difficulties. However, there are certain classes of regions for which effective solutions may be obtained by simple means. In plane elasticity, one such class comprises regions which may be mapped on to a circle by rational functions (one particular case has already been encountered in § 63). At first sight this class may appear to be too restricted; however, as will be explained in detail in § 89, regions of this type may be used to approximate to any desired accuracy simply connected regions of arbitrary shape.

This Part will be devoted almost entirely to the solution of boundary problems for regions of this kind. However, at the beginning (§ 79), there will be given the solutions of the first and second fundamental problems for arbitrary regions bounded by one contour, using a method closely connected with the method of solution for regions of the particular type described above. Finally, after a short introduction to other methods, the detailed solution of the above-mentioned problems will be given for the case of regions, bounded by an arbitrary number of contours (§ 102). This solution is due to D. I. Sherman. It will thus be seen that Cauchy integrals present very convenient means for the theoretical solution of general problems as well as for the effective deduction of practical results.

GENERAL SOLUTION OF THE FUNDAMENTAL PROBLEMS FOR REGIONS BOUNDED BY ONE CONTOUR

In this chapter a general method of solution of the first and second fundamental problems will be studied for regions bounded by a simple contour (§ 79). These solutions follow from integral equations which, for their part, are obtained directly from the functional equations deduced in § 78. These latter equations form the foundation for the practical methods studied in the remaining chapters of this Part and they may be investigated directly without recourse to integral equations. For this reason § 79 may be omitted by any reader not acquainted with the elements of the theory of integral equations, since the understanding of the subsequent chapters, containing solutions of problems for particular cases, does not require knowledge of that section.

§ 78. Reduction of the fundamental problems to functional equations.

1°. Let S be a finite or infinite region of the z plane, bounded by one simple contour L satisfying the conditions of § 47. Let S be mapped on to the circle $|\zeta| < 1$ of the ζ plane by the function

$$z = \omega(\zeta) \tag{78.1}$$

and let the circumference of that circle be denoted by γ.

In the case of finite regions S, it will be assumed that the point $\zeta = 0$ corresponds to the point $z = 0$, while in the case of infinite regions the point $\zeta = 0$ is to correspond to the point $z = \infty$. Thus, for finite regions

$$\omega(0) = 0 \tag{78.2}$$

and for infinite regions (cf. § 47)

$$\omega(\zeta) = \frac{c}{\zeta} + \text{a holomorphic function.} \tag{78.3}$$

It should also be remembered that $\omega'(\zeta) \neq 0$ inside and on γ (§ 47).

Further, it will be assumed (for the time being) that for infinite regions stresses as well as displacements remain bounded at infinity. This is equivalent (§ 36) to the supposition that both the stresses at infinity and the resultant vector of the external forces applied to the boundary vanish (conditions which must always hold true for finite regions).

Under these conditions and with the notation of § 50, the functions $\varphi_1(z)$ and $\psi_1(z)$ will be holomorphic in S (including the point $z = \infty$ in the case of infinite regions, cf. § 36). Hence the functions $\varphi(\zeta)$ and $\psi(\zeta)$ will be holomorphic inside the circle $|\zeta| < 1$. It will be assumed that $\varphi(\zeta)$, $\varphi'(\zeta)$, $\psi(\varphi)$ are continuous up to the circumference γ of the circle under consideration, i.e., that these functions have definite boundary values as ζ approaches points of γ along arbitrary paths; or, in other words, it will be assumed that the solutions are *regular* (§ 42) and only such solutions will be studied.

In addition, one may always assume (§ 41) $\varphi_1(0) = 0$ for finite regions and $\varphi_1(\infty) = 0$ for infinite regions, i.e., in both cases it may be assumed that

$$\varphi(0) = 0. \tag{78.4}$$

In the case of the first fundamental problem for finite regions the imaginary part of $\varphi_1'(0)$, i.e., of

$$\frac{\varphi'(0)}{\omega'(0)},$$

may also be fixed arbitrarily.

2°. The boundary condition of the *first fundamental problem* takes the form (cf. § 51)

$$\varphi(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)} \varphi'(\sigma) + \psi(\sigma) = f_1 + if_2 = f, \tag{78.5}$$

where $\sigma = e^{i\vartheta}$ is an arbitrary point of γ and $\varphi(\sigma)$, $\varphi'(\sigma)$, $\psi(\sigma)$ must be interpreted as boundary values for $\zeta \to \sigma$ from inside γ. This condition may be rewritten in conjugate complex form

$$\varphi(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)} \varphi'(\sigma) + \psi(\sigma) = f_1 - if_2 = f. \tag{78.6}$$

The quantity $f = f_1 + if_2$ is defined on L by the equation (§ 41)

$$f = f_1 + if_2 = i\int^{s} (X_n + iY_n)ds + \text{const.}, \tag{78.7}$$

where s is the arc coordinate of L and the constant may be fixed arbitrarily. This expression will be a given function of ϑ (because s is a known function of ϑ) or of σ.

It will not only be assumed that f is single-valued and continuous, but also that it has a continuous derivative with respect to ϑ, satisfying the H condition (§ 65, 3°). For this it will obviously be sufficient, if the functions X_n and Y_n satisfy the H condition.

It will be recalled that single-valuedness and continuity of $f = f_1 + if_2$ would be impossible, if the resultant vector (X, Y) of the external forces did not vanish, because in that case $f_1 + if_2$ would undergo an increase $i(X + iY)$ for every complete circuit of L, i.e., it would not revert to its original value (cf. § 42).

The following will now be noted. Provided $\varphi(\zeta)$ has been found in one way or another, the function $\psi(\zeta)$ can be calculated directly from the boundary condition. In fact, equation (78.6) gives the boundary value $\psi(\sigma)$ of $\psi(\zeta)$ which therefore is determined by

$$\psi(\zeta) = \frac{1}{2\pi i}\int \frac{\psi(\sigma)d\sigma}{\sigma - \zeta}$$

Introducing in this formula $\psi(\sigma)$, as determined by (78.6), and remembering that by (76.18)

$$\frac{1}{2\pi i}\int_{\gamma} \frac{\overline{\varphi(\sigma)}d\sigma}{\sigma - \zeta} = \overline{\varphi(0)} = 0\,,$$

one obtains

$$\psi(\zeta) = \frac{1}{2\pi i}\int_{\gamma} \frac{\bar{f}\,d\sigma}{\sigma - \zeta} - \frac{1}{2\pi i}\int_{\gamma} \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} \cdot \frac{\varphi'(\sigma)d\sigma}{\sigma - \zeta}\,. \tag{78.8}$$

There still remains the problem of finding $\varphi(\zeta)$. For this purpose a functional equation will be constructed which contains only $\varphi(\zeta)$ and which follows directly from the boundary condition. In fact, rewriting (78.5) as follows:

$$\overline{\psi(\sigma)} = f - \varphi(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi'(\sigma)} \tag{a}$$

and denoting the right-hand side for the time being by $F(\sigma)$, it is seen that the function $F(\sigma)$ must itself represent the boundary value of some function $\psi(\zeta)$, holomorphic inside γ. However, it is known (cf. § 76, 3°)

that the necessary and sufficient condition for this to be true is

$$\frac{1}{2\pi i}\int \frac{\overline{F(\sigma)}d\sigma}{\sigma - \zeta} = \bar{a} \text{ for all } \zeta \text{ inside } \gamma, \tag{b}$$

where $\bar{a}$ is a constant, i.e.,

$$a = \psi(0). \tag{78.9}$$

Substituting in (b) from the right-hand side of (a), one finds

$$\frac{1}{2\pi i}\int \frac{f d\sigma}{\sigma - \zeta} \quad \varphi(\zeta) \quad \frac{1}{2\pi i}\int \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \cdot \quad \overline{\varphi'(\sigma)}d\sigma = a$$

or, finally,

$$\varphi(\zeta) \quad \frac{1}{2\pi i}\int \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \quad \frac{\overline{\varphi'(\sigma)}d\sigma}{\sigma - \zeta} + \bar{a} = A(\zeta) \tag{78.10}$$

for all ζ inside γ, where

$$A(\zeta) = \frac{1}{2\pi i}\int \frac{f d\sigma}{\sigma - \zeta} \tag{78.11}$$

In deducing (78.10) use has been made of the fact that

$$\frac{1}{2\pi i}\int \varphi(\sigma)d\sigma = \varphi(\zeta).$$

The expression (78.10) is the functional equation from which the function $\varphi(\zeta)$ must be determined. It will be seen in the next section that this equation, in combination with the condition $\varphi(0) = 0$, completely determines the unknown function as well as the constant a, if, in the case of finite regions, the imaginary part of $\varphi'(0)/\omega'(0)$ is fixed. In the first place the constant a may be allowed to remain arbitrary; it can be fixed later by the condition $\varphi(0) = 0$.

Actually, if $\varphi(\zeta)$ is some solution of (78.10) for a given value $\bar{a}$ and if $\varphi(0) = a_0 \neq 0$, then $\varphi^*(\zeta) = \varphi(\zeta) - a_0$ will be a solution of the same equation, provided $\bar{a}$ is replaced by $\bar{a} + a_0$.

Hitherto it has been assumed that in the case of infinite regions the resultant vector (X, Y) of the external forces, applied to the contour L, and the stresses at infinity vanish. This assumption will now be relaxed.

In that case the functions $\varphi_1(z)$, $\psi_1(z)$ for infinite regions have the form (§ 36)

$$\begin{aligned}\varphi_1(z) &= \frac{X+iY}{2\pi(1+\varkappa)}\log z + \Gamma z + \varphi_1^0(z),\\ \psi_1(z) &= \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\log z + \Gamma' z + \psi_1^0(z),\end{aligned} \tag{78.12}$$

where $\varphi_1^0(z)$, $\psi_1^0(z)$ are functions, holomorphic in S (including the point $z = \infty$), and $\Re\,\Gamma$ and Γ' are given quantities; further, the imaginary part of Γ may be fixed arbitrarily. The quantities X and Y can be calculated beforehand, since the external stresses, acting on the boundary, are known.

By (78.3) these formulae may be written

$$\begin{aligned}\varphi(\zeta) &= -\frac{X+iY}{2\pi(1+\varkappa)}\log\zeta + \frac{\Gamma c}{\zeta} + \varphi_0(\zeta),\\ \psi(\zeta) &= -\frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\log\zeta + \frac{\Gamma' c}{\zeta} + \psi_0(\zeta),\end{aligned} \tag{78.13}$$

where $\varphi_0(\zeta)$, $\psi_0(\zeta)$ are functions, holomorphic inside and continuous up to γ. [Cf. (50.14) and (50.15), where it should not be forgotten that the region S has there been mapped on to the region outside the circle, while here it has been mapped on to the inside of γ.]

In the sequel, when solving the first fundamental problem, it will always be assumed that the imaginary part of Γ is zero, so that $\bar{\Gamma} = \Gamma$, i.e., it will be assumed that there is *no rotation at infinity*.

Substituting (78.13) in (78.5), it is seen that $\varphi_0(\zeta)$, $\psi_0(\zeta)$ must satisfy the same condition (78.5) as the functions $\varphi(\zeta)$, $\psi(\zeta)$, with the only difference that f has now to be replaced by f_0, where

$$f_0 = f - \frac{X+iY}{2\pi(1+\varkappa)}\log\sigma - \frac{\Gamma c}{\sigma} - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\left\{\frac{X-iY}{2\pi(1+\varkappa)}\cdot\frac{1}{\bar\sigma} - \frac{\Gamma\bar c}{\bar\sigma^2}\right\} + \\ + \frac{\varkappa(X+iY)}{2\pi(1+\varkappa)}\log\bar\sigma - \frac{\bar\Gamma'\bar c}{\bar\sigma}$$

or, noting that $\bar\sigma = 1/\sigma$,

$$f_0 = f - \frac{X+iY}{2\pi}\log\sigma - \frac{\Gamma c}{\sigma} - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\left\{\frac{X-iY}{2\pi(1+\varkappa)}\sigma - \Gamma\bar c\sigma^2\right\} - \bar\Gamma'\bar c\sigma. \tag{78.14}$$

In this expression $i\vartheta$ may be written instead of $\log \sigma$. It is easily seen that f_0 will be a single-valued, continuous function on γ the derivative of which with respect to ϑ satisfies the H condition, provided the given functions X_n and Y_n satisfy (as it has been assumed) that condition.

The single-valuedness of f_0 follows from the fact that f increases by $i(X + iY)$ and $\frac{(X + iY)\log \sigma}{2\pi}$ increases by the same quantity for every complete circuit of γ (in anti-clockwise direction which corresponds to a clockwise circuit of L, leaving the infinite region S on the left).

Thus $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ will be found from the same condition as $\varphi(\zeta)$ and $\psi(\zeta)$. Hence the more general case can always be reduced to that considered earlier.

3°. Next the *second fundamental problem* will be investigated. In this case the boundary condition has the form (§ 51)

$$\varkappa\varphi(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi'(\sigma)} - \overline{\psi(\sigma)} = 2\mu(g_1 + ig_2) = 2\mu g, \tag{78.15}$$

where g_1, g_2 are the known boundary values of the displacements u, v.

The almost complete analogy with the first fundamental problem is easily seen. Assuming at first that (in the case of infinite regions)

$$X = Y = 0, \quad \Gamma = \Gamma' = 0,$$

i.e., that $\varphi(\zeta)$ and $\psi(\zeta)$ are holomorphic, and proceeding as in the case of the first fundamental problem, one obtains the equation (analogous to 78.10)

$$\varkappa\varphi(\zeta) - \frac{1}{2\pi i}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\cdot\frac{\overline{\varphi'(\sigma)}\,d\sigma}{\sigma - \zeta} - \bar{a} = B(\zeta) \tag{78.16}$$

for all ζ inside γ, where

$$B(\zeta) = \frac{2\mu}{2\pi i}\int_\gamma \frac{g\,d\sigma}{\sigma - \zeta} \tag{78.17}$$

is a known function and $\bar{a}$ is the same constant as before.

Equation (78.16) is the functional equation which, in combination with the condition $\varphi(0) = 0$, completely determines the function $\varphi(\zeta)$ as well as the constant a; this will be proved in the next section. The remarks made earlier with regard to the determination of a will again be valid.

After the function $\varphi(\zeta)$ has been found, the function $\psi(\zeta)$ may be determined from the formula

$$\psi(\zeta) = -\frac{\mu}{\pi i} \frac{\bar{g}d\sigma}{\zeta} - \frac{1}{2\pi i}\int \frac{\omega(\sigma)}{\omega'(\sigma)} \cdot \frac{\varphi'(\sigma)d\sigma}{\sigma - \zeta}. \tag{78.18}$$

The case when X, Y, Γ, Γ' are not zero, but have arbitrarily fixed values (referring, of course, to the case of infinite regions) may be reduced to the preceding one in the same way as this was done for the first fundamental problem.

4°. The functional equation for $\varphi(\zeta)$ for the case of the ***mixed fundamental problem***, when the external forces are given for one part and the displacements for the remaining part of the boundary, may be constructed in an analogous manner. In this case the equation becomes somewhat more complicated and no consideration will be given to it here (cf. also end of § 79).

§ 79. Reduction to Fredholm equations. Existence theorems.

The proofs of the existence theorems, given in this section, have been taken, without essential changes from the Author's paper [11]. However, several simplifications and corrections of two elementary, but annoying blunders in the Author's reasoning in that paper have been introduced here. One of these had been kindly brought to the Author's notice by S. G. Mikhlin and the correction was already included in the first edition of this book. The other, despite its obvious and elementary nature (or rather because of it), remained unnoticed and was only discovered by the Author himself, while preparing the present edition. A short account of these proofs was also given in the Author's papers [9, 10].

The functional equations (78.10) and (78.16) represent a somewhat unusual type of integral equations which are, however, easily reduced to ordinary Fredholm equations of the second kind.

In the case of infinite regions, it will again be assumed that

$$X = Y = \Gamma = \Gamma' = 0,$$

since the problems can always be reduced to this case (cf. § 78).

A beginning will be made with the *first fundamental problem*. In order to reduce (78.10) to a Fredholm equation, it will be rewritten

$$\varphi(\zeta) + \frac{1}{2\pi i}\int \frac{\omega(\sigma) - \omega(\zeta)}{\omega'(\sigma)\,(\sigma - \zeta)}\,.\overline{\varphi'(\sigma)}d\sigma + k\omega(\zeta) + \bar{a} = A(\zeta), \tag{79.1}$$

where

$$k = \frac{\varphi'(0)}{\omega'(0)}; \tag{79.2}$$

it is easily seen, on the basis of (76.18), that the left-hand side of (79.1) is identical to the left-hand side of (78.10). In the case of infinite regions $\omega'(0) = \infty$ and hence $k = 0$.

Differentiating (79.1) with respect to ζ, one obtains

$$\varphi'(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\partial}{\partial\zeta}\left\{\frac{\omega(\sigma) - \omega(\zeta)}{\sigma - \zeta}\right\} \frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\, d\sigma + k\omega'(\zeta) = A'(\zeta), \tag{79.3}$$

whence, letting ζ tend to an arbitrary point σ_0 of γ, one finds

$$\varphi'(\sigma_0) + \frac{1}{2\pi i}\int_\gamma \frac{\partial}{\partial\sigma_0}\left\{\frac{\omega(\sigma) - \omega(\sigma_0)}{\sigma - \sigma_0}\right\} \frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\, d\sigma + k\omega'(\sigma_0) = A'(\sigma_0). \tag{79.4}$$

It is readily seen that this transition to the limit is completely justified under the conditions postulated earlier for the function f and the contour L. In fact, it has been assumed that f has a first derivative satisfying the H condition. Therefore $A'(\zeta)$ will be a function, continuous inside and up to γ; the function $A'(\sigma_0)$ in (79.4) denotes the boundary value of $A'(\zeta)$.

Further, the conditions assumed with regard to L ensure the continuity of $\omega(\zeta)$, $\omega'(\zeta)$, $\omega''(\zeta)$ up to γ, with the exclusion of the point $\zeta = 0$ in the case of infinite regions, and also that $\omega'(\zeta) \neq 0$; hence it follows that the function

$$K(\zeta, \sigma) = \frac{1}{\overline{\omega'(\sigma)}} \frac{\partial}{\partial\zeta} \frac{\omega(\sigma) - \omega(\zeta)}{\sigma - \zeta} = \frac{\omega(\sigma) - \omega(\zeta) - (\sigma - \zeta)\omega'(\zeta)}{\overline{\omega'(\sigma)}\,(\sigma - \zeta)^2} \tag{79.5}$$

is continuous for all values of σ and ζ inside and on γ, except for $\sigma = 0$, $\zeta = 0$ in the case of infinite regions.

In fact, by Taylor's formula with the remainder term in the form of a definite integral, one has

$$\omega(\sigma) - \omega(\zeta) - \omega'(\zeta)\,(\sigma - \zeta) = \int_\zeta^\sigma \omega''(t)\,(\sigma - t)dt,$$

where the integral may be taken along the segment of the straight line connecting

σ and ζ. Putting $t = \sigma - \lambda(\sigma - \zeta)$, one obtains

$$\omega(\sigma) - \omega(\zeta) - \omega'(\zeta)(\sigma - \zeta) = (\sigma - \zeta)^2 \int_0^1 \omega''[\sigma - \lambda(\sigma - \zeta)]\lambda d\lambda,$$

and so the continuity of $K(\zeta, \sigma)$ is proved.

Finally, by supposition (cf. § 78), the functions $\varphi(\zeta)$, $\varphi'(\zeta)$, $\psi(\zeta)$ are continuous up to the boundary.

The equation (79.4) may also be deduced from the equations obtained in a different way by V. A. Fok [1, 2] who, however, restricted consideration to finite regions. (cf. V. A. Fok and N. I. Muskhelishvili [1]).

The preceding formulae refer to the cases of finite as well as of infinite regions. However, in the latter case, they may be given a somewhat different form which will be more convenient. First of all, in that case $k = 0$. Hence (79.1) becomes

$$\varphi(\zeta) + \frac{1}{2\pi i}\int \frac{\omega(\sigma) - \omega(\zeta)}{\overline{\omega'(\sigma)}\,(\sigma - \zeta)}\,\overline{\varphi'(\sigma)}d\sigma + \bar{a} = A(\zeta). \qquad (79.1')$$

Further, noting that in accordance with the imposed conditions

$$\omega(\zeta) = \frac{c}{\zeta} + \omega_0(\zeta),$$

where $\omega_0(\zeta)$ is holomorphic inside γ, one has

$$\frac{\omega(\sigma) - \omega(\zeta)}{\sigma - \zeta} = \frac{\omega_0(\sigma) - \omega_0(\zeta)}{\sigma - \zeta} - \frac{c}{\sigma\zeta}.$$

Substituting this expression in (79.1'), one finds

$$\varphi(\zeta) + \frac{1}{2\pi i}\int \frac{\omega_0(\sigma) - \omega_0(\zeta)}{\overline{\omega'(\sigma)}\,(\sigma - \zeta)}\,\overline{\varphi'(\sigma)}d\sigma + \bar{a} = A(\zeta) \qquad (79.1'')$$

because, as is easily verified,

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\frac{d\sigma}{\sigma} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\,d\vartheta = 0;$$

in fact, the expression conjugate complex to the last integral

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\varphi'(\sigma)}{\omega'(\sigma)}\,d\vartheta = \frac{1}{2\pi i}\int \frac{\varphi'(\sigma)}{\omega'(\sigma)}\frac{d\sigma}{\sigma}$$

is zero, because $\varphi'(\sigma)/\sigma\omega'(\sigma)$ is obviously the boundary value of a function, holomorphic inside γ.

Differentiating (79.1'')

$$\varphi'(\zeta) + \frac{1}{2\pi i}\int_\gamma \frac{\partial}{\partial\zeta}\left\{\frac{\omega_0(\sigma) - \omega_0(\zeta)}{\overline{\omega'(\sigma)}\,(\sigma - \zeta)}\right\}\overline{\varphi'(\sigma)}d\sigma = A'(\zeta) \qquad (79.1''')$$

and taking, as before, the limit $\zeta \to \sigma_0$, one obtains the integral equation

$$\varphi'(\sigma_0) + \frac{1}{2\pi i}\int_\gamma \frac{\partial}{\partial\sigma_0}\left\{\frac{\omega_0(\sigma) - \omega_0(\sigma_0)}{\sigma - \sigma_0}\right\}\frac{\overline{\varphi'(\sigma)}}{\overline{\omega'(\sigma)}}\,d\sigma = A'(\sigma_0). \qquad (79.4')$$

Thus equation (79.4) which will now be written

$$\varphi'(\sigma_0) + \frac{1}{2\pi i}\int_\gamma K(\sigma_0, \sigma)\,\varphi'(\sigma)d\sigma + k\omega'(\sigma_0) = A'(\sigma_0) \qquad (79.6)$$

and which is applicable to both cases may in the case of infinite regions be replaced by (79.4'), i.e.,

$$\varphi'(\sigma_0) + \frac{1}{2\pi i}\int K_0(\sigma_0, \sigma)\,\overline{\varphi'(\sigma)}\,d\sigma = A'(\sigma_0), \qquad (79.6')$$

where

$$K_0(\zeta, \sigma) = \frac{1}{\overline{\omega'(\sigma)}}\frac{\partial}{\partial\zeta}\frac{\omega_0(\sigma) - \omega_0(\zeta)}{\sigma - \zeta} \qquad (79.5')$$

The integral equations (79.6) and (79.6') will now be investigated and a beginning will be made with the *case of infinite regions.*

Substituting in (79.6')

$$\varphi'(\sigma) = \varphi_1' + i\varphi_2'$$

and

$$K_0(\sigma_0, \sigma) = K_1 + iK_2$$

and separating real and imaginary parts, one obtains two real Fredholm equations; this system may be reduced by ordinary means to a single equation. This equation will not be written down, since it is sufficient to know that (79.6') reduces to a single Fredholm equation (of the second kind).

Suppose now that (79.6') has the (continuous) solution $\varphi'(\sigma)$. By

substituting this solution in the second term on the left-hand side of (79.1‴) one finds some function $\varphi'(\zeta)$ which is seen to be holomorphic inside γ and to have a definite boundary value on γ. Further, it is also seen that this boundary value coincides with the function $\varphi'(\sigma)$, figuring in the second term on the left-hand side of (79.1‴), or, in other words, that the holomorphic function $\varphi'(\zeta)$, thus determined, actually solves the functional equation (79.1‴). In fact, letting $\zeta \to \sigma_0$ in (79.1‴) and taking into consideration that $\varphi'(\sigma)$ in the second term on the left-hand side satisfies (79.4′), it is verified that $\varphi'(\zeta) \to \varphi'(\sigma_0)$. Integrating $\varphi'(\zeta)$ it is deduced that the holomorphic function $\varphi(\zeta)$ satisfies (79.1″) and hence also the original equation (79.1′), the constant $\bar{a}$ in this equation being completely determined by the condition $\varphi(0) = 0$.

Once $\varphi(\zeta)$ has been found the function $\psi(\zeta)$ is determined by (79.8), viz.,

$$\psi(\zeta) = \frac{1}{2\pi i}\int \frac{\bar{f}\,d\sigma}{\sigma - \zeta} - \frac{1}{2\pi i}\int \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\frac{\varphi'(\sigma)d\sigma}{(\sigma - \zeta)}. \tag{79.7}$$

Since $\varphi(\zeta)$, $\psi(\zeta)$ and $\varphi'(\zeta)$ are continuous up to γ, the functions $\varphi(\zeta)$ and $\psi(\zeta)$ give a regular solution of the problem. Thus a definite regular solution of the problem corresponds to every (continuous) solution $\varphi'(\sigma)$ of the integral equation (79.6′).

For the determination of the constant $\bar{a}$, referred to above, it is sufficient to put $\zeta = 0$ in (79.1″), or, better still, in (78.10) which is equivalent to (79.1″); in this way one obtains

$$\bar{a} = A(0) - \frac{1}{2\pi i}\int \frac{\quad}{\sigma\overline{\omega'(\sigma)}}\,\overline{\varphi'(\sigma)}d\sigma\,.$$

The properties of $\varphi'(\zeta)$, stated above, have been discussed earlier and those of $\varphi(\zeta)$ are then obvious; those of $\psi(\zeta)$ follow from the same reasoning and from the fact that (79.7) may be written

$$\psi(\zeta) = \frac{1}{2\pi i}\int \frac{\bar{f}\,d\sigma}{\sigma - \zeta} - \frac{1}{2\pi i}\int \frac{\overline{\omega(\sigma)} - \overline{\omega(\zeta)}}{\omega'(\sigma)\,(\sigma - \zeta)}\,\varphi'(\sigma)d\sigma - \frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\,\varphi'(\zeta). \tag{79.7′}$$

It will now be shown that the integral equation (79.6′) has always a unique solution. It is known that for this purpose it will be sufficient to prove that the corresponding homogeneous equation

$$\varphi'(\sigma_0) + \frac{1}{2\pi i}\int K_0(\sigma_0, \sigma)\overline{\varphi'(\sigma)}d\sigma = 0 \tag{79.6″}$$

has no non-zero solution. This is almost obvious on the basis of the earlier remarks. In fact, if this equation had a solution different from zero, one could by means of this solution obtain a solution of the fundamental problem for the case $f = 0$ with $\varphi'(\zeta) \neq 0$. However, this would mean that a non-zero solution exists for the case when no external forces act on the boundary and that the corresponding internal stresses in the body differ from zero. The impossibility of this is proved by the uniqueness theorem of § 40 (cf. also end of § 42). Thus the existence of the solution of the first fundamental problem has been proved for infinite regions.

The case of finite regions will be considered next. For the time being it will be assumed that the constant k in (79.1) has been fixed arbitrarily. In order to remove the term $k\omega(\zeta)$ in this equation, introduce the transformation

$$\varphi(\zeta) = -k\omega(\zeta) + \varphi_0(\zeta), \tag{79.8}$$

where $\varphi_0(\zeta)$ is a new unknown holomorphic function. One thus obtains from (79.1) the equation

$$\varphi_0(\zeta) + \frac{1}{2\pi i}\int \frac{\omega(\sigma) - \omega(\zeta)}{\overline{\omega'(\sigma)}\,(\sigma - \zeta)}\,\overline{\varphi_0'(\sigma)}\,d\sigma + \bar{a} = A(\zeta) \tag{79.9}$$

from which follows, as before,

$$\varphi_0'(\zeta) + \frac{1}{2\pi i}\int_\gamma K(\zeta, \sigma)\overline{\varphi_0'(\sigma)}\,d\sigma = A'(\zeta) \tag{79.10}$$

and, for $\zeta \to \sigma_0$,

$$\varphi_0'(\sigma_0) + \frac{1}{2\pi i}\int_\gamma K(\sigma_0, \sigma)\overline{\varphi_0'(\sigma)}\,d\sigma = A'(\sigma_0). \tag{79.11}$$

As in the case of (79.6′), this equation may be reduced to a system of two Fredholm integral equations from which one finally obtains a Fredholm equation (of the second kind). It will be shown below that (79.11) always has a (unique) solution.

First, however, consider the construction of the solution of the original problem, once any (continuous) solution $\varphi_0'(\sigma)$ of (79.11) has been found.

Substituting this solution in the second term on the left-hand side of (79.9) and taking into account the condition $\varphi_0(0) = 0$ which follows from (79.8) for $\varphi(0) = \omega(0) = 0$, one obtains the function $\varphi_0(\zeta)$ as well

as the constant $\bar{a}$. The function $\varphi(\zeta)$ will then be given by (79.8) and it will be the solution of the functional equation (79.1) for the given value of k. In order that the function $\varphi(\zeta)$, determined in this manner, lead to the solution of the original problem, it is, however, necessary and sufficient to select the value of k to satisfy (79.2), i.e.,

$$k = \frac{\varphi'(0)}{\omega'(0)}$$

or, using (79.8),

$$k + \bar{k} = \frac{\varphi_0'(0)}{\omega'(0)}. \tag{79.12}$$

This is obviously possible only if

$$\frac{\varphi_0'(0)}{\omega'(0)} = \text{a real number.} \tag{79.13}$$

Let it be assumed that (79.13) is fulfilled. Then (79.12) determines the real part of k. By fixing the imaginary part of k arbitrarily a definite expression will be obtained for the unknown function $\varphi(\zeta)$; after determining the corresponding function $\psi(\zeta)$ from (79.8) one finally finds a certain regular solution of the original problem.

The meaning of the condition (79.13) is easily explained. In fact, writing

$$\varphi_0(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)}\overline{\varphi_0'(\sigma)} + \overline{\psi_0(\sigma)} = f_1 + if_2 + \overline{\frac{\varphi_0'(0)}{\omega'(0)}}\,\omega(\sigma) \tag{79.14}$$

or

$$\overline{\varphi_0(\sigma)} + \frac{\omega(\sigma)}{\omega'(\sigma)}\varphi_0'(\sigma) + \psi_0(\sigma) = f_1 - if_2 + \frac{\varphi_0'(0)}{\omega'(0)}\,\overline{\omega(\sigma)}, \tag{79.15}$$

it is seen directly that (79.9) represents the condition for $\psi_0(\sigma)$, defined by (79.15), to be the boundary value of some function $\psi_0(\zeta)$, holomorphic inside γ. Thus the relations (79.14) and (79.15) must hold, where $\psi_0(\sigma)$ is the boundary value of $\psi_0(\zeta)$, holomorphic inside and continuous up to γ.

Multiplying (79.14) and (79.15) by $\overline{\omega'(\sigma)}d\bar{\sigma} = d\bar{z}$ and $\omega'(\sigma)d\sigma = dz$ respectively, adding, integrating around γ and noting that

$$\varphi_0(\sigma)\overline{\omega'(\sigma)}d\bar{\sigma} + \int_\gamma \overline{\omega(\sigma)}\varphi_0'(\sigma)d\sigma = \int_\gamma d[\varphi_0(\sigma)\overline{\omega(\sigma)}] = 0,$$

$$\overline{\varphi_0(\sigma)}\omega'(\sigma)d\sigma + \int_\gamma \omega(\sigma)\overline{\varphi_0'(\sigma)}d\bar{\sigma} = \int_\gamma d[\overline{\varphi_0(\sigma)}\omega(\sigma)] = 0,$$

$$\int \overline{\psi_0(\sigma)}\overline{\omega'(\sigma)}d\bar{\sigma} = \int \psi_0(\sigma)\omega'(\sigma)d\sigma = 0,$$

$$\int \omega(\sigma)\overline{\omega'(\sigma)}d\bar{\sigma} = \int zd\bar{z}, \qquad \omega(\sigma)\omega'(\sigma)d\sigma = \int \bar{z}dz,$$

$$\int \bar{z}dz = -\int zd\bar{z},$$

one obtains

$$0 = 2\int (f_1dx + f_2dy) + \left\{\frac{\overline{\varphi_0'(0)}}{\overline{\omega'(0)}} - \frac{\varphi_0'(0)}{\omega'(0)}\right\}\int zd\bar{z}.$$

But

$$\int_L zd\bar{z} = \int_L (xdx + ydy) + i\int_L (ydx - xdy) = -2iS,$$

where S is the area of the region inside L. Hence

$$\int_L (f_1dx + f_2dy) = iS\left\{\frac{\overline{\varphi_0'(0)}}{\overline{\omega'(0)}} - \frac{\varphi_0'(0)}{\omega'(0)}\right\}. \tag{79.16}$$

The expression in the curly brackets differs from the imaginary part of $\varphi_0'(0)/\omega'(0)$ only by a factor, and hence (79.13) is equivalent to

$$\int (f_1dx + f_2dy) = 0 \tag{79.17}$$

which is the *condition for the vanishing of the resultant moment of the external stresses* applied to L.

Now equation (79.11) will be considered and it will be shown that it has a unique solution. For this purpose the corresponding homogeneous

equation

$$\varphi_0'(\sigma_0) + \frac{1}{2\pi i}\int_\gamma K(\sigma_0, \sigma)\overline{\varphi_0'(\sigma)}d\sigma = 0 \qquad (79.11')$$

will be studied. This equation is obtained, if one wants to solve the first fundamental problem in the above-stated manner in the absence of external stresses, i.e., for $f_1 = f_2 = 0$ on L. As under this condition (79.17) will obviously be satisfied, the condition (79.13) will be fulfilled for any solution $\varphi_0'(\sigma)$ of (79.11′). Selecting the real part of k in accordance with (79.12) and fixing its imaginary part arbitrarily, the solution of the first fundamental problem for $f_1 = f_2 = 0$ may be constructed starting from $\varphi_0'(\sigma)$. If this function does not vanish everywhere on γ, the solution constructed in this manner will not correspond to the case of absence of stresses. In fact, the function $\varphi(\zeta)$ will be given by $\varphi(\zeta) = -k\omega(\zeta) + \varphi_0(\zeta)$ and, in the absence of stresses, one should have $\varphi(\zeta) = Ci\omega(\zeta)$. Hence, in this case, $\varphi_0(\zeta) = m\omega(\zeta)$, where m is some constant. Substituting this expression in (79.9) with $A(\zeta) = 0$, one obviously finds $m\omega(\zeta) = \text{const}$ which is only possible for $m = 0$, i.e., $\varphi_0(\zeta) = 0$. Thus the presence of a non-zero solution of (79.11′) implies a solution of the first fundamental problem, giving the state of stress in the absence of external forces, which is impossible by the uniqueness theorem. In this way it has been shown that the homogeneous equation, corresponding to (79.11), has no solution which is not identically zero, and therefore (79.11) has one and only one solution.

Solving (79.11) and assuming (79.17) to be satisfied, the function $\varphi(\zeta)$ will be found from (79.8) with the real part of k chosen in agreement with (79.12); the function $\psi(\zeta)$ can then be determined by (78.8) and the solution of the original problem obtained. The imaginary part of k remains arbitrary, as was to be expected, since a term of the form $Ci\omega(\zeta)$ in the expression for $\varphi(\zeta)$, where C is a real constant, does not affect the stress distribution.

It will be remembered that (79.17) is the condition that the resultant moment of the external forces must vanish. The condition for the vanishing of the resultant force vector is ensured by the continuity of the functions f_1 and f_2 on L; that is the reason why it does not appear in explicit form.

Thus the existence of the solution of the first fundamental problem has also been proved for the case of finite regions. At the same time

(theoretical) methods of solution have been given for this problem for the cases of finite as well as of infinite regions.

Next consider the *second fundamental problem*. This problem has been seen to reduce to the solution of the equation (78.16) which is quite analogous to the equation obtained for the first fundamental problem. The methods of solution of the first and second problems are so alike that there is no point in repeating the reasoning.

A certain difference occurs only in the case of the problem for finite regions; in fact, one will have now instead of (79.8)

$$\varphi(\zeta) = \frac{k}{\varkappa}\,\omega(\zeta) + \varphi_0(\zeta) \tag{79.18}$$

and k will have to be determined from the equation

$$k - \frac{\bar{k}}{\varkappa} = \frac{\overline{\varphi_0'(0)}}{\overline{\omega'(0)}} \tag{79.19}$$

which gives a definite value for k (remembering that $\varkappa > 1$) without any additional condition for the existence of the solution.

Thus the existence of the solution of the second fundamental problem has been proved and at the same time a (theoretical) method has been obtained for its solution.

The *mixed fundamental problem* may be solved by methods analogous to those above. In this case the stated procedure does not lead directly to a Fredholm equation, but to a so-called singular integral equation which is then easily reduced to a Fredholm equation. The mixed problem has been solved in this way by D. I. Sherman [10]. The solution may be considerably simplified, if use is made of the general theory of singular equations which has been developed recently.

A more detailed study of the integral equations for the first and second fundamental problems, obtained above, was likewise presented by D. I. Sherman [7]. In fact, Sherman introduced into these equations a parameter λ (not to be confused with the Lamé constant), similar to that occurring in the general theory of Fredholm equations, and proved that all the characteristic values of this parameter are real and distributed inside the region $-1 < \lambda \leqslant 1$. This fact is of practical value, since it shows that the above integral equations may be solved by iteration methods, i.e., that the Neumann series will converge for those values of λ to which these equations correspond; in fact, the integral equations

for the first and second fundamental problems correspond to the values $\lambda = 1$ and $\lambda = -1/\varkappa$ respectively (remembering that $\varkappa > 1$).

Apart from these results, Sherman deduced in the above paper a number of other results which are of independent interest.

In the later chapters of this Part remarks will be made with regard to the existence theorems for regions of more general shape and also with regard to some other general methods of solution of the fundamental problems.

§ 79a. On some other applications of the preceding integral equations. The integral equations of § 79 may also be applied to certain other important problems of the theory of elasticity, e.g. the (approximate) theory of bending of plates, loaded by forces normal to their plane. It has already been stated above that the case of *plates clamped along their edges* may be reduced to the so-called fundamental bi-harmonic problem, i.e., to the same boundary problem as the first fundamental problem of plane elasticity.

The case of *plates with free edges* has been found to reduce to the same boundary problem as the second fundamental problem of plane elasticity; the only difference is that the constant $\varkappa$ has to be replaced by some other constant, likewise larger than unity. This has been proved by S. G. Lekhnitzky [3[and, later on and independently, by I. N. Vekua [3].

CHAPTER 15

SOLUTION OF THE FUNDAMENTAL PROBLEMS FOR REGIONS MAPPED ON TO A CIRCLE BY RATIONAL FUNCTIONS. EXTENSION TO APPROXIMATE SOLUTION FOR REGIONS OF GENERAL SHAPE

As stated earlier, Cauchy type integrals provide the means for obtaining theoretical as well as practical solutions of the fundamental problems for certain fairly wide classes of regions. The starting points for this work are the formulae (78.10) or (78.16) or analogous formulae to be stated below. The case for which the mapping function $\omega(\zeta)$ is rational is particularly simple, since, as will be shown in this chapter, the solution in this case is obtained by quite elementary means. However, for the sake of clarity, a beginning will be made with the direct solution of the problems for some very simple regions.

The major part of the results stated in this chapter were contained in the Author's papers [4, 5, 7, 8.]

In their two papers [1, 2] D. M. Volkov and A. A. Nazarov gave a method which apparently permits solution by elementary means in the case of a wider class of regions. However, this class has not been specified by the authors with sufficient exactness, so that it cannot be stated beforehand in what cases, in addition to those stated by the Author here, a solution may be obtained by elementary means. In fact, in order to state the cases, where one can certainly obtain elementary solutions by applying completely definite methods, the Author, in those of his papers which were devoted to elementary methods, has limited consideration to cases where $\omega(\zeta)$ is a rational function. In addition, he should indicate that he does not agree with Volkov and Nazarov in their claim that their method leads to simpler calculations; cf. § 87*a*.

§ 80. Solution of the first fundamental problem for the circle.

It has already been stated in § 54 that many solutions of this problem are known. Among those mention will be made only of the solutions by G. V. Kolosov [1, 2], G. V. Kolosov and I. N. Muskhelishvili [1] and G. V. Kolosov [5] which was also published in 1931. In § 54 this problem was solved by the use of series. Cauchy type integrals achieve the object more rapidly and give a solution which is more convenient in applications.

Let R be the radius of the circle S with circumference L. In the present case

$$z = \omega(\zeta) = R\zeta, \tag{80.1}$$

where here and below the notation of § 78 will be used. In particular, γ will denote the unit circle $|\zeta| = 1$, $\sigma = e^{i\vartheta}$ a point on this circle.

The boundary condition now becomes

$$\varphi(\sigma) + \sigma\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = f_1 + if_2 = f \tag{80.2}$$

or

$$\overline{\varphi(\sigma)} + \bar{\sigma}\varphi'(\sigma) + \psi(\sigma) = f_1 - if_2 = \bar{f}. \tag{80.3}$$

Expressing the fact that the right-hand side of

$$\psi(\sigma) = \bar{f} - \overline{\varphi(\sigma)} - \bar{\sigma}\varphi'(\sigma) \tag{80.3'}$$

must be the boundary value of some function $\psi(\zeta)$, holomorphic inside γ, one obtains by (76.12)

$$\frac{1}{2\pi i}\int\limits_{\gamma} \frac{f\,d\sigma}{\sigma - \zeta} - \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\varphi(\sigma)d\sigma}{\sigma - \zeta} - \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\sigma\overline{\varphi'(\sigma)}d\sigma}{\sigma - \zeta} = \bar{a},$$

where $\bar{a}$ is some constant, and actually $a = \psi(0)$. Thus

$$\varphi(\zeta) + \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\sigma\overline{\varphi'(\sigma)}d\sigma}{\sigma - \zeta} + \bar{a} = \frac{1}{2\pi i}\int \frac{f\,d\sigma}{\sigma - \zeta}. \tag{80.4}$$

This equation is nothing else but the functional equation (78.10) for the case $\omega(\zeta) = R\zeta$.

In the case considered here the functional equation may be solved in a simple manner without transition to an integral equation, since the integral on the left-hand side can be immediately calculated in finite form.

In fact, consider the first three terms of the expansion for $\varphi(\zeta)$

$$\varphi(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + \ldots \tag{80.5}$$

from which follows

$$\varphi'(\zeta) = a_1 + 2a_2\zeta + \ldots,$$

and hence, by (76.17),

$$\frac{1}{2\pi i}\int\limits_{\gamma} \frac{\sigma\overline{\varphi'(\sigma)}d\sigma}{\sigma - \zeta} = \bar{a}_1\zeta + 2\bar{a}_2;$$

(80.4) may thus be written

$$\varphi(\zeta) + \bar{a}_1\zeta + 2\bar{a}_2 + \bar{a} = \frac{1}{2\pi i}\int_\gamma \frac{f\,d\sigma}{\sigma - \zeta}. \tag{80.6}$$

This last relation determines $\varphi(\zeta)$ apart from $\bar{a}_1\zeta + 2\bar{a}_2 + \bar{a}$, i.e., the unknown constants a_1, a_2, $\bar{a}$ have still to be found. For this purpose one has to express the condition that a_1, a_2 are coefficients of the expansion (80.5), since, if this condition is not fulfilled, the function $\varphi(\zeta)$, as determined by (80.5), will obviously not satisfy (80.4). In addition, one must put for definiteness

$$\varphi(0) = a_0 = 0. \tag{80.5'}$$

In order to express these conditions, it is sufficient to put $\zeta = 0$ in (80.6) and likewise in the equations, obtained from (80.6) by differentiating once and twice with respect to ζ, and to take into consideration that, by definition, $a_1 = \varphi'(0)$, $2a_2 = \varphi''(0)$. This process gives

$$2\bar{a}_2 + \bar{a} = \frac{1}{2\pi i}\int_\gamma f\frac{d\sigma}{\sigma}, \tag{a}$$

$$a_1 + \bar{a}_1 = \frac{1}{2\pi i}\int f\frac{d\sigma}{\sigma^2}, \tag{b}$$

$$a_2 = \frac{1}{2\pi i}\int_\gamma f\frac{d\sigma}{\sigma^3}. \tag{c}$$

The relations (a) — (c) may also be obtained by substituting in (80.6) the expansions

$$\varphi(\zeta) = a_1\zeta + a_2\zeta^2 + \ldots$$

and

$$\frac{1}{2\pi i}\int_\gamma \frac{f d\sigma}{\sigma - \zeta} = \frac{1}{2\pi i}\int_\gamma \frac{f d\sigma}{\sigma\left(1 - \frac{\zeta}{\sigma}\right)} = \frac{1}{2\pi i}\int_\gamma f\left(1 + \frac{\zeta}{\sigma} + \frac{\zeta^2}{\sigma^2} + \ldots\right)\frac{d\sigma}{\sigma} =$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f d\sigma}{\sigma} + \frac{\zeta}{2\pi i}\int_\gamma \frac{f d\sigma}{\sigma^2} + \frac{\zeta^2}{2\pi i}\int_\gamma \frac{f d\sigma}{\sigma^3} + \ldots$$

and by comparing coefficients of ζ^0, ζ^1, ζ^2.

The formula (c) is actually unnecessary (as was to be expected), since there is no need to know the quantity a_2 separately; it is sufficient to

know the sum $2a_2 + a$ which alone occurs in (80.6), determining $\varphi(\zeta)$.

Substituting $f = f_1 + if_2$, $\sigma = e^{i\vartheta}$, formula (b) gives

$$a_1 + \bar{a}_1 = \frac{1}{2\pi}\int_0^{2\pi} (f_1 + if_2)e^{-i\vartheta}d\vartheta.$$

This condition can only be fulfilled when the right-hand side is real, i.e., when

$$\int_0^{2\pi} (-f_1 \sin\vartheta + f_2 \cos\vartheta)d\vartheta = 0, \tag{80.7}$$

which expresses the necessity for the vanishing of the resultant moment of the external forces [cf. (54.3)]. If it is satisfied, the real part of a_1 is completely determined by (b), while its imaginary part, as expected, remains arbitrary; putting, for definiteness, $\mathfrak{J}(a) = 0$, one obtains from (b)

$$a_1 = \bar{a}_1 = \frac{1}{4\pi i}\int_\gamma \frac{f\,d\sigma}{\sigma^2}. \tag{80.8}$$

Finally, substituting in (80.6) for $2\bar{a}_2 + \bar{a}$ from (a), one finds

$$\varphi(\zeta) = \frac{1}{2\pi i}\int \frac{f\,d\sigma}{\sigma - \zeta} - \bar{a}_1\zeta - \frac{1}{2\pi i}\int \frac{f\,d\sigma}{\sigma}, \tag{80.9}$$

where $\bar{a}_1$ is given by (80.8).

Having found $\varphi(\zeta)$, the function $\psi(\zeta)$ may be immediately determined, because its boundary value $\psi(\sigma)$ is given by (80.3′). Determining $\psi(\zeta)$ from Cauchy's formula and taking into consideration that [cf. (76.18) and (70.3)]

$$\frac{1}{2\pi i}\int \frac{\varphi(\sigma)d\sigma}{\sigma - \zeta} = \varphi(0) = 0,$$

$$\frac{1}{2\pi i}\int \frac{\bar{\sigma}\varphi'(\sigma)d\sigma}{\sigma - \zeta} = \frac{1}{2\pi i}\int \frac{\varphi'(\sigma)}{\sigma}\cdot\frac{d\sigma}{\sigma - \zeta} = \frac{\varphi'(\zeta)}{\zeta} - \frac{a_1}{\zeta},$$

one obtains

$$\psi(\zeta) = \frac{1}{2\pi i}\int \frac{\bar{f}\,d\sigma}{\sigma - \zeta} - \frac{\varphi'(\zeta)}{\zeta} + \frac{a_1}{\zeta} \tag{80.10}$$

It is easily seen that the solution obtained will be regular (in the sense of § 42), if the function f given on L has a derivative satisfying the H condition.

Thus the problem has been solved. It will be noted that the last term on the right-hand side of (80.9) may be omitted, because a constant term in the expression for $\varphi(\zeta)$ does not influence the stress distribution. This constant has only been calculated with the "fundamental biharmonic problem" (§ 40) in mind, since the constant term has a meaning in that case.

Omitting the above-mentioned constant term, one has instead of (80.9) the simpler formula

$$\varphi(\zeta) = \frac{1}{2\pi i}\int \frac{f\,d\sigma}{\sigma - \zeta} - \bar{a}_1\zeta. \tag{80.9'}$$

In this case the boundary value of $\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}$ may differ from $f = f_1 + if_2$ by a constant term. If one takes (80.9), this boundary value will be exactly equal to f.

The solution deduced above is very convenient for applications, as will become apparent from the examples considered in the next section.

§ 80a. Examples.

1°. Circular disc under concentrated forces, applied to its boundary.

This problem was first solved by H. Hertz in 1883 and studied in detail by J. H. Michell [2], using methods quite different from those used in this book. Cf. also A. E. H. Love [1] § 155.

Let the concentrated forces

$$(X_1, Y_1), \quad (X_2, Y_2), \ \ldots, \quad (X_n, Y_n)$$

act at the points

$$z_1 = Re^{i\alpha_1}, \quad z_2 = Re^{i\alpha_2}, \ldots, z_n = Re^{i\alpha_n}$$
$$(0 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_n < 2\pi)$$

of the edge of the circular disc. The points

$$\sigma_1 = e^{i\alpha_1}, \ \ldots, \quad \sigma_n = e^{i\alpha_n}$$

correspond to these points in the ζ plane.

Under these conditions the expression $f = f_1 + if_2$ will be constant on each of the arcs $\sigma_1\sigma_2, \sigma_2\sigma_3, \ldots, \sigma_n\sigma_1$ (because these arcs are free from

external forces), but it will change discontinuously by $i(X_k + iY_k)$ for a passage through the point σ_k (cf. § 43). While it has been assumed in the earlier reasoning that the function $f = f_1 + if_2$, given on L, is continuous (and even has a derivative satisfying the H condition), it is easily verified that even in the present case the method leads to the correct solution of the problem.

For example, let $f = 0$ on $\sigma_n\sigma_1$. Then one will have on the arcs $\sigma_1\sigma_2$, $\sigma_2\sigma_3$, etc. that $f = i(X_1 + iY_1)$, $f = i(X_1 + iY_1) + i(X_2 + iY_2)$, etc. respectively. In order that f attain its original value 0 on $\sigma_n\sigma_1$ after one complete circuit, one must, obviously, have the conditions $X_1 + X_2 + \ldots + X_n = 0$, $Y_1 + Y_2 + \ldots + Y_n = 0$, i.e., the resultant vector of the applied forces must vanish. This condition is necessary, because $\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}$ must be single-valued inside γ, since the region under consideration is simply-connected.

Further,

$$\frac{1}{2\pi i}\int_\gamma \frac{f\,d\sigma}{\sigma-\zeta} = \frac{1}{2\pi i}\int_{\sigma_1}^{\sigma_2} + \ldots + \frac{1}{2\pi i}\int_{\sigma_{n-1}}^{\sigma_n} + \frac{1}{2\pi i}\int_{\sigma_n}^{\sigma_1} =$$

$$= \frac{X_1 + iY_1}{2\pi}\log\frac{\sigma_2-\zeta}{\sigma_1-\zeta} + \frac{(X_1 + iY_1) + (X_2 + iY_2)}{2\pi}\log\frac{\sigma_3-\zeta}{\sigma_2-\zeta} +$$

$$+ \ldots + \frac{(X_1 + iY_1) + \ldots + (X_n + iY_n)}{2\pi}\log\frac{\sigma_1-\zeta}{\sigma_n-\zeta}.$$

The last term is, of course, zero, but it has been written down for the sake of symmetry. After some obvious manipulations one finds

$$\frac{1}{2\pi i}\int \frac{f\,d\sigma}{\sigma-\zeta} = \frac{1}{2\pi}\{(X_1 + iY_1)\log(\sigma_1-\zeta) + (X_2 + iY_2)\log(\sigma_2-\zeta) + \ldots + (X_n + iY_n)\log(\sigma_n-\zeta)\}.$$

Similarly,

$$\frac{1}{2\pi i}\int \frac{\bar{f}\,d\sigma}{\sigma-\zeta} = \frac{1}{2\pi}\{(X_1 - iY_1)\log(\sigma_1-\zeta) + \ldots + (X_n - iY_n)\log(\sigma_n-\zeta)\}.$$

Finally, the constant a_1 has to be determined. For this purpose use may be made of (80.8); however, it will be simpler to adopt the following procedure. Since, by supposition, $a_1 = \bar{a}_1 = \varphi'(0)$, one finds from (80.9), replacing the integrals on the right-hand side by the expressions above,

integrating with respect to ζ and putting $\zeta = 0$,

$$2a_1 = \frac{1}{2\pi} \sum_{k=1}^{n} \frac{X_k + iY_k}{\sigma_k} = \frac{1}{2\pi} \sum_{k=1}^{n} (X_k + iY_k)\bar{\sigma}_k.$$

In order that a_1 may be real, the right-hand side of this expression must be a real quantity. This condition is easily seen to lead to

$$\sum_{k=1}^{n} (x_k Y_k - y_k X_k) = 0,$$

where $x_k + iy_k = z_k = \sigma_k R$, i.e., to the vanishing of the resultant moment.

If the above conditions (for the resultant force and moment) are satisfied, the solution of the problem is given by (80.9′) and (80.10)

$$\varphi(\zeta) = -\frac{1}{2\pi} \sum_{k=1}^{n} (X_k + iY_k) \log(\sigma_k - \zeta) - \frac{\zeta}{4\pi} \sum_{k=1}^{n} (X_k + iY_k)\bar{\sigma}_k, \quad (80.1a)$$

$$\psi(\zeta) = \frac{1}{2\pi} \sum_{k=1}^{n} (X_k - iY_k) \log(\sigma_k - \zeta) - \frac{1}{2\pi} \sum_{k=1}^{n} \frac{(X_k + iY_k)\bar{\sigma}_k}{\sigma_k - \zeta}. \quad (80.2a)$$

It is easily established that the stress function U will be continuous up to γ, so that one will actually have the concentrated forces at the specified points.

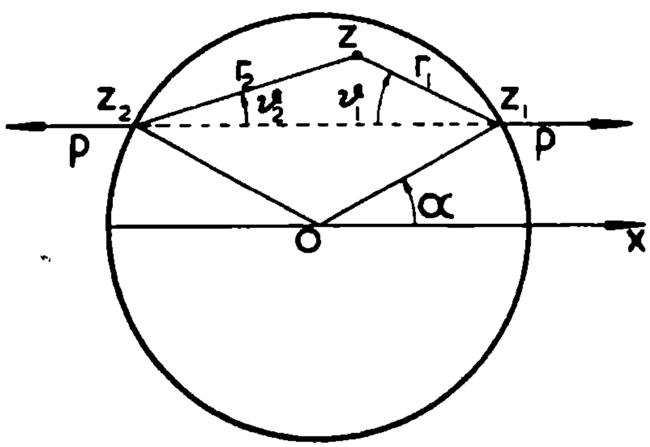


Fig. 35.

Consider, for example, the case when two equal and opposite forces $(p, 0)$ and $(-p, 0)$, parallel to the Ox axis, act on the disc at the points $z_1 = Re^{i\alpha}$ and $z_2 = Re^{i(\pi-\alpha)} = -Re^{-i\alpha}$ (Fig. 35). Then, reverting to the old variable $z = R\zeta$, one obtains from (80.1a) and (80.2a), omitting inessential constant terms,

$$\varphi_1(z) = -\frac{p}{2\pi}\left\{\log(z_1 - z) - \log(z_2 - z) + \frac{\bar{z}_1 - \bar{z}_2}{2R^2} z\right\},$$

$$\psi_1(z) = \frac{p}{2\pi}\left\{\log(z_1 - z) - \log(z_2 - z) - \frac{\bar{z}_1}{z_1 - z} + \frac{\bar{z}_2}{z_2 - z}\right\},$$

$$\Phi_1(z) = \varphi_1'(z) = \frac{p}{2\pi}\left\{\frac{1}{z_1 - z} - \frac{1}{z_2 - z} - \frac{\bar{z}_1 - \bar{z}_2}{2R^2}\right\},$$

$$\Psi_1(z) = \psi_1'(z) = \quad 2\pi\left\{\frac{1}{z_1 - z} - \quad z \quad (z_1 - z)^2 - \frac{\bar{z}_2}{(z_2 - z)^2}\right\}.$$

The stresses will be given by the formulae

$$X_x + Y_y = 4\Re\,\Phi_1(z),$$

$$Y_y - X_x + 2iX_y = 2[\bar{z}\Phi_1'(z) + \Psi_1(z)].$$

Substituting for Φ_1 and Ψ_1 and noting that

$$z_1 = Re^{i\alpha}, \quad z_2 = -Re^{-i\alpha},$$

$$z_1 \quad z = r_1 e^{-i\vartheta_1}, \quad z_2 - z \quad r_2 e^{i\vartheta_2}$$

(see Fig. 35, ϑ_1 and ϑ_2 being positive or negative, when the point z lies above or below the line of action of the forces), one finds

$$X_x + Y_y = \frac{2p}{\pi}\left\{\frac{\cos\vartheta_1}{r_1} + \frac{\cos\vartheta_2}{r_2} - \frac{\cos\alpha}{R}\right\},$$

$$X_x - Y_y = \quad p \left\{\cos 3\vartheta_1 + \cos\vartheta_1 \quad \cos 3\vartheta_2 + \cos\vartheta_2\right\},$$

$$2X_y = -\frac{p}{\pi}\left\{\frac{\sin 3\vartheta_1 + \sin\vartheta_1}{r_1} - \frac{\sin 3\vartheta_2 + \sin\vartheta_2}{r_2}\right\};$$

hence

$$X_x = \frac{2p}{\pi}\left\{\frac{\cos^3\vartheta_1}{r_1} + \frac{\cos^3\vartheta_2}{r_2}\right\} - \frac{p}{\pi R}\cos\alpha,$$

$$Y_y = \frac{2p}{\pi}\left\{\frac{\sin^2\vartheta_1\cos\vartheta_1}{r_1} + \frac{\sin^2\vartheta_2\cos\vartheta_2}{r_2}\right\} - \frac{p}{\pi R}\cos\alpha,$$

$$X_y = -\frac{2p}{\pi}\left\{\frac{\sin\vartheta_1\cos^2\vartheta_1}{r_1} - \frac{\sin\vartheta_2\cos^2\vartheta_2}{r_2}\right\}.$$

The displacements u and v are also easily obtained, using the formula

$$2\mu(u + iv) = \varkappa\varphi_1(z) - z\varphi'(z) - \psi_1(z)$$

which gives

$$2\mu(u+iv) =$$

$$= \frac{p}{2\pi}\left\{\varkappa \log \frac{z_2 - z}{z_1 - z} + \log \frac{\bar{z}_2 - \bar{z}}{\bar{z}_1 - \bar{z}} + \frac{z_1 - z}{\bar{z}_1 - \bar{z}} - \frac{z_2 - z}{\bar{z}_2 - \bar{z}} - \frac{(\varkappa - 1)\cos\alpha}{R} z\right\}.$$

The values of the multi-valued functions, occuring in this formula, have to be restricted to one definite branch. If another branch is used, the resulting displacements will differ from the first by a rigid body displacement. Separating real and imaginary parts and replacing $\varkappa$ by its value $(\lambda + 3\mu)/(\lambda + \mu)$, one finds

$$u = \frac{p}{4\mu\pi}\left\{\frac{2(\lambda + 2\mu)}{\lambda + \mu}\log\frac{r_2}{r_1} + \cos 2\vartheta_1 - \cos 2\vartheta_2 - \frac{2\mu\cos\alpha}{\lambda + \mu}\cdot\frac{x}{R}\right\},$$

$$v = \frac{p}{4\mu\pi}\left\{\frac{2\mu}{\lambda + \mu}(\vartheta_1 + \vartheta_2) - \sin 2\vartheta_1 - \sin 2\vartheta_2 - \frac{2\mu\cos\alpha}{\lambda + \mu}\cdot\frac{y}{R}\right\}.$$

In these formulae $x + iy = z$. In the last formula one may write $y - l$ instead of y, where l is the distance of the centre from the line of action of the forces (obviously this amounts to the addition of a rigid body displacement). In that case all points, lying on the line of action of the forces, will remain there after deformation.

If one is not dealing with plane deformations, but with a thin disc, one must use λ^* instead of λ and p must be conceived as the quantity $F/2h$, where F is the concentrated force and $2h$ is the thickness of the plate. (Actually, in the above work, p denotes a force which does not act at a point, but on a straight line, perpendicular to the Oxy plane, and which is estimated per unit length of this line.)

A large number of examples of a similar kind may be treated which are of interest for technical applications. In particular, it is very simple to deduce solutions for all those cases which were considered by J. H. Michell [2] using other, artificial methods (cf. G. V. Kolosov and I. N. Muskhelishvili [1]).

2°. Disc under concentrated forces and couples acting at internal points. The solution of this problem is likewise obtained with extraordinary simplicity from the general formulae of § 80. For this purpose it is sufficient to introduce into the functions φ, ψ definite singularities at the points of application of the concentrated forces and couples, as stated in § 57. It will be left to the reader to find the general solution. For the sake of brevity, consideration will be restrict-

ed here to the example of two exactly opposite forces one of which acts at the centre, while the other acts at an arbitrary point of the disc. Without affecting generality, it may be assumed that the second force is applied at a point of the Ox axis (and directed along it). Thus one has the two concentrated forces $(-p, 0)$ acting at O and $(+p, 0)$ at z_0, where z_0 is real.

In the case under consideration the functions $\varphi_1(z)$, $\psi_1(z)$ will have the following forms (cf. § 57):

$$\varphi_1(z) = \frac{p}{2\pi(1+\varkappa)} \log z - \frac{p}{2\pi(1+\varkappa)} \log (z - z_0) + \varphi_1^0(z),$$

$$\psi_1(z) = -\frac{\varkappa p}{2\pi(1+\varkappa)} \log z + \frac{\varkappa p}{2\pi(1+\varkappa)} \log (z - z_0) + \tag{80.3a}$$

$$+ \frac{p}{2\pi(1+\varkappa)} \cdot \frac{z_0}{z - z_0} + \psi_1^0(z)$$

or, in terms of $\zeta = z/R$,

$$\varphi(\zeta) = \frac{p}{2\pi(1+\varkappa)} \{\log \zeta - \log (\zeta - \zeta_0)\} + \varphi_0(\zeta),$$

$$\psi(\zeta) = -\frac{\varkappa p}{2\pi(1+\varkappa)} \{\log \zeta - \log (\zeta - \zeta_0)\} + \tag{80.4a}$$

$$+ \frac{p}{2\pi(1+\varkappa)} \cdot \frac{\zeta_0}{\zeta - \zeta_0} + \psi_0(\zeta),$$

where $\varphi_0(\zeta)$, $\psi_0(\zeta)$ are holomorphic inside γ and $\zeta_0 = z_0/R$.

The boundary condition (assuming the edge of the disc to be free) may be written

$$\varphi(\sigma) + \sigma\varphi'(\sigma) + \psi(\sigma) = 0$$

or, substituting from (80.4*a*),

$$\varphi_0(\sigma) + \sigma\varphi_0'(\sigma) + \psi_0(\sigma) = f_0, \tag{80.5a}$$

where

$$f_0 = \frac{p}{2\pi(1+\varkappa)} \log \frac{\sigma - \zeta_0}{\sigma} - \frac{\varkappa p}{2\pi(1+\varkappa)} \log (1 - \zeta_0\sigma) +$$

$$+ \frac{p}{2\pi(1+\varkappa)} \left\{ \frac{\sigma - \zeta_0}{1 - \sigma\zeta_0} \sigma - \sigma^2 \right\}, \tag{80.6a}$$

and hence

$$\overline{f}_0 = \frac{p}{2\pi(1+\varkappa)} \log(1 - \sigma\zeta_0) - \frac{\varkappa p}{2\pi(1+\varkappa)} \log \frac{\sigma - \zeta_0}{\sigma} +$$

$$+ \frac{p}{2\pi(1+\varkappa)} \left\{ \frac{1}{\sigma} \cdot \frac{1 - \sigma\zeta_0}{\sigma - \zeta_0} - \frac{1}{\sigma^2} \right\}. \qquad (80.6'a)$$

The functions $\varphi_0(\zeta)$, $\psi_0(\zeta)$ will be found from (80.9′) and (80.10), where one has to replace φ, ψ, f by φ_0, ψ_0, f_0. Calculation of the integrals occurring in these formulae presents no difficulty.

The choice of the branches of the multi-valued functions $\log\left(1 - \frac{\zeta_0}{\zeta}\right)$ and $\log(1 - \zeta\zeta_0)$ is arbitrary. However, they must be chosen in such a way that they represent on γ conjugate quantities. For the first function, a branch will be chosen which is holomorphic outside γ and zero for $\zeta = \infty$; for the second function, a branch will be taken which is holomorphic inside γ and zero for $\zeta = 0$.

Noting the properties of the branches of the functions $\log\left(1 - \frac{\zeta_0}{\zeta}\right)$ and $\log(1 - \zeta . \zeta_0)$, as chosen above, one has from the formulae of § 70 and by Cauchy's formula

$$\frac{1}{2\pi i} \int_\gamma \log \frac{\sigma - \zeta_0}{\sigma} \cdot \frac{d\sigma}{\sigma - \zeta} = 0, \quad \frac{1}{2\pi i} \int \log(1 - \zeta_0\sigma) \frac{d\sigma}{\sigma - \zeta} = \log(1 - \zeta_0\zeta).$$

Also, by the same formulae,

$$\frac{1}{2\pi i} \int \left\{ \frac{\sigma - \zeta_0}{1 - \sigma\zeta_0} \sigma - \sigma^2 \right\} \frac{d\sigma}{\sigma - \zeta} = \frac{\zeta - \zeta_0}{1 - \zeta_0\zeta} . \zeta - {}^{r_2}$$

$$i \int \left\{ \frac{1}{\sigma} \cdot \frac{1 - \sigma\zeta_0}{\sigma - \zeta_0} - \frac{1}{\sigma^2} \right\} \frac{d\sigma}{\sigma - \zeta_0} = 0.$$

Hence

$$\frac{1}{2\pi i} \int_\gamma \frac{f_0 \, d\sigma}{\sigma - \zeta} = - \frac{\varkappa p}{2\pi(1+\varkappa)} \log(1 - \zeta_0\zeta) + \frac{p\zeta}{2\pi(1+\varkappa)} \left\{ \frac{\zeta - \zeta_0}{1 - \zeta_0\zeta} - \zeta \right\},$$

$$\frac{1}{2\pi i} \int \frac{\overline{f}_0 \, d\sigma}{\sigma - \zeta} \qquad \frac{p}{2\pi(1+\varkappa)} \log(1 - \zeta_0\zeta).$$

For the calculation of a_1 it will be noted that $2a_1$ equals the value of the derivative of the last but one expression for $\zeta = 0$ (cf. § 80). Thus,

$$a_1 = \frac{(\varkappa - 1)p\zeta_0}{4\pi(1+\varkappa)},$$

and, by (80.9′) and (80.10), one finds

$$\varphi_0(\zeta) = \frac{\varkappa p}{2\pi(1+\varkappa)} \log (1 - \zeta_0\zeta) + \frac{p\zeta}{2\pi(1+\varkappa)} \left\{ \frac{\zeta - \zeta_0}{1 - \zeta_0\zeta} - \zeta \right\} - \frac{(\varkappa - 1)p\zeta_0\zeta}{4\pi(1+\varkappa)},$$

$$\psi_0(\zeta) = \frac{\bar{p}}{2\pi(1+\varkappa)} \log (1 - \zeta_0\zeta) - \frac{p(\varkappa - 1)\zeta_0^2 + 1}{2\pi(1+\varkappa)(1 - \zeta_0\zeta)} - \frac{p}{2\pi(1+\varkappa)} \cdot \frac{1 - \zeta_0^2}{(1 - \zeta_0\zeta)^2}.$$

A constant term has been omitted in the last expression. Finally, by (80.4*a*),

$$\varphi(\zeta) = \frac{p}{2\pi(1+\varkappa)} \log \frac{\zeta}{\zeta - \zeta_0} - \frac{\varkappa p}{2\pi(1+\varkappa)} \log (1 - \zeta_0\zeta) + \frac{p\zeta}{2\pi(1+\varkappa)} \left\{ \frac{\zeta - \zeta_0}{1 - \zeta_0\zeta} - \zeta \right\} - \frac{(\varkappa - 1)p\zeta_0\zeta}{4\pi(1+\varkappa)},$$

$$\psi(\zeta) = \frac{\varkappa p}{2\pi(1+\varkappa)} \log \frac{\zeta}{\zeta - \zeta_0} + \frac{p}{2\pi(1+\varkappa)} \cdot \frac{\zeta_0}{\zeta - \zeta_0} + \frac{\bar{p}}{2\pi(1+\varkappa)} \log (1 - \zeta_0\zeta) - \frac{p}{2\pi(1+\varkappa)} \cdot \frac{(\varkappa - 1)\zeta_0^2 + 1}{1 - \zeta_0\zeta} - \frac{p}{2\pi(1+\varkappa)} \cdot \frac{1 - \zeta_0^2}{(1 - \zeta_0\zeta)^2}, \tag{80.7a}$$

and the problem is solved. When dealing with a thin plate one has to replace $\varkappa$ by $\varkappa^*$.

The problem for systems of arbitrarily distributed forces can be solved just as simply.

3°. Rotating disc with attached discrete masses. Let the thin elastic plate rotate about its centre with angular velocity Ω and let there be arbitrary discrete masses attached to points

of the plate. It is sufficient to find the solution for the case of one mass m, because the solution of the general case may be obtained by superposition of several such solutions.

The effect of a concentrated mass obviously reduces to the action of a concentrated centrifugal force in a radial direction and of magnitude $F = ml\Omega^2$, where l is the distance of the mass from the axis of rotation; a reaction, equal in magnitude and opposite in direction to the force F, will act at the axis of rotation. Thus the solution of this problem will be obtained by adding to the solution of the problem of a rotating disc without discrete masses (cf. end of § 59a) the solution of the problem, considered in the preceding example. In the present case $p = F/2h = ml\Omega^2/2h$, where $2h$ is the thickness of the plate (because p is calculated per unit thickness).

The solution of the problem of a disc rotating about an eccentric axis may be obtained in a similar manner.

§ 81. Solution of the second fundamental problem for the circle. In the notation of the preceding sections the boundary condition in this case takes the form

$$\varkappa\varphi(\sigma) - \sigma\overline{\varphi'(\sigma)} - \overline{\psi(\sigma)} = 2\mu(g_1 + ig_2) = 2\mu g \tag{81.1}$$

or

$$\varkappa\overline{\varphi(\sigma)} - \bar{\sigma}\varphi'(\sigma) - \psi(\sigma) = 2\mu(g_1 - ig_2) = 2\mu\bar{g}, \tag{81.2}$$

where g_1, g_2 are the given displacements of points of the boundary.

In view of the complete analogy with the problem of § 80, only the final solution will be recorded here:

$$\varkappa\varphi(\zeta) = \frac{\mu}{\pi i}\int_\gamma \frac{g\,d\sigma}{\sigma - \zeta} + \bar{a}_1\zeta + 2\bar{a}_2 + \bar{a}, \tag{81.3}$$

$$\psi(\zeta) = -\frac{\mu}{\pi i}\int_\gamma \frac{\bar{g}\,d\sigma}{\sigma - \zeta} - \frac{1}{\zeta}\varphi'(\zeta) + \frac{a_1}{\zeta}, \tag{81.4}$$

where

$$2\bar{a}_2 + \bar{a} = -\frac{\mu}{\pi i}\int_\gamma g\frac{d\sigma}{\sigma}, \tag{81.5}$$

$$(\varkappa^2 - 1)a_1 = \frac{\mu\varkappa}{\pi i}\int_\gamma \frac{g d\sigma}{\sigma^2} + \frac{\mu}{\pi i}\int_\gamma \bar{g}\,d\sigma. \tag{81.6}$$

This solution will be regular, provided the function g, given on the boundary, has a derivative satisfying the H condition.

§ 82. Solution of the first fundamental problem for the infinite plane with an elliptic hole *). Use will be made here of the transformation of the region S under consideration on to the region $|\zeta| > 1$, i.e., on the infinite plane with a circular hole. (In the Author's earlier work the transformation on to the circle has been used instead.)

The relevant transformation is (cf. § 48, 5°)

$$z = \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right) \quad (R > 0, \quad 0 \quad m \quad 1). \tag{82.1}$$

The circle $|\zeta| = 1$ corresponds to the ellipse L with centre at the origin and semi-axes

$$a = R(1 + m), \quad b = R(1 - m).$$

By suitable choice of R and m one may obtain ellipses of any dimension and shape. If $m = 0$, the ellipse becomes a circle. In the limiting case $m = 1$, it is the segment of the Ox axis between the points $x = \pm 2R$ and the region S is the infinite plane with a straight cut.

In the present case

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} = \frac{1}{\sigma}\,\frac{\sigma^2 + m}{1 - m\sigma^2}, \qquad \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)} = \sigma\,\frac{1 + m\sigma^2}{\sigma^2 - m}$$

and the boundary condition takes the form

$$\varphi(\sigma) + \frac{1}{\sigma}\,\frac{\sigma^2 + m}{1 - m\sigma^2}\,\overline{\varphi'(\sigma)} + \overline{\psi(\sigma)} = f \tag{82.2}$$

or

$$\overline{\varphi(\sigma)} + \sigma\,\frac{1 + m\sigma^2}{\sigma^2 - m}\,\varphi'(\sigma) + \psi(\sigma) = \bar{f}. \tag{82.3}$$

First assume that

$$X = Y = 0, \quad \Gamma = \Gamma' = 0$$

[the notation being the same as in § 50; cf. (50.14) and (50.15)], i.e., that the resultant vector of the external forces applied to the contour is zero and that the stresses as well as the rotation vanish at infinity.

*) N.I. Muskhelishvili [4].

Then $\varphi(\zeta)$ and $\psi(\zeta)$ will be holomorphic outside γ, including the point at infinity. In addition, one may assume $\varphi(\infty) = 0$.

The statement that $\psi(\sigma)$ must represent on γ the boundary value of some function $\psi(\zeta)$, holomorphic outside γ, takes by (76.13) the form

$$\frac{1}{2\pi i}\int_\gamma \frac{f\,d\sigma}{\sigma-\zeta} - \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\sigma)d\sigma}{\sigma-\zeta} - \frac{1}{2\pi i}\int_\gamma \frac{1}{\sigma}\cdot\frac{\sigma^2+m}{1-m\sigma^2}\overline{\varphi'(\sigma)}\frac{d\sigma}{\sigma-\zeta} = 0,$$

where ζ is a point outside γ; noting that by (70.1′), i.e., by Cauchy's formula for infinite regions,

$$\frac{1}{2\pi i}\int_\gamma \frac{\varphi(\sigma)d\sigma}{\sigma-\zeta} = -\varphi(\zeta) + \varphi(\infty) = -\varphi(\zeta),$$

one obtains

$$-\varphi(\zeta) + \frac{1}{2\pi i}\int \frac{1}{\sigma}\cdot\frac{\sigma^2+m}{1-m\sigma^2}\overline{\varphi'(\sigma)}\frac{d\sigma}{\sigma-\zeta} = \frac{1}{2\pi i}\int_\gamma \frac{f\,d\sigma}{\sigma-\zeta}. \qquad (a)$$

This equation corresponds to the functional equation (78.10) for the general case (the absence of the constant $\bar{a}$ on the left-hand side being explained by the fact that use has been made of the transformation on to $|\zeta| > 1$ and not on to $|\zeta| < 1$); this equation can immediately be solved, because

$$\frac{1}{\sigma}\cdot\frac{\sigma^2+m}{1-m\sigma^2}\overline{\varphi'(\sigma)}$$

is the boundary value of the function

$$\frac{1}{\zeta}\cdot\frac{\zeta^2+m}{1-m\zeta^2}\bar{\varphi}'\left(\frac{1}{\zeta}\right),$$

holomorphic inside γ, as a result of which the integral on the left-hand side of (a) becomes zero.

Since $\varphi(\zeta)$ is holomorphic outside γ and $\varphi(\infty) = 0$,

$$\varphi(\zeta) = \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \dots \quad \text{for } |\zeta| > 1.$$

Hence

$$\varphi'(\zeta) = -\frac{a_1}{\zeta^2} - \frac{2a_2}{\zeta^3} - \dots \quad \text{for } |\zeta| > 1,$$

so that

$$\overline{\varphi}'\left(\frac{1}{\zeta}\right) = -\bar{a}_1\zeta^2 - 2\bar{a}_2\zeta^3 \dots \quad \text{for } |\zeta| < 1,$$

which proves the above statement.

Thus one obtains the very simple formula

$$\varphi(\zeta) = \frac{1}{2\pi i}\int \frac{f\,d\sigma}{\sigma - \zeta} \tag{82.4}$$

which determines $\varphi(\zeta)$. In this way the boundary value $\psi(\sigma)$ of the function $\psi(\zeta)$ will be known by (82.3), and therefore $\psi(\zeta)$ is given by Cauchy's formula [(70.1')]

$$\psi(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\psi(\sigma)d\sigma}{\sigma - \zeta} + \psi(\infty);$$

substituting for $\psi(\sigma)$ from (82.3) and noting that (see remarks below)

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{\varphi(\sigma)}d\sigma}{\sigma - \zeta} = 0, \quad \frac{1}{2\pi i}\int_\gamma \sigma\frac{1 + m\sigma^2}{\sigma^2 - m}\,\varphi'(\sigma)\frac{d\sigma}{\sigma - \zeta} = -\zeta\frac{1 + m\zeta^2}{\zeta^2 - m}\,\varphi'(\zeta),$$

one finds finally, omitting the constant $\psi(\infty)$ which does not influence the stress distribution,

$$\psi(\zeta) = -\frac{1}{2\pi i}\int_\gamma \frac{\bar{f}\,d\sigma}{\sigma - \zeta} - \zeta\,\frac{1 + m\zeta^2}{\zeta^2 - m}\cdot\varphi'(\zeta). \tag{82.5}$$

It is easily seen that the formulae obtained give a regular solution of the problem, provided f has a derivative satisfying the H condition.

For the deduction of the formulae preceding (82.5), it is sufficient to note that $\overline{\varphi(\sigma)}$ is the boundary value of $\overline{\varphi}(1/\zeta)$, holomorphic inside γ, and that

$$\sigma\,\frac{1 + m\sigma^2}{\sigma^2 - m}\,\varphi'(\sigma)$$

is the boundary value of a function, holomorphic outside γ and vanishing at infinity (cf. also preceding note).

The constant $\psi(\infty)$ may be determined by (76.15); in fact,

$$\psi(\infty) = \frac{1}{2\pi i}\int_\gamma \frac{\psi(\sigma)d\sigma}{\sigma}.$$

Substituting for $\psi(\sigma)$ from (82.3) and taking into consideration that

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\varphi(\sigma)}}{\sigma}\,d\sigma = 0, \quad \frac{1}{2\pi i}\int_{\gamma}\frac{1+m\sigma^2}{\sigma^2-m}\,\varphi'(\sigma)d\sigma = 0,$$

one obtains

$$\psi(\infty) = \frac{1}{2\pi i}\int_{\gamma}\bar{f}\,\frac{d\sigma}{\sigma}.$$

Next, the general case will be considered and it will be treated i accordance with the rule of § 78. By (50.14) and (50.15)

$$\varphi(\zeta) = \Gamma R\zeta - \frac{X+iY}{2\pi(1+\varkappa)}\log\zeta + \varphi_0(\zeta), \tag{82.6}$$

$$\psi(\zeta) = \Gamma' R\zeta + \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\log\zeta + \psi_0(\zeta), \tag{82.7}$$

where $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ are holomorphic for $|\zeta| > 1$ and where one can assume

$$\varphi_0(\infty) = 0;$$

in addition, as always for the solution of the first fundamental problem, it will be assumed that there is no rotation at infinity, i.e., that $\Gamma = \bar{\Gamma}$.

Substituting these expressions in (82.2), it is seen that $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ satisfy the same boundary condition (82.2), the only difference being that f must be replaced by f_0, where

$$f_0 = f - \Gamma R\left(\sigma + \frac{\sigma^2+m}{\sigma(1-m\sigma^2)}\right) - \frac{\bar{\Gamma}' R}{\sigma} +$$
$$+ \frac{X+iY}{2\pi}\log\sigma + \frac{X-iY}{2\pi(1+\varkappa)}\cdot\frac{\sigma^2+m}{1-m\sigma^2}. \tag{82.8}$$

It will be remembered that f_0 will be single-valued on γ, because the increase of f for a complete circuit of γ will be compensated by the increase of the logarithmic term.

The functions $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ may be found by help of the formulae, stated above,

$$\varphi_0(\zeta) = -\frac{1}{2\pi i}\int_{\gamma}\frac{f_0\,d\sigma}{\sigma-\zeta}, \tag{82.4'}$$

$$\psi_0(\zeta) = \frac{1}{2\pi i} \quad \frac{\bar{f}_0\, d\sigma}{\sigma - \zeta} \quad -\zeta \frac{1 + m\zeta^2}{\zeta^2 - m} \varphi_0'(\zeta), \tag{82.5'}$$

and the problem is completely solved.

§ 82a. Examples.

1°. Stretching of a plate with an elliptic hole. Let the edge of the hole be free from external stresses and let the state of stress at infinity be tension of magnitude p in a direction forming an angle α with the Ox axis. Then $X = Y = 0$ and by (36.10) [putting $N_1 = p,\ N_2 = 0$]

$$\Gamma = \bar{\Gamma} = \frac{p}{} \quad \Gamma' = -\frac{p}{2} e^{-2i\alpha}. \tag{82.1a}$$

Also in the present case $f = 0$. Hence (82.8) gives

$$f_0 = -\frac{pR}{4}\left(\sigma + \frac{\sigma^2 + m}{\sigma(1 - m\sigma^2)}\right) + \frac{pRe^{2i\alpha}}{2\sigma},$$

$$\bar{f}_0 = -\frac{pR}{4}\left(\frac{1}{\sigma} + \sigma \frac{1 + m\sigma^2}{\sigma^2 - m}\right) \quad pRe^{-2i\alpha}\sigma$$

The function $(\zeta^2 + m)/\zeta(1 - m\zeta^2)$ is holomorphic inside γ, except at $\zeta = 0$, where it has a pole with the principal part m/ζ; the function $\zeta(1 + m\zeta^2)/(\zeta^2 - m)$ is holomorphic outside γ, except at $\zeta = \infty$, where it has the form $m\zeta + O(1/\zeta)$. Hence, by the formulae of § 70,

$$\frac{1}{2\pi i} \quad \frac{\sigma^2 + m}{\sigma(1 - m\sigma^2)} \cdot \frac{d\sigma}{\sigma - \zeta} \qquad \frac{m}{\zeta},$$

$$\frac{1}{2\pi i} \int \sigma \frac{1 + m\sigma^2}{\sigma^2 - m} \cdot \frac{d\sigma}{\sigma - \zeta} = -\zeta \frac{1 + m\zeta^2}{\zeta^2 - m} + m\zeta = -\frac{(1 + m^2)\zeta}{\zeta^2 - m}.$$

Further, it is obvious that

$$\frac{1}{2\pi i} \int_\gamma \frac{\sigma d\sigma}{\sigma - \zeta} = 0, \quad \frac{1}{2\pi i} \int_\gamma \frac{d\sigma}{\sigma(\sigma - \zeta)} = -\frac{1}{\zeta}$$

Thus (82.4′) and (82.5′) give

$$\varphi_0(\zeta) = -\frac{mpR}{4\zeta} + \frac{pRe^{2i\alpha}}{2\zeta} = \frac{pR(2e^{2i\alpha} - m)}{4\zeta},$$

$$\psi_0(\zeta) = -\frac{pR}{4\zeta} - \frac{pR(1+m^2)\zeta}{4(\zeta^2-m)} - \zeta\frac{1+m\zeta^2}{\zeta^2-m}\varphi_0'(\zeta),$$

and, finally, by (82.6) and (82.7)

$$\varphi(\zeta) = \frac{pR}{4}\left(\zeta + \frac{2e^{2i\alpha}-m}{\zeta}\right),$$

$$\psi(\zeta) = -\frac{pR}{2}\left\{e^{-2i\alpha}\zeta + \frac{e^{2i\alpha}}{m\zeta} - \frac{(1+m^2)(e^{2i\alpha}-m)}{m}\cdot\frac{\zeta}{\zeta^2-m}\right\}, \quad (82.24)$$

and the problem is solved.

A solution of this problem, by a quite different method, was given by C. E. Inglis [1], and it was found again in 1921 by T. Pöschl [1]. It has been seen that this solution is a very particular case of the general solution of the first fundamental problem for the infinite plane with an elliptic hole which was published by the Author [4] in 1919; cf. also the Author's paper [7]. A particular case of this problem (tension in the direction of the major axis of the hole) was solved in 1909 by G. V. Kolosov [1].

In 1931, L. Föppl published a (very complicated) solution of the above-mentioned particular case which he considered as an illustration of his method of solution of problems by the help of conformal mapping. The general method of Föppl (studied in the same paper) is very difficult to understand (at least the Author has not succeeded in doing so).

The calculation of the components of stress and displacement is not difficult. Only the sum

$$\widehat{\rho\rho} + \widehat{\vartheta\vartheta} = 4\,\Re\,\Phi(\zeta)$$

will be determined here, where by the preceding formulae

$$4\Phi(\zeta) = \frac{4\varphi'(\zeta)}{\omega'(\zeta)} = p\frac{\zeta^2+m-2e^{2i\alpha}}{\zeta^2-m} = p\frac{(\rho^2e^{2i\vartheta}+m-2e^{2i\alpha})(\rho^2e^{-2i\vartheta}-m)}{(\rho^2e^{2i\vartheta}-m)(\rho^2e^{-2i\vartheta}-m)}.$$

The denominator of the last fraction is real and equal to

$$\rho^4 - 2m\rho^2\cos 2\vartheta + m^2.$$

Separating the real part in the numerator one finds

$$\widehat{\rho\rho} + \widehat{\vartheta\vartheta} = p\frac{\rho^4 - 2\rho^2\cos 2(\vartheta-\alpha) - m^2 + 2m\cos 2\alpha}{\rho^4 - 2m\rho^2\cos 2\vartheta + m^2}.$$

On the boundary of the hole $\rho = 1$ and $\widehat{\rho\rho} = 0$. Hence the value of $\widehat{\vartheta\vartheta}$

along the edge of the hole is given by

$$\widehat{\vartheta\vartheta} = p\,\frac{1 - m^2 + 2m\cos 2\alpha - 2\cos 2(\vartheta - \alpha)}{1 - 2m\cos 2\vartheta + m^2}$$

this formula, apart from the notation, agrees with that given by T. Pöschl [1] (Note that his formula contains a misprint).

In the case of *bi-axial tension*, when at infinity

$$N_1 = N_2 = p, \quad \Gamma = \frac{p}{2}, \quad \Gamma' = 0,$$

one obtains either directly or by superposition of the preceding solutions for $\alpha = 0$ and $\alpha = \frac{\pi}{2}$

$$\varphi(\zeta) = \frac{pR}{2}\left(\zeta - \frac{m}{\zeta}\right), \quad \psi(\zeta) = -\frac{pR(1 + m^2)\zeta}{\zeta^2 - m}.$$

2°. Elliptic hole the edge of which is subject to uniform pressure. In this case

$$X_n = -P\cos(n, x),$$
$$Y_n = -P\cos(n, y),$$

where P is the magnitude of the pressure; hence

$$(X_n + iY_n)ds = -P(dy - idx) = Pidz.$$

Therefore

$$f = i\int (X_n + iY_n)ds = -Pz = -PR\left(\sigma + \frac{m}{\sigma}\right),$$
$$\bar{f} = -PR\left(\frac{1}{\sigma} + m\sigma\right).$$

Substituting these values in (82.4) and (82.5) (assuming the stresses to vanish at infinity) one finds

$$\varphi(\zeta) = -\frac{PRm}{\zeta}, \quad \psi(\zeta) = -\frac{PR}{\zeta} - \frac{PRm}{\zeta}\cdot\frac{1 + m\zeta^2}{\zeta^2 - m}$$

and the problem is solved.

The displacements and stresses will only be calculated here for the limiting case $m = 1$ (i.e., the straight cut; cf. Fig. 36.); their calculation in the general case is likewise not difficult.

In fact, one finds by (50.7), (50.9) and (50.10)

$$\widehat{\rho\rho} = -P + \frac{P(\rho^2 - 1)^3(\rho^2 + 1)}{(\rho^4 - 2\rho^2 \cos 2\vartheta + 1)^2},$$

$$\widehat{\vartheta\vartheta} = -P + \frac{P(\rho^4 - 1)(1 + 2\rho^2 + \rho^4 - 4\rho^2 \cos 2\vartheta)}{(\rho^4 - 2\rho^2 \cos 2\vartheta + 1)^2},$$

$$\widehat{\rho\vartheta} = \frac{2P\rho^2(\rho^2 - 1)^2 \sin 2\vartheta}{(\rho^4 - 2\rho^2 \cos 2\vartheta + 1)^2},$$

$$v_\rho = -\frac{PR}{2\mu\rho} \cdot \frac{(1 + \varkappa)\rho^2 \cos 2\vartheta + 1 - \varkappa - 2\rho^2}{\sqrt{\rho^4 - 2\rho^2 \cos 2\vartheta + 1}},$$

$$v_\vartheta = -\frac{PR\rho}{2\mu} \cdot \frac{(1 - \varkappa) \sin 2\vartheta}{\sqrt{\rho^4 - 2\rho^2 \cos 2\vartheta + 1}}.$$

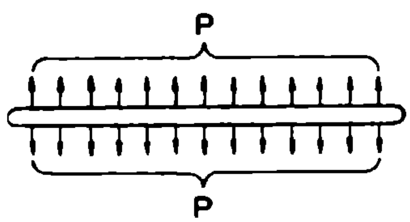


Fig. 36.

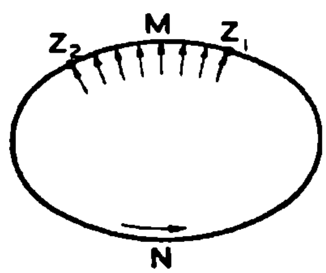


Fig. 37*a*.

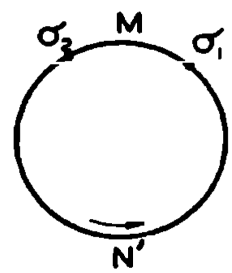


Fig. 37*b*.

3°. Elliptic hole the edge of which is subject to uniform tangential stress T.

In this case

$$(X_n + iY_n)ds = Tdz,$$

$$f = iTz = iTR\left(\sigma + \frac{m}{\sigma}\right), \quad \bar{f} = -iTR\left(\frac{1}{\sigma} + m\sigma\right).$$

As in the preceding example, one obtains (assuming the stresses to vanish at infinity)

$$\varphi(\zeta) = \frac{TRmi}{\zeta}, \quad \psi(\zeta) = -\frac{TRi}{\zeta} + \frac{TRmi}{\zeta} \cdot \frac{1 + m\zeta^2}{\zeta^2 - m}.$$

4°. Elliptic hole (or straight cut) part of the edge of which is subject to uniform pressure.

Consider now the case when the uniform pressure P acts only on

the part z_1Mz_2 of the boundary (Fig. 37*a*) and when the stresses, as before, vanish at infinity.

In this case (cf. example 2°.) one may take (beginning the circuit at z_1)

$$f = -Pz = PR\left(\sigma + \frac{m}{\sigma}\right) \text{ on the arc } z_1Mz_2,$$

$$f = -Pz_2 \text{ on the arc } z_2Nz_1.$$

Following around the contour (*in anti-clockwise direction*) and returning to z_1, the expression f undergoes the increase

$$-P(z_2 - z_1) = P(z_1 - z_2);$$

the same increase will result for any subsequent circuit (cf. § 42).

By (82.8)

$$f_0 = f + \frac{X + iY}{2\pi} \log \sigma + \frac{X - iY}{2\pi(1 + \varkappa)} \cdot \frac{\sigma^2 + m}{1 - m\sigma^2} =$$

$$= f - \frac{P(z_1 - z_2) \log \sigma}{2\pi i} + \frac{P(\bar{z}_1 - \bar{z}_2)}{2\pi i(1 + \varkappa)} \cdot \frac{\sigma^2 + m}{1 - m\sigma^2},$$

because, by (33.1), $X + iY = iP(z_1 - z_2)$; it must not be forgotten that in the above-mentioned formula the direction of the circuit is that which leaves the region occupied by the body on the left, i.e., in the present case this direction is clockwise.

The value of the multi-valued function $\log \sigma$ may be fixed arbitrarily at any point (e.g. at the point $\sigma_1 = e^{i\vartheta_1}$, corresponding to the point z_1); for a circuit along γ the function $\log \sigma$ must vary continuously, so that for a complete circuit (in clockwise direction) $\log \sigma$ undergoes an increase $2\pi i$ and f_0 reverts to its original value. Hence f_0 will be single-valued and continuous on the entire contour.

If f_0 had discontinuities, this would correspond to concentrated forces at the locations of these jumps; however, by supposition, no concentrated forces are to be present. Note that the derivative of f_0 has discontinuities at the points, corresponding to z_1 and z_2, but it is easily verified that the formulae, deduced below, give the solution of the problem.

There remains the determination of $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ from (82.4'), (82.5'). Denoting by σ_2 the point of γ which corresponds to z_2 (Fig. 37*b*),

one finds

$$\varphi_0(\zeta) = -\frac{1}{2\pi i}\int_\gamma \frac{f_0\, d\sigma}{\sigma-\zeta} = \frac{PR}{2\pi i}\int_{\sigma_1}^{\sigma_2}\left(\sigma+\frac{m}{\sigma}\right)\frac{d\sigma}{\sigma-\zeta} + \frac{Pz_2}{2\pi i}\int_{\sigma_2}^{\sigma_1}\frac{d\sigma}{\sigma-\zeta} +$$

$$+\frac{P(z_1-z_2)}{2\pi i}\cdot\frac{1}{2\pi i}\int_\gamma\frac{\log\sigma}{\sigma-\zeta}\,d\sigma -$$

$$-\frac{P(\bar z_1-\bar z_2)}{2\pi i(1+\varkappa)}\cdot\frac{1}{2\pi i}\int_\gamma\frac{\sigma^2+m}{1-m\,\sigma^2}\cdot\frac{d\sigma}{\sigma-\zeta}.$$

But

$$\int_{\sigma_1}^{\sigma_2}\left(\sigma+\frac{m}{\sigma}\right)\frac{d\sigma}{\sigma-\zeta} = \sigma_2-\sigma_1-\frac{m}{\zeta}\log\frac{\sigma_2}{\sigma_1}+\left(\zeta+\frac{m}{\zeta}\right)\log\frac{\sigma_2-\zeta}{\sigma_1-\zeta},$$

where $\log\frac{\sigma_2}{\sigma_1}$ must be understood as the quantity $i\Theta$, with Θ being the angular distance of the points σ_1 and σ_2, measured from σ_1 in anti-clockwise direction; further,

$$\int_{\sigma_2}^{\sigma_1}\frac{d\sigma}{\sigma-\zeta} = \log\frac{\sigma_1-\zeta}{\sigma_2-\zeta}, \quad \frac{1}{2\pi i}\int_\gamma\frac{\sigma^2+m}{1-m\sigma^2}\cdot\frac{d\sigma}{\sigma-\zeta} = 0,$$

the latter result following from the fact that the integrand is holomorphic inside γ.

There remains the calculation of the integral

$$I(\zeta) = \frac{1}{2\pi i}\int_\gamma\frac{\log\sigma}{\sigma-\zeta}\,d\sigma$$

which is most easily achieved in the following manner. One has

$$\frac{dI}{d\zeta} = \frac{1}{2\pi i}\int_\gamma\frac{\log\sigma}{(\sigma-\zeta)^2}\,d\sigma = -\frac{1}{2\pi i}\int_\gamma\log\sigma\, d\frac{1}{\sigma-\zeta} =$$

$$= -\frac{1}{2\pi i}\left[\frac{\log\sigma}{\sigma-\zeta}\right]_{\sigma=\sigma_1}^{\sigma=\sigma_1} + \frac{1}{2\pi i}\int_\gamma\frac{d\sigma}{\sigma(\sigma-\zeta)}.$$

But by Cauchy's formula for infinite regions

$$\frac{1}{2\pi i}\int \frac{d\sigma}{\sigma(\sigma-\zeta)} = -\frac{1}{\zeta}$$

and

$$\left[\frac{\log\sigma}{\sigma-\zeta}\right]_{\sigma=\sigma_1}^{\sigma=\sigma_1} = \frac{2\pi i}{\sigma_1-\zeta},$$

because for a circuit of γ the function $\log\sigma$ increases by $2\pi i$. Hence

$$\frac{dI}{d\zeta} = -\frac{1}{\zeta} - \frac{1}{\sigma_1-\zeta};$$

consequently

$$I(\zeta) = \log(\sigma_1-\zeta) - \log\zeta + \text{const.}$$

Thus, omitting constant terms, one has

$$\varphi_0(\zeta) = \frac{P}{2\pi i}\left\{-\frac{mR}{\zeta}\log\frac{\sigma_2}{\sigma_1} + \left[R\left(\zeta+\frac{m}{\zeta}\right)-z_2\right]\log(\sigma_2-\zeta) - \right.$$

$$\left. -\left[R\left(\zeta+\frac{m}{\zeta}\right)-z_1\right]\log(\sigma_1-\zeta) - (z_1-z_2)\log\zeta\right\},$$

where

$$z_1 = R\left(\sigma_1+\frac{m}{\sigma_1}\right),\quad z_2 = R\left(\sigma_2+\frac{m}{\sigma_2}\right).$$

The function $\psi_0(\zeta)$ may be obtained in the same manner; one thus finds finally

$$\varphi(\zeta) = \frac{P}{2\pi i}\left\{-\frac{mR}{\zeta}\log\frac{\sigma_2}{\sigma_1} + \left[R\left(\zeta+\frac{m}{\zeta}\right)-z_2\right]\log z(_2-\zeta) - \right.$$

$$\left. -\left[R\left(\zeta+\frac{m}{\zeta}\right)-z_1\right]\log(\sigma_1-\zeta) - \frac{\varkappa(z_1-z_2)}{\varkappa+1}\log\zeta\right\},$$

$$\psi(\zeta) = \frac{P}{2\pi i}\left\{-\frac{R(1+m^2)}{\zeta^2-m}\zeta\log\frac{\sigma_2}{\sigma_1} + R(\sigma_1-\sigma_2)\frac{1+m\zeta^2}{\zeta^2-m} - \right.$$

$$\left. -\bar z_2\log(\sigma_2-\zeta) + \bar z_1\log(\sigma_1-\zeta) - \frac{\bar z_1-\bar z_2}{\varkappa+1}\log\zeta - \frac{z_1-z_2}{\varkappa+1}\cdot\frac{1+m^2}{\zeta^2-m}\right\}.$$

If the entire contour is loaded,

$$z_1 = z_2, \quad \sigma_1 = \sigma_2, \quad \log \frac{\sigma_2}{\sigma_1} = 2\pi i,$$

and one obtains the simple formula deduced directly for this case (cf. example 2°).

On the other hand, letting the arc $z_1 M z_2$ tend to zero and increasing P in such a way that $\lim P \mid z_2 - z_1 \mid = F$ is a finite quantity, one obtains in the limit the case of a concentrated normal force applied to the edge of the hole.

It is also easy to find directly the solution for the case of any number of arbitrary concentrated forces applied to points of the contour or to internal points of the body. (Cf. the analogous solution for the circular disc.)

5°. Approximate solution of the problem of bending of a strip (beam) with an elliptic hole. The stress function

$$U = -\frac{1}{6} A y^3$$

corresponds to the following state of stress:

$$X_x = -Ay, \quad Y_y = X_y = 0. \tag{82.3a}$$

Hence, cutting from the body a strip bounded by the straight lines $y = \pm a$, the edges of this strip will be free from external stresses, while purely normal forces $X_x = -Ay$ will act on any transverse (i.e., parallel to Oy) section of the strip. These forces are statically equivalent to a couple with moment

$$M = 2h \int_{-a}^{+a} A y^2 dy = \frac{4}{3} A h a^3, \tag{82.4a}$$

where $2h$ is the thickness of the plate (normal to the plane Oxy). Thus the above function solves the problem of bending of continuous strips (beams) by couples, applied to their ends (Fig. 38). The functions $\varphi_1(z)$, $\psi_1(z)$ corresponding to U are easily seen to be

$$\varphi_1(z) = \frac{Aiz^2}{8}, \quad \psi_1(z) = -\frac{Aiz^2}{8}. \tag{82.5a}$$

It will now be assumed that an elliptic hole with centre at the origin has been cut out of the strip. The problem of bending of such a beam

will now be solved approximately, subject to the conditions that the edge of the hole is free from external forces and that, at large distances from the hole, the state of stress tends to that given by (82.3a); thus it will be assumed that the dimensions of the hole are small compared with the length of the beam (see also the penultimate paragraph of this section) and the problem will be solved as if the elliptic hole were in an unbounded plate. Under these circumstances one must have

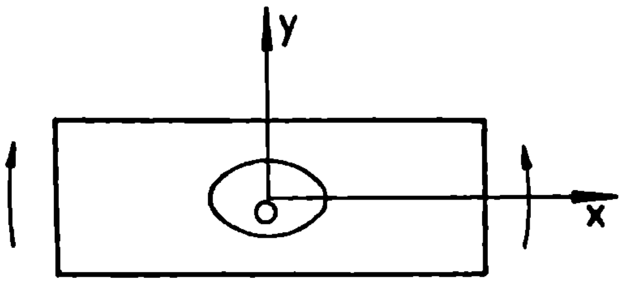


Fig. 38.

$$\begin{aligned} \varphi_1(z) &= \varphi_1^0(z) + \frac{Aiz^2}{8}, \\ \psi_1(z) &= \psi_1^0(z) - Aiz^2 \end{aligned} \qquad (82.6a)$$

where φ_1^0, ψ_1^0 are functions, holomorphic outside the ellipse including the point at infinity.

For simplicity, it will be assumed that the major axis of the ellipse is directed along the axis of the beam. The solution of the general problem would only be slightly more complicated. The solution for the particular case when the major axis is perpendicular to the axis of the beam was found by A. S. Lokshin [1] using different methods.

Introducing the variable ζ, one has in an obvious notation

$$\begin{aligned} \varphi(\zeta) &= \varphi_0(\zeta) + \frac{Ai}{8} R^2 \left(\zeta + \frac{m}{\zeta}\right)^2, \\ \psi(\zeta) &= \psi_0(\zeta) - \frac{Ai}{8} R^2 \left(\zeta + \frac{m}{\zeta}\right)^2. \end{aligned} \qquad (82.7a)$$

Substituting from (82.7a) in the boundary conditions (82.2) or (82.3) with $f = 0$, it is seen that $\varphi_0(\zeta)$, $\psi_0(\zeta)$ satisfy the same conditions, provided one takes instead of f or $\bar{f}$

$$\begin{aligned} f_0 &= -\frac{AiR^2(1-m)^2}{8}\left(\sigma - \frac{1}{\sigma}\right)^2, \\ \bar{f}_0 &= \frac{AiR^2(1-m)^2}{8}\left(\sigma - \frac{1}{\sigma}\right)^2 \end{aligned} \qquad (82.8a)$$

respectively. Substituting these expressions in (82.4') and (82.5'), noting that the right-hand sides of the preceding formulae represent functions holomorphic inside γ with the exception of the point $\sigma = 0$, where they have poles with the principal parts

$$-\frac{R^2A(1-m)^2 i}{8\sigma^2}, \quad \frac{R^2A(1-m)^2 i}{8\sigma^2}$$

respectively, and applying (70.4), one finds immediately

$$\varphi_0(\zeta) = -\frac{1}{2\pi i}\int_\gamma \frac{f_0\, d\sigma}{\sigma - \zeta} = -\frac{R^2A(1-m)^2 i}{8\zeta^2},$$

$$\psi_0(\zeta) = -\frac{1}{2\pi i}\int_\gamma \frac{\bar{f}_0\, d\sigma}{\sigma - \zeta} - \zeta\frac{1+m\zeta^2}{\zeta^2 - m}\varphi_0'(\zeta) = \tag{82.9a}$$

$$= \frac{R^2A(1-m)^2 i}{8\zeta^2} - \frac{R^2A(1-m)^2 i}{4\zeta^2}\cdot\frac{1+m\zeta^2}{\zeta^2 - m},$$

and the problem is solved.

For $m = 0$, one obtains the solution for the circular hole, while $m = 1$ gives that for the case of a straight cut; it is easily verified that in the latter case $\varphi_0(\zeta) = \psi_0(\zeta) = 0$, i.e., a longitudinal cut does not influence the state of stress.

The problem of bending by transverse forces and other analogous problems may be solved in the same manner. A number of such problems for the case of circular, elliptic and some other types of holes (in fact, holes bounded by hypotrochoids approximating to triangles and squares, cf. § 48, 4°) were solved and studied in detail by M. I. Naiman [1] using the methods of this book. Many problems, important from the point of view of application, were solved by G. N. Savin [2] who reduced them to numerically convenient formulae and gave a number of tables which enabled comparison of the deduced results with experiments; more will be said about Savin's work in § 89. S. G. Lekhnitzky [2] studied several cases of bending of beams with circular holes at a somewhat earlier stage. Even earlier than this, Z. Tuzi [1] gave the solution of the problem of pure bending of a beam with a circular hole (which can be obtained from the solution given above by putting $m = 0$).

Experiments with models have shown that the solution remains sufficiently exact from the practical point of view, when the dimensions

of the hole are not small compared with the width of the strip, provided they are not larger than 3/5 of the width of the beam for circular holes (Z. Tuzi) and 1/3 for square holes (G. N. Savin).

All the above solutions are approximate and based on the consideration of an infinite plane with the corresponding holes. There also exists a (fairly complicated) exact solution of the first fundamental problem of the theory of elasticity for an infinite strip (of finite width) with symmetrically distributed circular holes which was given by R. C. I. Howland and A. C. Stevenson [1].

§ 83. Solution of the second fundamental problem for the infinite plane with an elliptic hole. In this case the boundary condition has the form

$$\varkappa\varphi(\sigma) - \frac{1}{\sigma}\frac{\sigma^2+m}{1-m\sigma^2}\overline{\varphi'(\sigma)} - \overline{\psi(\sigma)} = 2\mu(g_1+ig_2) = 2\mu g \tag{83.1}$$

or

$$\varkappa\overline{\varphi(\sigma)} - \sigma\frac{1+m\sigma^2}{\sigma^2-m}\varphi'(\sigma) - \psi(\sigma) = 2\mu(g_1-ig_2) = 2\mu\bar{g}, \tag{83.2}$$

where g_1, g_2 are the given components of displacement of points on the ellipse.

At first, it will be assumed that the displacements remain bounded at infinity (i.e., $X = Y = \Gamma = \Gamma' = 0$); one then obtains in the same manner as in § 82

$$\varphi(\zeta) = -\frac{2\mu}{\varkappa}\frac{1}{2\pi i}\int_\gamma\frac{g\,d\sigma}{\sigma-\zeta}, \tag{83.3}$$

$$\psi(\zeta) = \frac{\mu}{\pi i}\int_\gamma\frac{\bar{g}\,d\sigma}{\sigma-\zeta} - \zeta\frac{1+m\zeta^2}{\zeta^2-m}\varphi'(\zeta) + \psi(\infty) \tag{83.4}$$

If one leaves the value of $\psi(\infty)$ arbitrary, the boundary condition will be fulfilled apart from a constant term. In order to determine $\psi(\infty)$, multiply both sides of (83.2) by $\frac{1}{2\pi i}\frac{d\sigma}{\sigma}$ and integrate around γ. This is easily seen to give [cf. remarks following (82.5)]

$$\psi(\infty) = -\frac{\mu}{\pi i}\int_\gamma\bar{g}\frac{d\sigma}{\sigma}, \tag{83.5}$$

and the problem is solved for the case when the displacements are to be bounded at infinity.

In the general case, assuming as before that $\Gamma = \bar{\Gamma}$, i.e., that *the rotation at infinity is zero*, one has

$$\varphi(\zeta) = \Gamma R\zeta - \frac{X + iY}{2\pi(1 + \varkappa)} \log \zeta + \varphi_0(\zeta), \tag{83.6}$$

$$\varphi(\zeta) = \Gamma' R\zeta + \frac{\varkappa(X - iY)}{2\pi(1 + \varkappa)} \log \zeta + \psi_0(\zeta). \tag{83.7}$$

Substituting these expressions in (83.1), one sees that $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ satisfy the same boundary condition as $\varphi(\zeta)$ and $\psi(\zeta)$, except that $2\mu g$ must now be replaced by

$$2\mu g_0 = 2\mu g - \Gamma R\left(\varkappa\sigma - \frac{1}{\sigma}\,\frac{\sigma^2 + m}{1 - m\sigma^2}\right) + \\ + \frac{\bar{\Gamma}' R}{\sigma} - \frac{X - iY}{2\pi(\varkappa + 1)} \cdot \frac{\sigma^2 + m}{1 - m\sigma^2}, \tag{83.8}$$

whence follows

$$2\mu \bar{g}_0 = 2\mu \bar{g} - \Gamma R\left(\frac{\varkappa}{\sigma} - \sigma\,\frac{1 + m\sigma^2}{\sigma^2 - m}\right) + \\ + \Gamma' R\sigma - \frac{X + iY}{2\pi(\varkappa + 1)} \cdot \frac{1 + m\sigma^2}{\sigma^2 - m}. \tag{83.9}$$

The values of $\varphi_0(\zeta)$ and $\psi_0(\zeta)$ will be obtained from (83.3) and (83.4) by replacing φ, ψ, g, $\bar{g}$ by φ_0, ψ_0, g_0, $\bar{g}_0$ respectively. Thus (cf. § 82*a*, example 1°)

$$\varphi_0(\zeta) = -\frac{2\mu}{\varkappa}\,\frac{1}{2\pi i}\int_{\gamma} \frac{g\,d\sigma}{\sigma - \zeta} + (\Gamma m + \bar{\Gamma}')\frac{R}{\varkappa\zeta}, \tag{83.10}$$

$$\psi_0(\zeta) = \frac{\mu}{\pi i}\int_{\gamma} \frac{\bar{g}\,d\sigma}{\sigma - \zeta} + \Gamma R\left(\frac{\varkappa}{\zeta} - \frac{(1 + m^2)\zeta}{\zeta^2 - m}\right) + \\ + \frac{X + iY}{2\pi(\varkappa + 1)} \cdot \frac{1 + m^2}{\zeta^2 - m} - \zeta\,\frac{1 + m\zeta^2}{\zeta^2 - m}\,\varphi_0'(\zeta) + \psi_0(\infty), \tag{83.11}$$

where $\psi_0(\infty)$ is determined by the following formula, obtained from

(83.5) and (83.9),

$$\psi_0(\infty) = \frac{\mu}{\pi i}\int_\gamma \frac{\bar{g}_0\,d\sigma}{\sigma} = -\frac{\mu}{\pi i}\int_\gamma \frac{\bar{g}\,d\sigma}{\sigma} + \frac{m(X+iY)}{2\pi(\varkappa+1)}. \quad (83.12)$$

It is easily verified that the above solution will be regular, if the function g, given on the contour, has a derivative satisfying the H condition.

In the limiting case $m = 1$, one obtains the solution of the second fundamental problem for the infinite plane with a straight cut.

§ 83a. Examples.

1°. Uni-directional tension of an infinite plate with a rigid elliptic centre. Let the infinite plate with

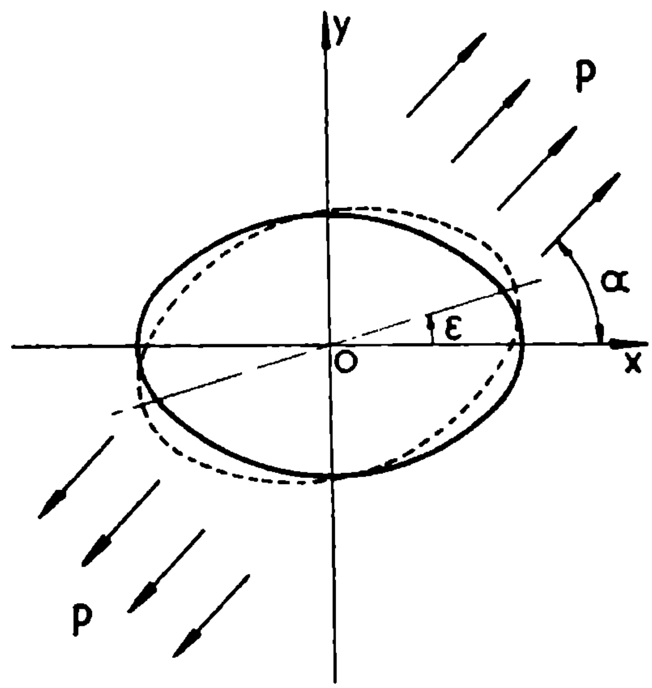


Fig. 39.

a rigid elliptic centre, *welded* into it, be subject to uni-directional tension, as in § 82*a*, example 1° (Cf. Fig. 39). It will be assumed that no external forces, apart from those exerted by the surrounding material, act on the rigid kernel, and hence $X = Y = 0$. In the notation of § 82*a*, example 1°, one will have

$$\Gamma = \bar{\Gamma} = \frac{p}{4}, \quad \Gamma' = -\tfrac{1}{2}pe^{-2i\alpha}$$

The applied tension may cause a (rigid) translation and rotation of the kernel. Since the translation may be eliminated by a rigid displacement of the entire system, it may be neglected and it may be assumed that the kernel rotates about its centre by an (unknown) angle ε. The boundary values of the displacement components will thus be

$$g_1 = -\varepsilon y, \quad g_2 = +\varepsilon x, \tag{83.1a}$$

so that

$$g = i\varepsilon(x + iy) = i\varepsilon z = i\varepsilon R\left(\sigma + \frac{m}{\sigma}\right), \quad \bar{g} = -i\varepsilon R\left(\frac{1}{\sigma} + m\sigma\right).$$

Further, since

$$\frac{1}{2\pi i}\int_\gamma \frac{g\,d\sigma}{\sigma - \zeta} = i\varepsilon R \cdot \frac{1}{2\pi i}\int_\gamma \left(\sigma + \frac{m}{\sigma}\right)\frac{d\sigma}{\sigma - \zeta} = -\frac{i\varepsilon Rm}{\zeta},$$

$$\frac{1}{2\pi i}\int_\gamma \frac{\bar{g}\,d\sigma}{\sigma - \zeta} = \frac{i\varepsilon R}{\zeta}$$

and, by (83.12), $\psi_0(\infty) = 0$, one obtains from (83.6), (83.7), (83.10) and (83.11), putting $X = Y = 0$,

$$\varphi(\zeta) = \Gamma R\zeta + (2\mu m\varepsilon i + \Gamma m + \bar{\Gamma}')\frac{R}{\varkappa\zeta},$$

$$\psi(\zeta) = \Gamma' R\zeta + \frac{2\mu\varepsilon Ri}{\zeta} + \Gamma R\left(\frac{\varkappa}{\zeta} - \frac{(1 + m^2)\zeta}{\zeta^2 - m}\right) + \tag{83.2a}$$

$$+ (2\mu m\varepsilon i + \Gamma m + \bar{\Gamma}')\frac{1 + m\zeta^2}{\zeta^2 - m} \cdot \frac{R}{\varkappa\zeta}.$$

There remains the determination of the angle ε from the condition that the resultant moment of the forces, acting on the elliptic centre from the surrounding material, must vanish. This moment will be calculated by the help of (33.3).

Since in the present case $\varphi(\zeta)$, $\psi(\zeta)$, and hence $\varphi_1(z)$, $\psi_1(z)$, are single-valued, the resultant moment M_0 of the forces, acting on the side of the rigid centre, will be equal to the increase of $\Re\chi_1(z)$ for a complete circuit of the ellipse (in clockwise direction). Thus it will be sufficient to calculate the multi-valued term of

$$\Re\chi_1(z) = \Re\int \psi_1(z)dz = \Re\int \psi(\zeta)\omega'(\zeta)d\zeta.$$

The second formula of (83.2a) shows that, putting $\Gamma' = B' + iC'$, this multi-valued term is

$$\Re\left\{i2\mu\varepsilon R^2\left(1+\frac{m^2}{\varkappa}\right)\log\zeta - iC'mR^2\left(1+\frac{1}{\varkappa}\right)\log\zeta\right\}.$$

Hence

$$M_0 = 4\pi\mu\varepsilon R^2\left(1+\frac{m^2}{\varkappa}\right) - 2\pi mR^2C'\left(1+\frac{1}{\varkappa}\right). \tag{83.3a}$$

The condition $M_0 = 0$ thus gives

$$\varepsilon = \frac{m(1+\varkappa)C'}{2\mu(m^2} = \frac{pm(1+\varkappa)\sin 2\alpha}{4\mu(m^2+\varkappa)} \tag{83.4a}$$

and the problem is solved. In the case of a circular centre $(m = 0)$ the rotation is zero, while in the limiting case of the segment of a straight line, i.e., for a straight rigid reinforcement $(m = 1)$, it is

$$\varepsilon = \frac{p\sin 2\alpha}{4\mu} \tag{83.4'a}$$

In the case of *bi-axial tension*, when

$$\Gamma = \overline{\Gamma} = \frac{p}{2}, \quad \Gamma' = 0,$$

one obviously will have $\varepsilon = 0$ and

$$\varphi(\zeta) = \frac{pR}{2}\left(\zeta+\frac{m}{\varkappa\zeta}\right), \quad \psi(\zeta) = \frac{pR}{2}\left(\frac{\varkappa}{\zeta} - \frac{(1+m^2)\zeta}{\zeta^2-m} + \frac{1+m\zeta^2}{\zeta^2-m}\cdot\frac{m}{\varkappa\zeta}\right). \tag{83.5a}$$

2°. Case when the elliptic centre is not allowed to rotate.

If under the conditions of the preceding example (uni-axial tension) the rigid elliptic kernel is restrained in its original position by a couple, then $\varepsilon = 0$ and (83.2a) gives

$$\begin{aligned}\varphi(\zeta) &= \Gamma R\zeta + (\Gamma m + \overline{\Gamma}')\frac{R}{\varkappa\zeta},\\ \psi(\zeta) &= \Gamma' R\zeta + \Gamma R\left(\frac{\varkappa}{\zeta} - \frac{(1+m^2)\zeta}{\zeta^2-m}\right) + (\Gamma m + \overline{\Gamma}')\frac{1+m\zeta^2}{\zeta^2-m}\cdot\frac{R}{\varkappa\zeta}.\end{aligned} \tag{83.6a}$$

The moment M_0 of the couple restraining the kernel is, by (83.3*a*),

$$M_0 = -2\pi m\, R^2 C'\left(1 + \frac{1}{\varkappa}\right) = -\pi p m R^2 \left(1 + \frac{1}{\varkappa}\right)\sin 2\alpha. \quad (83.7a)$$

3°. Case when a couple with given moment acts on the elliptic kernel.

It will be assumed that the stresses vanish at infinity. Then (83.2*a*) gives

$$\varphi(\zeta) = \frac{2\mu m\varepsilon Ri}{\varkappa\zeta}, \quad \psi(\zeta) = \frac{2\mu\varepsilon Ri}{\varkappa\zeta}\left(\varkappa + m\,\frac{1 + m\zeta^2}{\zeta^2 - m}\right), \quad (83.8a)$$

where by (83.3*a*)

$$\varepsilon = \frac{M_0\varkappa}{4\pi\mu R^2(m + \varkappa)}. \quad (83.9a)$$

4°. Case when a force acts on the centre of the elliptic kernel.

It will be assumed that the stresses vanish at infinity. It is easily seen that the kernel does not rotate in this case. In fact, this is obvious from symmetry considerations for the cases when the force acts along one of the axes of the ellipse; the general case is then obtained as a combination of these two cases. Further, it may be assumed that the kernel, in general, remains in its original position (because this may always be brought about by a rigid displacement of the entire system). Hence one has in (83.10), (83.11) and (83.12): $g = 0$, $\Gamma = \Gamma' = 0$, whence it follows that

$$\varphi_0(\zeta) = 0, \quad \psi_0(\zeta) = \frac{X + iY}{2\pi(\varkappa + 1)}\cdot\frac{1 + m^2}{\zeta^2 - m} + \frac{m(X + iY)}{2\pi(\varkappa + 1)},$$

where (X, Y) is the applied force; therefore, by (83.6) and (83.7),

$$\varphi(\zeta) = -\frac{X + iY}{2\pi(\varkappa + 1)}\log\zeta,$$

$$\psi(\zeta) = \frac{\varkappa(X - iY)}{2\pi(\varkappa + 1)}\log\zeta + \frac{X + iY}{2\pi(\varkappa + 1)}\cdot\frac{1 + m^2}{\zeta^2 - m} + \frac{m(X + iY)}{2\pi(\varkappa + 1)}.$$

§ 84. General solution of the fundamental problems for regions, mapped on to the circle by the help of polynomials. It is not accidental that the solutions for the regions considered in the preceding

sections (§ 80—83) have been so simple and elementary. Actually, it has been shown that the solution of the fundamental problems may always be obtained in elementary form and that, in fact, it may be expressed by Cauchy type integrals, provided the mapping function $\omega(\zeta)$ is rational.

The solution of the fundamental, biharmonic problem for the case when $\omega(\zeta)$ is a *polynomial* was first given by E. Almansi [1]. T. Boggio [1, 2] stated a method of solution of the second fundamental problem for the case when $\omega(\zeta)$ is a rational function. The present method is quite different from the methods employed by these authors and is, in the Author's opinion, much simpler. It was first studied in the Author's paper [4] and given in greater detail in his paper [5].

A beginning will be made with the *first fundamental problem* and, in particular, with the case when $\omega(\zeta)$, mapping S on to the circle $|\zeta| < 1$, is a polynomial

$$\omega(\zeta) = c_1\zeta + c_2\zeta^2 + \ldots + c_n\zeta^n \quad (c_1 \neq 0, \quad c_n \neq 0) \tag{84.1}$$

(i.e., the region S must be finite); c_1 cannot be zero, because $\omega'(\zeta)$ would vanish inside the circle and the transformation would not be single-valued and invertible. No generality is lost by omitting a constant term, i.e., it may be assumed that $\zeta = 0$ corresponds to $z = 0$.

In this case the functional equation (78.10), viz.,

$$\varphi(\zeta) \quad \frac{1}{2\pi i} \quad \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \frac{\overline{\varphi'(\sigma)}d\sigma}{\sigma - \zeta} + \bar{a} \quad A(\zeta), \tag{84.2}$$

which determines the function $\varphi(\zeta)$, may be solved in an elementary and very simple manner. It will be remembered that the function $A(\zeta)$ is given by

$$A(\zeta) = \frac{1}{2\pi i} \int_{\gamma} \frac{f\,d\sigma}{\sigma - \zeta}; \tag{84.3}$$

as in § 78, it will be assumed that $f = f_1 + if_2$, given on γ, has a derivative satisfying the H condition. Further, a denotes a constant $[a = \psi(0)]$ which has to be determined at the same time as $\varphi(\zeta)$. In fact, this constant will be determined by the condition $\varphi(0) = 0$. If this condition is omitted (which may be done without influencing the stress distribution), the constant may be left arbitrary.

In the present case $\omega(\sigma)/\overline{\omega'(\sigma)}$ is the boundary value of the rational

function

$$\frac{\omega(\zeta)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)} = \frac{c_1\zeta + \ldots + c_n\zeta^n}{\bar{c}_1 + 2\bar{c}_2\zeta^{-1} + \ldots + n\bar{c}_n\zeta^{-n+1}} = \zeta^n \frac{c_1 + \ldots + c_n\zeta^{n-1}}{\bar{c}_1\zeta^{n-1} + \ldots + n\bar{c}_n}, \quad (84.4)$$

holomorphic outside γ (cf. § 63), except at $\zeta = \infty$ where it has a pole of order n. Hence this function may be represented outside γ in the form

$$\frac{\omega(\zeta)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)} = b_n\zeta^n + b_{n-1}\zeta^{n-1} + \ldots + b_1\zeta + b_0 + \sum_{k=1}^{\infty} b_{-k}\zeta^{-k}. \quad (84.5)$$

It should be noted that it is unnecessary for the deduction of the solution up to the boundary to calculate all the coefficients of (84.5): *it is sufficient to determine only b_0, b_1, ..., b_n*, and this is known *to require only elementary algebraic operations.*

As a result of the fact that $\omega(\sigma)/\overline{\omega'(\sigma)}$ has the above stated form the integral on the left-hand side of (84.2) can be calculated by elementary methods. In fact, $\overline{\varphi'(\sigma)}$ is the boundary value of $\bar{\varphi}'(1/\zeta)$, holomorphic outside γ (§ 76, 2°). Hence

$$\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\, \overline{\varphi'(\sigma)}$$

is the boundary value of

$$\frac{\omega(\zeta)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)}\, \bar{\varphi}'\left(\frac{1}{\zeta}\right),$$

holomorphic outside γ except at $\zeta = \infty$ where it has a pole of order not greater than n. Its principal part there will now be found. Remembering the condition $\varphi(0) = 0$, one has

$$\varphi(\zeta) = a_1\zeta + a_2\zeta^2 + \ldots + a_n\zeta^n + \ldots \quad (|\zeta| < 1) \quad (84.6)$$

(*where only those terms which have been written down will, in fact, be required*) and hence

$$\varphi'(\zeta) = a_1 + 2a_2\zeta + \ldots + na_n\zeta^{n-1} + \ldots,$$

$$\bar{\varphi}'\left(\frac{1}{\zeta}\right) = \bar{a}_1 + \frac{2\bar{a}_2}{\zeta} + \ldots + \frac{n\bar{a}_n}{\zeta^{n-1}} + \ldots \quad (|\zeta| > 1).$$

Thus, for $|\zeta| > 1$,

$$\frac{\omega(\zeta)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)}\,\bar{\varphi}'\left(\frac{1}{\zeta}\right) = K_0 + K_1\zeta + \ldots + K_n\zeta^n + O\left(\frac{1}{\zeta}\right), \tag{84.7}$$

where

$$\begin{aligned} K_1 &= \bar{a}_1 b_1 + 2\bar{a}_2 b_2 + \ldots + n\bar{a}_n b_n, \\ K_2 &= \bar{a}_1 b_2 + \ldots + (n-1)\,\bar{a}_{n-1} b_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ K_n &= \bar{a}_1 b_n \end{aligned} \tag{84.8}$$

(the expression for K_0 is not required).

Therefore, by (70.4′),

$$\frac{1}{2\pi i}\int \frac{\omega(\sigma)\varphi'(\sigma)d\sigma}{\omega'(\sigma)\,(\sigma-\zeta)} = K_0 + K_1\zeta + \ldots + K_n\zeta^n, \tag{84.9}$$

and (84.2) gives directly

$$\varphi(\zeta) + \bar{a} + K_0 + K_1\zeta + \ldots + K_n\zeta^n = \frac{1}{2\pi i}\int \frac{f\,d\sigma}{\sigma-\zeta} \tag{84.10}$$

In this expression the constants

$$\bar{a} + K_0, \quad K_1, \ldots, K_n$$

are, at present, still unknown; they must be determined from the conditions that $\varphi(0) = 0$ and that the constants $a_1, a_2, \ldots, a_n$, occurring implicitly in the expression (84.10) for $\varphi(\zeta)$ by means of $K_0, K_1, \ldots, K_n$, must represent the coefficients of (84.6). These conditions will now be formulated. Noting that

$$\frac{1}{\sigma-\zeta} = \frac{1}{\sigma} + \frac{\zeta}{\sigma^2} + \frac{\zeta^2}{\sigma^3} + \ldots,$$

one has

$$A(\zeta) = \frac{1}{2\pi i}\int \frac{f\,d\sigma}{\sigma-\zeta} = A_0 + A_1\zeta + A_2\zeta^2 + \ldots, \tag{84.11}$$

where

$$A_k = \frac{1}{2\pi i}\int f\,\sigma^{-k-1}d\sigma = \frac{1}{2\pi}\int_0^{2\pi} f\,e^{-ik\vartheta}d\vartheta, \quad k = 0, 1, 2, \ldots. \tag{84.12}$$

Comparing coefficients of $\zeta^0, \zeta^1, \ldots, \zeta^n$ on both sides of (84.10), one obtains the $(n+1)$ equations

$$\bar{a} + K_0 = A_0, \tag{84.13}$$

$$a_k + K_k = A_k, \quad k = 1, 2, \ldots, n. \tag{84.14}$$

The equation (84.13) determines the value of $(\bar{a} + K_0)$. Further, substituting in (84.14) from (84.8), one finds the system

$$\begin{aligned} a_1 + \bar{a}_1 b_1 + \ldots + n\bar{a}_n b_n &= A_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_n + \bar{a}_1 b_n \qquad\qquad &= A_n, \end{aligned} \tag{84.14'}$$

which agrees with the system (63.6″). Putting $a_k = \alpha_k + i\beta_k$ and separating real and imaginary parts, one obtains from (84.14′) $2n$ linear equations with the $2n$ unknowns

$$\alpha_k, \beta_k, \quad k = 1, 2, \ldots, n.$$

(Instead of this system of equations one may construct the conjugate complex system in terms of $\bar{a}_k$.)

This system must have a solution, provided the condition for the vanishing of the resulting moment of the external forces is satisfied, i.e., the condition

$$\int_L (f_1 dx + f_2 dy) = 0; \tag{84.15}$$

in fact, it is known that under this condition the original problem has a solution, and hence the system (84.14′) cannot be incompatible. In addition, it is clear on the basis of the uniqueness theorem that the system under consideration completely determines the unknown constants, except for a_1 which is not completely determined. Actually, only the real part of $a_1/\omega'(0)$ is determined, while its imaginary part remains arbitrary, since it is known that the imaginary part of $\varphi_1'(0) = \varphi'(0)/\omega'(0) = a_1/\omega'(0)$ is quite arbitrary.

Thus one obtains $\varphi(\zeta)$, satisfying (84.2) for a suitable value of the constant a, by fixing (arbitrarily) the imaginary part of $a_1/\omega'(0)$, by finding the quantities $a_1, \ldots, a_n$ from (84.14′), by substituting these in the expressions for $K_1, \ldots, K_n$ and by introducing these last values together with that of $\bar{a} + K_0$ in (84.10). (The value of a will be given by (84.13) in which

K_0 has a definite value, since $\varphi(\zeta)$ has been found; however, it is not necessary to calculate this constant.)

After this the function $\psi(\zeta)$ may be calculated from (78.8), viz.,

$$\psi(\zeta) = \frac{1}{2\pi i}\int \frac{\bar{f}\,d\sigma}{\sigma - \zeta} - \frac{1}{2\pi i}\int \frac{\overline{\omega(\sigma)}}{\overline{\omega'(\sigma)}}\,\frac{\varphi'(\sigma)d\sigma}{\sigma - \zeta}\,; \tag{a}$$

in the present case the second integral on the right-hand side is expressed in finite form by $\varphi(\zeta)$. In fact,

$$\frac{\overline{\omega(\sigma)}}{\overline{\omega'(\sigma)}}\,\varphi'(\sigma)$$

is the boundary value of

$$\frac{\bar{\omega}\left(\frac{1}{\zeta}\right)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)}\,\varphi'(\zeta),$$

holomorphic inside γ, except at $\zeta = 0$; on the basis of (84.7), it is easily seen that one has inside γ

$$\frac{\bar{\omega}\left(\frac{1}{\zeta}\right)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)}\,\varphi'(\zeta) = \bar{K}_1\frac{1}{\zeta} + \ldots + \bar{K}_n\frac{1}{\zeta^n} + \text{a holomorphic function.}$$

Hence, by (70.3),

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega(\sigma)}\varphi'(\sigma)d\sigma}{\overline{\omega'(\sigma)}\,(\sigma - \zeta)} = \frac{\bar{\omega}\left(\frac{1}{\zeta}\right)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)}\,\varphi'(\zeta) - \frac{\bar{K}_1}{\zeta} - \ldots - \frac{\bar{K}_n}{\zeta^n},$$

and (*a*) gives

$$\psi(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\bar{f}\,d\sigma}{\sigma - \zeta} - \frac{\bar{\omega}\left(\frac{1}{\zeta}\right)}{\bar{\omega}'\left(\frac{1}{\zeta}\right)}\,\varphi'(\zeta) + \frac{\bar{K}_1}{\zeta} + \ldots + \frac{\bar{K}_n}{\zeta^n}. \tag{84.16}$$

The preceding results apply, with obvious insignificant modifications, to *the case of an infinite region* S mapped on to the circle $|\zeta| < 1$ by a function of the form

$$\omega(\zeta) = \frac{c_0}{\zeta} + c_1\zeta + \ldots + c_n\zeta^n. \tag{84.1'}$$

The problem must first be reduced to the case when $\varphi(\zeta)$ and $\psi(\zeta)$ are holomorphic inside γ (§ 78). Afterwards, the procedure stated above may be followed. In this case the solution becomes even simpler, because the system, analogous to (84.14′), will always have a (unique) solution without the supplementary condition (84.15).

§ 85. Generalization to the case of transformations by means of rational functions. The cases of regions, mapped on to the circle by the help of polynomials and of functions of the form (84.1′), are particular cases of regions mapped by means of rational functions of general form. In this more general case the solution may be obtained by the same method as above; the only difference is that in this case one has, in general, to calculate roots of a certain algebraic equation.

Consider again the functional equation (78.10), viz.,

$$\varphi(\zeta) + \frac{1}{2\pi i}\int_{\gamma} \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}} \cdot \frac{\overline{\varphi'(\sigma)}d\sigma}{\sigma - \zeta} + \bar{a} = \frac{1}{2\pi i}\int_{\gamma} \frac{f\,d\sigma}{\sigma - \zeta} = A(\zeta); \quad (85.1)$$

although the notation used here is the same as in the preceding section, the function $\omega(\zeta)$ is now a rational function of general form which transforms conformally the given region S on to the circle $|\zeta| < 1$. In the case when S is infinite it will be assumed that the point $z = \infty$ corresponds to the point $\zeta = 0$.

Also in the present case the integral on the left-hand side of (85.1) may be evaluated by elementary means, as it will now be proved.

Consider the expression $\overline{\omega(\sigma)}/\omega'(\sigma)$ which is the boundary value of the rational function

$$\frac{\overline{\omega}\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)}.$$

Since $\omega(\zeta)$ may now have outside γ poles other than at $\zeta = \infty$, $\overline{\omega}(1/\zeta)$ may have poles inside γ, and not only at the point $\zeta = 0$, as it was the case in § 84. The function $\overline{\omega}(1/\zeta)$ cannot have poles outside and on γ except at $\zeta = \infty$, because $\omega(\zeta)$ must be continuous inside and on γ, except at $\zeta = 0$ in the case when the region is infinite. Similarly, it will be remembered that $\omega'(\zeta)$ cannot have zeros inside or on γ.

Denote the poles of $\omega(\zeta)$, other than the pole $\zeta = \infty$ (if it exists), by $\zeta_1, \zeta_2, \ldots, \zeta_n$; these poles are the roots of the algebraic equation

$1/\omega(\zeta) = 0$ to which reference has been made earlier, and all of them will lie outside γ. Then the poles of the function $\overline{\omega}(1/\zeta)$, other than the pole $\zeta = 0$, will be

$$\zeta_1' = \frac{1}{\zeta_1}, \zeta_2' = \frac{1}{\zeta_2}, \ldots, \zeta_n' = \frac{1}{\zeta_n},$$

all of which lie inside γ. These points and, generally speaking, the point $\zeta = 0$ will also be poles of the function $\overline{\omega}(1/\zeta)/\omega'(\zeta)$ which lie inside γ. Hence this function may obviously be represented in the following manner:

$$\frac{\overline{\omega}(\zeta^{-1})}{\omega'(\zeta)} = c_0 + \sum_{l=1}^{m_0} \frac{c_l}{\zeta^l} + \sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{c_{kl}}{(\zeta - \zeta_k')^l} + R(\zeta), \tag{85.2}$$

where $c_0, \ldots, c_l, c_{kl}$ are known constants, $R(\zeta)$ is a rational function, holomorphic inside and on γ and vanishing at $\zeta = 0$, and $m_0, m_1, \ldots, m_n$ are the orders of the poles $0, \zeta_1', \ldots, \zeta_n'$, respectively.

Consider now the product

$$\Omega(\zeta) = \frac{\overline{\omega}\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)} \varphi'(\zeta).$$

Obviously, this product represents a function, holomorphic inside γ with the exclusion of the points $0, \zeta_1', \ldots, \zeta_n'$, where it may have poles of order $m_0, m_1, \ldots, m_n$ (but not higher), and hence it may be written in a form, analogous to (85.2),

$$\Omega(\zeta) = C_0 + \sum_{l=1}^{m_0} \frac{C_l}{\zeta^l} + \sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{C_{kl}}{(\zeta - \zeta_k')^l} + \Omega_0(\zeta), \tag{85.3}$$

where C_0, C_l, C_{kl} are constants and $\Omega_0(\zeta)$ is a function, holomorphic inside γ and vanishing for $\zeta = 0$.

It is easily seen that the following important statement is true: the constants C_l $(l = 1, 2, \ldots, m_0)$ are linear combinations (with known constant coefficients) of the quantities

$$\varphi'(0), \quad \varphi''(0), \ldots, \varphi^{(m_0)}(0), \tag{a}$$

and the constants C_{kl} are similar combinations of

$$\varphi'(\zeta_k'), \quad \varphi''(\zeta_k'), \ldots, \varphi^{(m_k)}(\zeta_k'), \quad k = 1, 2, \ldots, n. \tag{b}$$

These relations are readily written down. There exists an analogous combination for C_0, but this will not be required.

Now consider the expression

$$\frac{\omega(\sigma)}{\omega'(\sigma)}\overline{\varphi'(\sigma)}$$

on the left-hand side of (85.1) which is the boundary value of

$$\overline{\Omega}\left(\frac{1}{\zeta}\right)=\frac{\omega(\zeta)}{\overline{\omega}'\left(\frac{1}{\zeta}\right)}\overline{\varphi}'\left(\frac{1}{\zeta}\right).$$

By (85.3), this may be written in the form (remembering that $\overline{\zeta}_k'=1/\zeta_k$)

$$\overline{\Omega}\left(\frac{1}{\zeta}\right)=\overline{C}_0+\sum_{l=1}^{m_0}\overline{C}_l\zeta^l+\sum_{k=1}^{n}\sum_{l=1}^{m_k}\frac{\overline{C}_{kl}\zeta_k^l\zeta^l}{(\zeta_k-\zeta)^l}+\overline{\Omega}_0\left(\frac{1}{\zeta}\right), \tag{85.4}$$

where $\overline{\Omega}_0\left(\frac{1}{\zeta}\right)$ is holomorphic outside γ and vanishes at $\zeta=\infty$. The constants $\overline{C}_l$, $\overline{C}_{kl}$ are obviously linear combinations of quantities, conjugate complex to (*a*) and (*b*).

Applying (70.4′), one immediately obtains

$$\frac{1}{2\pi i}\int_\gamma\frac{\omega(\sigma)\overline{\varphi'(\sigma)}d\sigma}{\overline{\omega'(\sigma)}\,(\sigma-\zeta)}=\overline{C}_0+\sum_{l=1}^{m_0}\overline{C}_l\zeta^l+\sum_{k=1}^{n}\sum_{l=1}^{m_k}\frac{\overline{C}_{kl}\zeta_k^l\zeta^l}{(\zeta_k-\zeta)^l}.$$

It is easily seen that the expression

$$\overline{C}_0+\sum_{l=1}^{m_0}\overline{C}_l\zeta^l+\sum_{k=1}^{n}\sum_{l=1}^{m_k}\frac{\overline{C}_{kl}\zeta_k^l\zeta^l}{(\zeta_k-\zeta)^l}$$

may be written in the form

$$C_0'+\sum_{l=1}^{m_0}\overline{C}_l\zeta^l+\sum_{k=1}^{n}\sum_{l=1}^{m_k}\frac{C_{kl}'}{(\zeta-\zeta_k)^l},$$

where C_0', C_{kl}' are constants, i.e., using the notation of § 70, in the form

$$G_\infty(\zeta)+G_1(\zeta)+\ldots+G_n(\zeta).$$

Substituting this expression in the left-hand side of (85.1), one finally obtains

$$\varphi(\zeta)+\bar{a}+\overline{C}_0+\sum_{l=1}^{m_0}\overline{C}_l\zeta^l+\sum_{k=1}^{n}\sum_{l=1}^{m_k}\frac{\overline{C}_{kl}\zeta_k^l\zeta^l}{(\zeta_k-\zeta)^l}=A(\zeta). \tag{85.5}$$

From this follows an expression for $\varphi(\zeta)$ as a function holomorphic inside γ (and continuous up to γ), because the points ζ_k lie outside γ.

It remains to formulate the condition that the quantities (*a*) and (*b*), linear combinations of which are the coefficients C_l and C_{kl}, actually are the relevant derivatives of $\varphi(\zeta)$, as determined by (85.5), at the corresponding points and that $\varphi(0) = 0$. The last condition gives

$$\bar{a} + \bar{C}_0 = A(0). \tag{85.6}$$

The other conditions are also easily found by differentiating (85.5) correspondingly often and by substituting for ζ the values $0, \zeta_1', \zeta_2', \ldots, \zeta_n'$. For example, one must have

$$\varphi'(0) + \bar{C}_1 + \sum_{k=1}^{n} \bar{C}_{k1} = A'(0),$$

etc. Thus one deduces a system of linear equations (with constant coefficients) in the unknown quantities (*a*), (*b*) and their conjugate complex values. This system (cf. § 84) will have a unique solution, if, in the case of finite regions, the imaginary part of $\varphi'(0)/\omega'(0)$ is fixed arbitrarily and if, in the same case, the following condition is satisfied:

$$\int (f_1 dx + f_2 dy) = 0, \tag{85.7}$$

which is necessary for the existence of the solution of the problem.

The coefficients $\bar{a}$ and $\bar{C}_0$ only occur in (85.6) which determines the sum $\bar{a} + \bar{C}_0$; there is no need to determine these quantities separately, since (85.5) only contains their sum.

Having found $\varphi(\zeta)$, the function $\psi(\zeta)$ may be found from (78.8), viz.,

$$\psi(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{f\,d\sigma}{\sigma-\zeta} - \frac{1}{2\pi i}\int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\,\frac{\varphi'(\sigma)d\sigma}{\sigma-\zeta},$$

whence, taking into consideration that $\dfrac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\varphi'(\sigma)$ is the boundary value of the function $\Omega(\zeta)$, one obtains directly

$$\psi(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{f\,d\sigma}{\sigma-\zeta} - \frac{\bar{\omega}\left(\dfrac{1}{\zeta}\right)}{\omega'(\zeta)}\varphi'(\zeta) + \sum_{l=1}^{m_0} \frac{C_l}{\zeta^l} + \sum_{k=1}^{n}\sum_{l=1}^{m_k} \frac{C_{kl}}{(\zeta-\zeta_k')^l}. \tag{85.8}$$

Thus the problem is solved. In the case of infinite regions, it is sometimes

more convenient (mainly for the sake of clarity) to use the transformation on to the region $|\zeta| > 1$; but this is not always so, since the method used here is equally applicable to finite and infinite regions. With obvious minor modifications, the above reasoning will also apply to that method of conformal mapping.

NOTE. The method of solution, discussed above, may be somewhat modified, and in some concrete cases this may lead to considerable simplifications. For example, it will sometimes be profitable to multiply beforehand both sides of the boundary condition by a conveniently chosen polynomial. One such method was stated in the earlier editions of this book; generally speaking, it leads in the end to about the same amount of computations as the present method, although in certain individual cases it simplifies the calculations.

§ 86. Solution of the second fundamental problem. On the solution of the mixed fundamental problem. In the preceding sections the first fundamental problem has been considered for the sake of definiteness. However, comparing the boundary conditions of the first and second fundamental problems in the form given in § 78, it becomes clear that the above methods of solution may, almost without any change, be transferred to the case of the second fundamental problem. Hence there is no necessity to restate the method separately for applications to the second fundamental problem.

The solution of the mixed fundamental problem is somewhat more complicated; however, in this case too, effective solution can be achieved by elementary means when, as in the preceding sections, the transforming function $\omega(\zeta)$ is rational. Such a solution was stated by D. I. Sherman [10]. However, no time will be devoted to this problem here, since a simpler method will be studied in the next Part.

§ 87. Other methods of solution of the fundamental problems. Reverting to the first fundamental problem, attention will now be drawn to the fact that in some cases it is practically more convenient to start from the boundary condition (51.3) rather than from (51.1); this condition may be written

$$[\Phi(\sigma) + \Phi(\sigma)]\omega'(\sigma) - \sigma^2[\omega(\sigma)\,\Phi'(\sigma) + \omega'(\sigma)\Psi(\sigma)] = [\rho\rho - i\,\rho\vartheta]\overline{\omega'(\sigma)} \quad (87.1)$$

or

$$[\Phi(\sigma)+\overline{\Phi(\sigma)}]\omega'(\sigma)-\bar{\sigma}^2[\omega(\sigma)\,\overline{\Phi'(\sigma)}+\overline{\omega'(\sigma)}\,\overline{\Psi(\sigma)}]=[\widehat{\rho\rho}+i\,\widehat{\rho\vartheta}]\omega'(\sigma). \quad (87.2)$$

When $\omega(\zeta)$ is a rational function, the method of § 85 again leads to an elementary solution. On the basis of the work of that section, its application is so obvious that no space will be wasted on details (cf. N. I. Muskhelishvili [5] for a detailed study); however, a simple example will be presented in the next section. It should only be noted that this method is particularly convenient in the case when the region, occupied by the body, is infinite, because in that case $\varphi(\zeta)$ and $\psi(\zeta)$ are not single-valued, whereas $\Phi(\zeta)$ and $\Psi(\zeta)$ are holomorphic throughout the region under consideration.

Similarly, the method of solution of the second fundamental problem may be modified by replacing the boundary condition (78.15), which may be rewritten

$$\varkappa\varphi(\sigma)-\omega(\sigma)\overline{\Phi(\sigma)}-\overline{\psi(\sigma)}=2\mu(g_1+ig_2), \quad (87.3)$$

by a condition obtained by differentiation of (87.3) with respect to ϑ; noting that $\sigma=e^{i\vartheta}$ and multiplying the differentiated equation by $(-ie^{-i\vartheta})$, one thus finds

$$[\varkappa\Phi(\sigma)-\overline{\Phi(\sigma)}]\omega'(\sigma)+\bar{\sigma}^2[\omega(\sigma)\overline{\Phi'(\sigma)}+\overline{\omega'(\sigma)}\,\overline{\Psi(\sigma)}]=$$

$$=2\mu\left[\frac{dg_1}{d\sigma}+i\frac{dg_2}{d\sigma}\right]. \quad (87.4)$$

§ 87a. Example.

Solution of the first fundamental problem for an infinite plane with a circular hole*).

In this case let

$$z=\omega(\zeta)=R\zeta, \quad (87.1a)$$

where R is the radius of the hole, i.e., the region is mapped on $|\zeta|>1$. The boundary condition (87.1) then takes the form

$$\Phi(\sigma)+\overline{\Phi(\sigma)}-\bar{\sigma}\overline{\Phi'(\sigma)}-\bar{\sigma}^2\overline{\Psi(\sigma)}=N-iT, \quad (87.2a)$$

where N and T are the normal and tangential external stresses with the same sign convention as in § 56 (in fact, N is the projection of the external stress on the normal n to the circle, *directed towards the centre*, while T

*) This problem has been solved by another method in § 56.

is the projection on the tangent, directed *to the left* when looking along n).

For the sake of simplicity, it will be assumed that the stresses vanish at infinity. Then $\Phi(\zeta)$, $\Psi(\zeta)$ are not only holomorphic outside γ, including the point at infinity, but they also vanish at infinity, if it is assumed that the rotation there is zero. Thus, for large $|\zeta|$,

$$\Phi(\zeta) = \frac{a_1}{\zeta} + O\left(\frac{1}{\zeta^2}\right), \quad \Phi'(\zeta) = O\left(\frac{1}{\zeta^2}\right), \quad \Psi(\zeta) = \frac{a_1'}{\zeta} + O\left(\frac{1}{\zeta^2}\right); \tag{87.3a}$$

in addition, if the displacements are to be single-valued, the condition

$$\varkappa \bar{a}_1 + a_1' = 0 \tag{87.3'a}$$

must also be satisfied. [cf. (56.6); a_1 and a_1' may be determined beforehand (cf. § 56), but this will not be done here.]

Calculations may be somewhat simplified by multiplying both sides of (87.2a) by $\sigma^{-1} = e^{-i\vartheta}$; this gives

$$\frac{1}{\sigma}\,\Phi(\sigma) + \frac{1}{\sigma}\,\overline{\Phi(\sigma)} - \Phi'(\sigma) - \sigma\Psi(\sigma) = -(X_n - iY_n) \tag{87.4a}$$

or, going to the conjugate complex expression,

$$\sigma\Phi(\sigma) + \sigma\Phi(\sigma) - \Phi'(\sigma) - \bar{\sigma}\Psi(\sigma) = -(X_n + iY_n), \tag{87.5a}$$

because, as is easily verified,

$$N - iT = -(X_n - iY_n)e^{i\vartheta}. \tag{87.6a}$$

Formulating now the condition that $\sigma\Psi(\sigma)$, as determined by (87.4a), is the boundary value of $\zeta\Psi(\zeta)$, holomorphic outside γ, one obtains by (76.13)

$$\frac{1}{2\pi i}\int_\gamma \frac{\sigma\Phi(\sigma)d\sigma}{\sigma-\zeta} + \frac{1}{2\pi i}\int \frac{\sigma\overline{\Phi(\sigma)}d\sigma}{\sigma-\zeta} - \frac{1}{2\pi i}\int \frac{\overline{\Phi'(\sigma)}d\sigma}{\sigma-\zeta} + \\ + \frac{1}{2\pi i}\int_\gamma \frac{X_n + iY_n}{\sigma-\zeta}\,d\sigma = 0, \tag{87.7a}$$

where ζ is some point outside γ, or

$$-\zeta\Phi(\zeta) + a_1 + \frac{1}{2\pi i}\int \frac{X_n + iY_n}{\sigma-\zeta}\,d\sigma = 0,$$

whence finally

$$\Phi(\zeta) = \frac{1}{2\pi i\zeta} \int \frac{X_n + iY_n}{\sigma - \zeta} d\sigma + \frac{a_1}{\zeta}. \tag{87.8a}$$

In transforming (87.7a), use has been made of the formulae of § 70 and of the fact that $\sigma\Phi(\sigma)$ is the boundary value of $\zeta\Phi(\zeta)$, holomorphic outside γ and equal to a_1 for $\zeta = \infty$, and that $\sigma\overline{\Phi(\sigma)}$, $\overline{\Phi'(\sigma)}$ are the boundary values of $\zeta\overline{\Phi}(1/\zeta)$, $\overline{\Phi}'(1/\zeta)$, holomorphic inside γ.

The constant a_1 is not determined by the functional equation (87.7a) which will be satisfied by the expression found for $\Phi(\zeta)$ for any value of a_1; in fact, expressing that a_1 on the right-hand side of (87.8a) is the coefficient of ζ^{-1} in the expansion for $\Phi(\zeta)$, one obtains the identity $a_1 = a_1$. However, this was to be expected, since, for the present, no consideration has been given to the condition of single-valuedness of the displacements.

Next, the function $\zeta\Psi(\zeta)$ will be determined. Its boundary value is known from (87.4a), if $\Phi(\zeta)$ is replaced by (87.8a). In order to find $\zeta\Psi(\zeta)$ from Cauchy's formula, one has to know its value a_1' for $\zeta = \infty$; this value may be found from (87.4a) by multiplying it by $d\sigma/2\pi i\sigma$ and by integrating around γ which gives

$$\bar{a}_1 - a_1' = -\frac{1}{2\pi i} \int (X_n - iY_n) \frac{d\sigma}{\sigma} = -\frac{1}{2\pi} \int_0^{2\pi} (X_n - iY_n) d\vartheta = -\frac{X - iY}{2\pi R}, \tag{a}$$

where (X, Y) is the resultant vector of the external forces, applied to the edge of the hole.

The following considerations lead to (a): $\frac{1}{\sigma^2}\Phi(\sigma)$ and $\frac{1}{\sigma}\Phi'(\sigma)$ are the boundary values of functions, holomorphic outside γ and vanishing at infinity like ζ^{-2}; hence the corresponding integrals are zero. Further, $\frac{1}{\sigma^2}\overline{\Phi(\sigma)}$ is the boundary value of $\frac{1}{\zeta^2}\overline{\Phi}\left(\frac{1}{\zeta}\right)$ holomorphic inside γ, except for $\zeta = 0$, where it has a simple pole with the principal part $\bar{a}_1/\zeta$. Finally, $\Psi(\sigma)$ is the boundary value of a function, holomorphic outside γ and having, for large $|\zeta|$, the form

$$\frac{a_1'}{\zeta} + O\left(\frac{1}{\zeta^2}\right).$$

The quantities a_1 and a_1' may now be determined from (87.3'a) and

(a); in fact,

$$a_1 = -\frac{X + iY}{2\pi R(1+\varkappa)}, \quad a_1' = \frac{\varkappa(X - iY)}{2\pi R(1+\varkappa)}. \tag{87.9a}$$

Applying Cauchy's formula to determine $\zeta\Psi(\zeta)$ from its boundary value as given by (87.4a), one finds

$$\Psi(\zeta) = -\frac{1}{2\pi i\zeta}\int_\gamma \frac{X_n - iY_n}{\sigma - \zeta}\, d\sigma + \frac{\Phi(\zeta)}{\zeta^2} - \frac{\Phi'(\zeta)}{\zeta} + \frac{a_1'}{\zeta}. \tag{87.10a}$$

Thus the problem is solved by (87.8a) and (87.10a), where a_1 and a_1' have the values given by (87.9a).

As a simple example consider the case when a uniformly distributed force, parallel to the Ox axis, acts on the right-hand half $(-\pi/2 \leqslant \vartheta \leqslant \pi/2)$ of the hole, while the other half is free from stresses. This problem was solved directly in the above-mentioned paper [1] by D. M. Volkov and A. N. Nazarov who did not deduce general formulae of the type (87.8a) and (87.10a).

In this case

$$X_n = p \qquad \text{for } -\frac{\pi}{2} \leqslant \vartheta \leqslant \frac{\pi}{2},$$

$$X_n = 0 \qquad \text{for the remaining values of } \vartheta,$$

$$Y_n = 0 \qquad \text{for all } \vartheta.$$

Hence

$$X = R\int_{-\pi/2}^{+\pi/2} X_n d\vartheta = \pi R p, \quad Y = 0,$$

and therefore

$$a_1 = -\frac{p}{2(1+\varkappa)}, \quad a_1' = \frac{\varkappa p}{2(1+\varkappa)}.$$

Since $Y_n = 0$, the integrals in (87.8a) and (87.10a) are equal to

$$\int_\gamma \frac{X_n d\sigma}{\sigma - \zeta} = p\int_{-i}^{+i} \frac{d\sigma}{\sigma - \zeta} = -p\int_{-i}^{+i} \frac{d\sigma}{\zeta - \sigma},$$

where the last integral must be taken along the right-hand semi-circle in

the positive direction, so that

$$\int_{\gamma} \frac{X_n d\sigma}{\sigma - \zeta} = p[\log (\zeta - \sigma)]_{\sigma = -i}^{\sigma = +i} = p \log \frac{\zeta - i}{\zeta + i}$$

for a suitably chosen branch of the logarithmic function.

Substituting this value in (87.8*a*) and (87.10*a*), closed expressions are obtained for $\Phi(\zeta)$ and $\Psi(\zeta)$, but it is unnecessary to write them down here.

§ 88. Further examples. Application to some other boundary problems.

1°. The method of solution of §§ 84—87 is applicable, in particular, to all those simply-connected regions for which the conformal transformations on to the circle have been stated in § 48.

The case of the infinite plane with an elliptic hole which is one of the examples of § 48 has been considered in detail in §§ 82, 83.

The case of finite regions, bounded by Pascal's limaçon, has been studied in § 63 as an application of the method of series expansions; application of the method of § 82 leads much faster to the final results. It will be left to the reader to solve the fundamental problems for this case by use of the method of § 82.

The case of the infinite plane with hypotrochoidal holes (cf. § 48, 4°) has been studied in detail by G. S. Shapiro [1] who applied the method of § 82 to several practically important problems (cf. also reference in § 89 to the work of G. N. Savin).

The solution of the first fundamental problem for regions, bounded by Booth's lemniscates (§ 48,6°), was obtained by G. N. Bukharinov [1] by means of the method stated in § 85.

Several other examples which are of greater interest from the point of view of application will be stated in the next section.

2°. The problem of frictionless contact between an elastic and a rigid body under mutual pressure may also be easily solved by a method, analogous to that of §§ 84—87, provided the region, occupied by the elastic body, can be mapped on to the circle by means of a rational function. The solution of the problem by such a method was given by the Author in his paper [19] and it was studied in detail in the earlier editions of this book. A simpler solution of this problem will be presented in the next Part (§ 128).

3°. As has already been stated in § 79*a*, the problem of bending of plates under the influence of lateral loads reduces, in the case when the edges of the plate are clamped, to the fundamental biharmonic problem, i.e., to the same boundary problem as the first fundamental problem of the plane theory of elasticity, while in the case of free edges it leads to the same boundary problem as the second fundamental problem. Hence, if the region occupied by the plate is mapped on the circle by means of a rational function, the present method of solution may immediately be applied to the above two problems.

A. I. Lourie [1] has shown that the method of the precedin sections will also apply to plates with supported edges, if the region occupied by the plate can be mapped on to the circle by means of polynomials. It is not difficult to generalize his method to the case of transformation by means of rational functions.

In another paper, A. I. Lourie [2] gives the solution of the problem of bending of circular plates for all the three above-mentioned edge conditions.

Note also the recently published note by M. M. Friedman [1] on the bending of plates with curvilinear holes.

Finally, it should be mentioned that it has recently been shown by L. A. Galin [3] that the method of complex representation in conjunction with complex transformation also allows the effective solution of some boundary problems in those cases where parts of the body are subject to plastic deformation; generally speaking, problems of this type are very complicated, since the line of division of the elastic and plastic regions is not known beforehand.

§ 89. Application to the approximate solution of the general case. The above method of solution may also be applied successfully to the approximate solution of the fundamental problems for simply-connected regions, bounded by practically arbitrary contours. It will now be indicated how this can be done and it will be necessary for this purpose to repeat some of the statements made at the end of § 63.

Let S be a given region bounded by one simple contour L and let

$$z = \omega(\zeta) \tag{89.1}$$

map the region S on to the circle $|\zeta| < 1$. It will first be assumed that S is finite. Then $\omega(\zeta)$ is holomorphic for $|\zeta| < 1$, and hence it may be

expanded, for the stated values of ζ, in a series of the form

$$\omega(\zeta) = c_1\zeta + c_2\zeta^2 + \dots; \tag{89.2}$$

it has been assumed here that $c_0 = \omega(0) = 0$, but this is, of course, not essential.

If one only retains the first n terms of (89.2), i.e., if one takes instead of $\omega(\zeta)$ the polynomial

$$\omega_n(\zeta) = c_1\zeta + c_2\zeta^2 + \dots + c_n\zeta^n, \tag{89.3}$$

then

$$z = \omega_n(\zeta) \tag{89.4}$$

will map on to the circle $|\zeta| < 1$ some region S_n, and not S. If one takes n sufficiently large, the region S_n will be as close as one pleases to the region S; it has already been indicated in § 63 that this will be so for known, very general conditions referring to the contour L. In practice, it is usually sufficient to retain only a small number of terms in the expansion (89.2), in order to obtain a region, sufficiently close to S.

In very many cases even a crude approximation is sufficient. For example, if one is dealing with frequently occurring practical applications of the equations of the theory of elasticity to bodies such as rock stratas which are far from being homogeneous, it is clear that in such cases great accuracy is unjustifiable. Thus one may practically solve a given problem for the regions S_n by retaining a number of terms in (89.2) which will be sufficient for the stated purpose, and the solution will represent an approximate solution for the original region S.

In the case of infinite regions, one has instead of (89.2) an expansion of the form

$$\omega(\zeta) = \frac{c}{\zeta} + c_1\zeta + c_2\zeta^2 + \dots \tag{89.2'}$$

(assuming $c_0 = 0$) and instead of (89.3)

$$\omega_n(\zeta) = \frac{c}{\zeta} + c_1\zeta + c_2\zeta^2 + \dots + c_n\zeta^n, \tag{89.3'}$$

and the earlier statements will again apply.

Naturally, any other expansion in a series of rational functions can be used for $\omega(\zeta)$ instead of these power series. It may be proved that

under known, general assumptions with regard to the contour L and to the selected method of expansion, the solutions for the regions S_n will tend to the solution for the given region S, when $n \to \infty$; the proof is given in the Author's paper [6] and, for more general conditions, in a paper by D. I. Sherman [5].

It will only be noted that the method of approximate solution gives good results even in cases when L is not smooth, but has corners (angular points), e.g. when L is a polygon. In order to transform regions, bounded by straight line segments, on to the circle, the known Schwarz-Christoffel formula may be used.

The above method was applied successfully by G. N. Savin to the solution of a number of practically important problems. Referring the reader to the books by G. N. Savin [1, 2], and also to the paper by A. N. Dinnik, A. B. Morgaevski and G. N. Savin [1], only two examples taken from Savin [1] will be discussed here which show clearly the practical usefulness of this interesting method.

As a first example consider the region, represented by the infinite plane with a hole in the form of an equilateral triangle. In this case the mapping function may be written in the form

$$\omega(\zeta) = -A \int_1^\zeta (1 + t^3)^{\frac{2}{3}} \frac{dt}{t^2} + \text{const.},$$

where A is a real constant determining the dimensions of the triangle. Expanding in a power series, one finds for a suitable choice of the arbitrary constant

$$\omega(\zeta) = A\left(\frac{1}{\zeta} - \frac{\zeta^2}{3} + \frac{\zeta^5}{45} - \ldots\right).$$

By retaining the first two or three terms of this expansion one obtains instead of the triangle the contours illustrated in Figs. 40 or 41 respectively.

As a second example consider the infinite plane with a square hole. In this case

$$\omega(\zeta) = -A \int_1^\zeta (1 + t^4)^{\frac{1}{2}} \frac{dt}{t^2} + \text{const.},$$

where A determines the dimensions of the square. Expanding in a power series and choosing the appropriate value for the arbitrary constant, one finds

$$\omega(\zeta) = A\left(\frac{1}{\zeta} - \frac{1}{6}\zeta^3 + \frac{1}{56}\zeta^7 - \frac{1}{176}\zeta^{11} + \ldots\right).$$

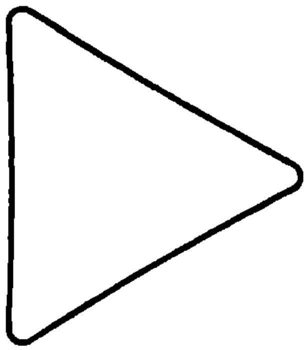


Fig. 40.

$$\omega(\zeta) = A\left(\frac{1}{\zeta} - \frac{\zeta^2}{3}\right)$$

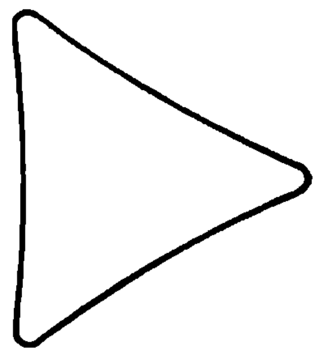


Fig. 41.

$$\omega(\zeta) = A\left(\frac{1}{\zeta} - \frac{\zeta^2}{3} + \frac{\zeta^5}{45}\right)$$

The contours corresponding to the retention of the first 2, 3 and 4 terms of this expansion are shown in Figs. 42, 43, 44.

It is seen that an approximation, sufficiently good for most purposes, is given by three terms. By a slight modification of the coefficients of the terms retained even better approximations may be obtained; practical methods for deducing better approximations have been evolved by M. I. Naiman in a paper which has not yet been published. G. N. Savin also considers in detail the case of holes with straight sides of different lengths.

In order to avoid any later reference to this problem, it will be noted here that the above methods may likewise be applied to the semi-infinite regions considered in the next chapter.

Finally, it should be stated that these methods of approximate solution can also be applied successfully to regions bounded by several contours, if they are combined with the so-called "alternating method" ("Schwarz algorithm") or the *method of successive approximation*, analogous to that used by Schwarz in solving Dirichlet's problem. This method admits reduction of a given boundary problem for regions, bounded by several contours, to the successive solution of the same problem for several regions, each bounded by a single contour, for successively varying boundary conditions. An infinite number of such operations is required

for the exact solution, but practically useful approximate solutions may be obtained after a finite number of steps. Each separate problem for a region bounded by one contour may likewise be solved approximately, using the above method.

It should be noted that the method of successive approximation has been developed by S. G. Mikhlin [5, 9, 13] and by D. I. Sherman [5];

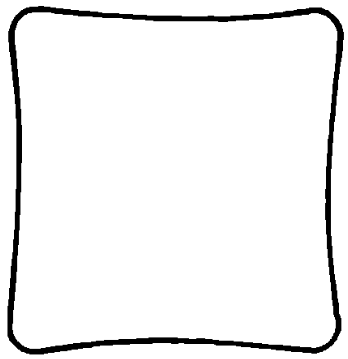


Fig. 42.

$$\omega(\zeta) = A\left(\frac{1}{\zeta} - \frac{1}{6}\zeta^3\right)$$

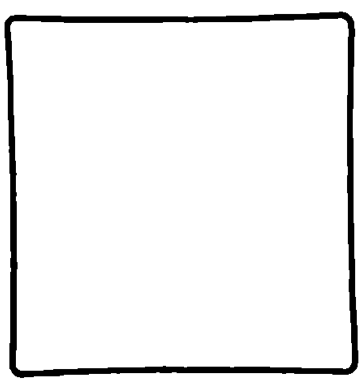


Fig. 43.

$$\omega(\zeta) = A\left(\quad -\frac{1}{6}\zeta^3 + \frac{1}{56}\zeta^7\right)$$

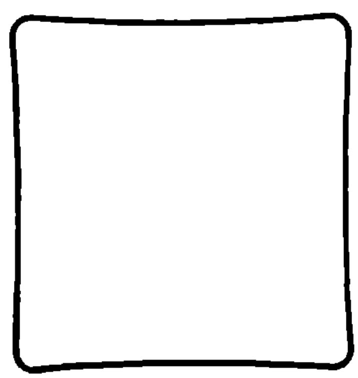


Fig. 44.

$$\omega(\zeta) \quad A\left(\frac{1}{\zeta} - \frac{1}{6}\zeta^3 + \frac{1}{56}\zeta^7 - \frac{1}{176}\zeta^{11}\right)$$

a study of their results may be found in Mikhlin's book [13]. Note also the work of A. Ya. Gorgidze [1, 2]. A convergence proof of the Schwarz algorithm for very general conditions has been given by S. L. Sobolev [2].

The method of successive approximation was applied by S. G. Mikhlin [4] to the solution of the first fundamental problem for a half-plane with an elliptic hole. This problem has been solved by D. I. Sherman [4] using a different method.

Chapter 16

SOLUTION OF THE FUNDAMENTAL PROBLEMS FOR THE HALF-PLANE AND FOR SEMI-INFINITE REGIONS

Hitherto consideration has been restricted to regions bounded by (finite) contours. The study of the case where the boundary is an open line, extending to infinity in both directions ("semi-infinite region"), does not meet with any essentially new difficulties. In certain cases it is convenient to map the region under consideration on to the half-plane rather than on to the circle (there being, of course, no essential difference between these two methods). The general case will not be considered in the present chapter; only the solution of the fundamental problems for the half-plane and for certain definite semi-infinite regions will be treated. The general case of semi-infinite regions has been studied by S. G. Mikhlin [7].

§ 90. General formulae and propositions for the half-plane.

Let the region S, occupied by the body, consist of the "lower" half-plane (Fig. 45) bounded by the Ox axis, i.e., of the points $y < 0$. In §§ 90, 91 temporary use will be made of the notation of Chap. 5, i.e.,

$$\varphi(z),\quad \psi(z),\quad \Phi(z),\quad \Psi(z)$$

will again be written instead of $\varphi_1(z)$, $\psi_1(z)$ etc.

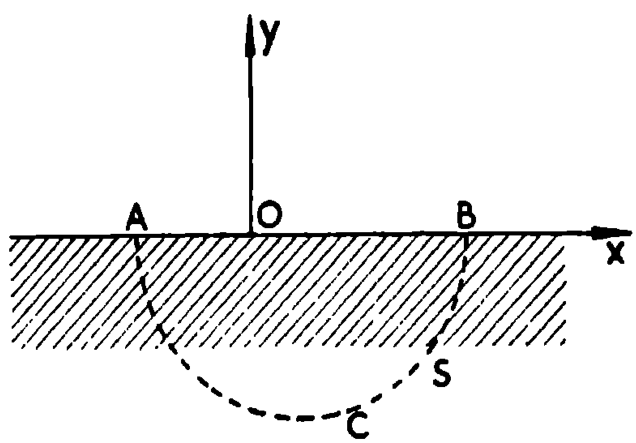


Fig. 45.

It will be assumed that the stress components satisfy those conditions of continuity and differentiability which apply throughout the preceding work, and that they tend to zero as $z \to \infty$. If the boundary of the region S did not extend to infinity, but were instead a circle, it would follow from these

conditions that Φ and Ψ would have the forms

$$\Phi(z) = \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \ldots, \quad \Psi(z) = \frac{\gamma_1'}{z} + \frac{\gamma_2'}{z^2} + \ldots$$

for large $|z|$ (*assuming here and in what follows that the rotation vanishes at infinity*, cf. § 36.)

In the present case the condition will be imposed that the functions Φ and Ψ may be represented for large $|z|$ by

$$\Phi(z) = \frac{\gamma}{z} + o\left(\frac{1}{z}\right), \quad \Psi(z) = \frac{\gamma'}{z} + o\left(\frac{1}{z}\right), \qquad \Phi'(z) = -\frac{\gamma}{z^2} + o\left(\frac{1}{z^2}\right), \tag{90.1}$$

where γ and γ' are constants. (With regard to this choice, see also the Note at the end of § 93.)

In addition, the functions $\Phi(z)$ and $\Psi(z)$ will be holomorphic in every finite region, contained in S.

The following conditions may be added to (90.1):

$$\begin{aligned} \varphi(z) &= \gamma \log z + o(1) + \text{const.}, \\ \psi(z) &= \gamma' \log z + o(1) + \text{const.}; \end{aligned} \tag{90.2}$$

in these formulae one definite branch of the multi-valued function $\log z$ must be selected, e.g. $\log |z| + i\vartheta$, where ϑ (argument of z) varies from $-\pi$ to $+\pi$.

It will be remembered that the symbols $o(1/z)$ and $o(1)$ denote quantities such that

$$\left|o\left(\frac{1}{z}\right)\right| < \frac{\varepsilon}{|z|}, \quad |o(1)| < \varepsilon,$$

where ε only depends on $|z|$ and $\varepsilon \to 0$ as $|z| \to \infty$. The condition (90.2) would follow from (90.1) by an integration, provided one had on the right-hand sides of (90.1) $o(1/z^{1+\mu})$ instead of $(1/z)$, where μ is a positive constant (which is arbitrarily small).

Finally, the further condition will be introduced: *the resultant vector of the external forces, applied to a segment AB of the Ox axis, tends to a definite limit as A and B move to infinity* (A towards the left and B towards the right). This condition will always be satisfied, if only a finite part of the boundary is loaded.

This last condition will now be formulated mathematically. If X', Y' are the components of the resultant vector of the external forces, applied to AB, then one has by (33.1)

$$X' + iY' = +i\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right]_A^B, \tag{90.3}$$

where

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = \varphi(z) + z\overline{\Phi(z)} + \overline{\psi(z)}. \tag{90.4}$$

(The positive sign has been chosen on the right-hand side of (90.3), since the region S lies on the right, and not on the left, as one moves from A to B.)

When calculating the increase of the left-hand side of (90.3) for the transition from A to B, one may, instead of moving along the straight line AB, follow any curve which lies in S, e.g. the semi-circle ACB (cf. Fig. 45.).

If A and B lie sufficiently far away and on different sides of O, then (90.1), (90.2) and (90.4) give

$$\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right]_A^B = \gamma \log\frac{r''}{r'} + \gamma\pi i + \bar{\gamma}' \log\frac{r''}{r'} - \bar{\gamma}'\pi i + \varepsilon, \tag{90.3'}$$

where r' and r'' are the distances of A and B from O and ε is arbitrarily small (and tends to zero as r', r'' increase). In order that the preceding expression will remain finite for any abitrarily large r' and r'' (independent of each other), it is obviously necessary and sufficient that

$$\gamma + \bar{\gamma}' = 0. \tag{90.5}$$

Under this condition the vector (X, Y) of the external forces, applied to the whole of the Ox axis, will be given by

$$X + iY = i\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right]_{-\infty}^{+\infty} = -\pi(\gamma - \bar{\gamma}'). \tag{90.6}$$

It follows from (90.5) and (90.6) that

$$\gamma = -\frac{X + iY}{2\pi}, \quad \gamma' = \frac{X - iY}{2\pi} \tag{90.7}$$

Hence one has finally for large $|z|$:

$$\Phi(z) = -\frac{X+iY}{2\pi z} + o\left(\frac{1}{z}\right), \quad \Psi(z) = \frac{X-iY}{2\pi z} + o\left(\frac{1}{z}\right),$$
$$\Phi'(z) = \frac{X+iY}{2\pi z^2} + o\left(\frac{1}{z^2}\right), \tag{90.1'}$$

$$\varphi(z) = -\frac{X+iY}{2\pi}\log z + o(1) + \text{const.},$$
$$\psi(z) = \frac{X-iY}{2\pi}\log z + o(1) + \text{const.} \tag{90.2'}$$

Note also that under the above conditions the stress components

$$X_x, \quad X_y, \quad Y_y$$

will be of order $O(1/z)$, while the displacements will have for large $|z|$ the form

$$2\mu(u+iv) = \varkappa\gamma\log z - \bar{\gamma}'\log\bar{z} - \bar{\gamma}\frac{z}{\bar{z}} + o(1) + \text{const.} =$$
$$= -\frac{\varkappa(X+iY)}{2\pi}\log z - \frac{X+iY}{2\pi}\log\bar{z} + \frac{X-iY}{2\pi}\frac{z}{\bar{z}} + o(1) + \text{const.} \tag{90.8}$$

If $X = Y = 0$, then X_x, Y_y, X_y will be of order $o(1/z)$ and $u + iv$ will be bounded.

The same fundamental problem may be set for the region S as for the regions considered in the earlier chapters of this Part. One has only to give special consideration to the fact that the behaviour of those quantities which are given on the boundaries should be in agreement at infinity with the conditions imposed above.

In the *first fundamental problem* the quantities $Y_y = N(t)$ and $X_y = T(t)$ will be given on the axis Ox as functions of the abscissae t. On the basis of (90.1'), it is easily found that for large $|t|$

$$N = o\left(\frac{1}{t}\right), \quad T = o\left(\frac{1}{t}\right). \tag{90.9}$$

In the *second fundamental problem* the functions u and v, given on Ox,

must satisfy by (90.8) the conditions

$$2\mu(u + iv) = -\frac{\varkappa + 1}{2\pi}(X + iY)\log t + c + o(1) \quad \text{for } t > 0,$$

$$2\mu(u + iv) = -\frac{\varkappa + 1}{2\pi}(X + iY)\log|t| + c + \tag{90.10}$$

$$+ \frac{i(\varkappa - 1)}{2\pi}(X + iY) + o(1) \qquad \text{for } t < 0,$$

where c is a constant.

The reader will easily establish analogous conditions for the *mixed fundamental problem.*

In the cases of the second fundamental and of the mixed problems the quantities X, Y will be assumed known.

It is easily proved that under these conditions the fundamental problems have a unique solution. The proof is quite analogous to that given in § 40 for the case of infinite regions, and for this reason it will not be repeated.

NOTE. 1. The formulae (90.1), (90.2) or (90.1′), (90.2′) may be replaced by others which are more convenient for the study of the behaviour of the functions under consideration near the boundary. For example, one may obviously write instead of (90.2′)

$$\begin{aligned} \varphi(z) &= -\frac{X + iY}{2\pi}\log(z - z_0) + \varphi^*(z) + \text{const.}, \\ \psi(z) &= \frac{X - iY}{2\pi}\log(z - z_0) + \psi^*(z) + \text{const.}, \end{aligned} \tag{90.2″}$$

where z_0 is an arbitrarily fixed point outside S (i.e., a point of the upper half-plane) and $\varphi^*(z)$, $\psi^*(z)$ are functions, holomorphic in S and of order $o(1)$ for large $|z|$.

NOTE. 2. On concentrated forces applied to the boundary.

If one retains only the first terms in the formulae (90.2′), i.e., if one writes

$$\varphi(z) = -\frac{X + iY}{2\pi}\log z, \quad \psi(z) = \frac{X - iY}{2\pi}\log z \tag{90.11}$$

and applies them to the whole half-plane S, then it is easily seen that they correspond to the effect of a concentrated force (X, Y) applied to the boundary at the origin. In fact, for a circuit along an infinitely small semi-circle below O, the expression $\partial U/\partial x + i\,\partial U/\partial y$ increases by $i(X+iY)$, and hence the resultant vector of the forces, applied (from above) to this semi-circle, equals (X, Y); further, it may be shown that the resultant moment of the same forces about the origin is zero.

The components of stress and displacement corresponding to these functions φ and ψ, i.e., to the effect of concentrated forces, may be calculated by means of the general formulae of § 32 or § 39. For example, one has by § 39 for the polar components of stress

$$\widehat{rr} + \widehat{\vartheta\vartheta} = 4\Re\varphi'(z) = -4\Re\frac{X+iY}{2\pi r}e^{-i\vartheta} = -\frac{2}{\pi r}(X\cos\vartheta + Y\sin\vartheta),$$

$$\widehat{\vartheta\vartheta} - \widehat{rr} + 2i\widehat{r\vartheta} = 2[\bar{z}\varphi''(z) + \psi'(z)]e^{2i\vartheta} = \frac{2}{\pi r}(X\cos\vartheta + Y\sin\vartheta),$$

whence

$$\widehat{rr} = -\frac{2}{\pi r}(X\cos\vartheta + Y\sin\vartheta), \quad \widehat{\vartheta\vartheta} = 0, \quad \widehat{r\vartheta} = 0. \tag{90.12}$$

This solution of the problem of the effect of a concentrated force on the boundary agrees in essence with the solution found by Flamant (Cf. A. E. H. Love [1] §§ 149, 150). This problem is the two-dimensional analogue of the problem of the effect of a concentrated force on the boundary of a body, occupying the half-space (the boundary of which is an unbounded plane), i.e., of the so-called problem of Boussinesq.

§ 91. The general formulae for semi-infinite regions. Next consider *semi-infinite regions* and the generalization of the formulae of the preceding section to this case. Let L be the boundary of the region S which is a simple open line extending in both directions to infinity. The line L divides the plane into two parts one of which is S; the second part will be denoted by S'. The positive direction on L will be chosen in such a way that it leaves S *on the left*.

It will be assumed that the line L has the following property: if M_0 is some fixed point, the rays M_0A, M_0B linking M_0 to two points A and B of L tend to definite positions as A and B move to infinity along L in opposite directions.

Let $\Pi(M_0)$ be the (signed) angle covered by the ray M_0M as the point M, moving along L in the positive direction, describes the entire line. It will be said that $\Pi(M_0)$ is the angle subtended by L at M_0. It is easily seen that the magnitude of $\Pi(M_0)$ will be the same for all points on one side of L and that the angles Π, Π' subtended by L at points lying in S and S' respectively are related by the condition

$$\Pi - \Pi' = 2\pi. \tag{91.1}$$

For example, if S is the lower half-plane, its boundary L is the Ox axis, but the positive direction of L is the negative Ox direction (because S must lie on the left for motion in that direction). Then, for points M_0 of the lower half-plane, one has $\Pi = \pi$, while for points M_0 of the upper half-plane $\Pi' = -\pi$.

In the present case of semi-infinite regions it will be assumed that the functions $\Phi(z)$, $\Psi(z)$, $\Phi'(z)$ are also subject to the condition that for large $|z|$ (cf. § 90)

$$\Phi(z) = \frac{\Upsilon}{z} + o\left(\frac{1}{z}\right), \quad \Psi(z) = \frac{\Upsilon'}{z} + o\left(\frac{1}{z}\right), \quad \Phi'(z) = -\frac{\Upsilon}{z^2} + o\left(\frac{1}{z^2}\right)$$

or, what is the same thing,

$$\begin{gathered} \Phi(z) = \frac{\Upsilon}{z - z_0} + o\left(\frac{1}{z}\right), \quad \Psi(z) = \frac{\Upsilon'}{z - z_0} + o\left(\frac{1}{z}\right), \\ \Phi'(z) = -\frac{\Upsilon}{(z - z_0)^2} + o\left(\frac{1}{z^2}\right), \end{gathered} \tag{91.2}$$

where z_0 is some (arbitrarily) fixed point of S' (i.e., not in S); in (91.2) $o(1/z)$, $o(1/z^2)$ are symbols for functions, holomorphic in S and having for large $|z|$ the indicated order.

The following may still be added to these conditions (cf. § 90):

$$\begin{aligned} \varphi(z) &= \gamma \log (z - z_0) + o(1) + \text{const.}, \\ \psi(z) &= \gamma' \log (z - z_0) + o(1) + \text{const.}; \end{aligned} \tag{91.3}$$

here $o(1)$ is the symbol for a function, holomorphic in S and tending to zero as $|z| \to \infty$.

As in the preceding section, it will also be assumed that the resultant vector of the external forces, applied to an arc AB of L, tends to the definite limit (X, Y) as the points A and B move to infinity in opposite directions.

In the present case one finds instead of (90.3′) the formula

$$\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right]_A^B = \gamma \log \frac{r''}{r'} + \gamma \Pi' i + \bar{\gamma}(e^{2i\beta} - e^{2i\alpha}) + \\ + \bar{\gamma}' \log \frac{r''}{r'} - \bar{\gamma}'\Pi' i + \varepsilon \qquad (91.3')$$

where one may move along L, since z_0 does not lie on this line, and where, as above, Π' is the angle subtended by L at points of S', while α, β are the (signed) angles between the Ox axis and the limiting positions of the rays M_0A and M_0B, drawn from some fixed point M_0, as A and B move to infinity along L, the first in the negative and the second in the positive direction; r' and r'' are the distances of z_0 from A and B respectively and ε is a quantity tending to zero as A and B move to infinity. Clearly, one may assume

$$\beta - \alpha = \Pi'. \qquad (91.4)$$

As in the preceding section, the conclusion is drawn that one must have

$$\gamma + \bar{\gamma}' = 0 \qquad (91.5)$$

and that

$$X + iY = i\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right]_L = (\gamma - \bar{\gamma}')\Pi' - i(e^{2i\beta} - e^{2i\alpha})\bar{\gamma}$$

or, using (91.5),

$$X + iY = 2\Pi'\gamma - i(e^{2i\beta} - e^{2i\alpha})\bar{\gamma}. \qquad (91.6)$$

In the analogous formulae of the preceding section it had been assumed that the boundary of S was described in the negative direction; hence one had to take

$$+ i\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right]_{-\infty}^{+\infty} \text{ instead of } - i\left[\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y}\right]_L.$$

It will now be assumed that $\Pi' \neq 0$, i.e., that $\Pi \neq 2\pi$. Then, adding to (91.6) the relation obtained by transition to the conjugate complex expression and solving for γ and $\bar{\gamma}$, one finds

$$\gamma = \frac{2\Pi'(X + iY) + i(e^{2i\beta} - e^{2i\alpha})(X - iY)}{4(\Pi'^2 - \sin^2\Pi')} \qquad (91.7)$$

and, by (91.5),

$$\gamma' = -\frac{2\Pi'(X - iY) - i(e^{-2i\beta} - e^{-2i\alpha})(X + iY)}{4(\Pi'^2 - \sin^2\Pi')}. \qquad (91.8)$$

In deducing these formulae it has been assumed that $\Pi' \neq 0$. If $\Pi' = 0$, then (91.6) gives $X = Y = 0$. This shows that for $\Pi' = 0$ it is necessary for the existence of a solution under the above conditions that the resultant vector of the external forces, applied to the boundary, be zero.

§ 92. Basic formulae, connected with conformal transformation on to the half-plane. It is convenient for the solution of problems of the theory of elasticity for semi-infinite regions to make use of transformations on the half-plane rather than on the circle. (As stated earlier, there is no difference in principle between the two approaches.)

As for transformations on to the circle, it is advantageous to introduce on the z plane of the elastic body curvilinear coordinates, related to the transformation.

As before, denote the region under consideration by S and its boundary by L. Let

$$z = \omega(\zeta), \quad (z = x + iy, \quad \zeta = \xi + i\eta) \tag{92.1}$$

be the function mapping S on to the lower half of the ζ plane, i.e., on to the half-plane $\eta < 0$, so that finite points correspond to finite points.

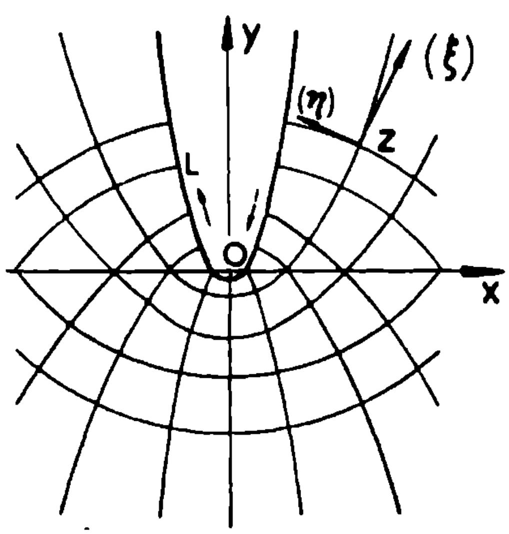


Fig. 46*a*.

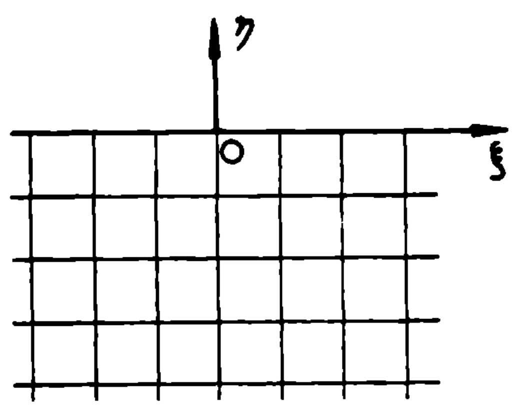


Fig. 46*b*.

Straight lines $\eta =$ const., lying in this half-plane, obviously correspond in the S region to some open lines which go to infinity at both ends; these lines will be denoted by (ξ). Similarly, the semi-infinite straight lines $\xi =$ const. in the lower half of the ζ plane correspond in S to lines (η) which begin on L and go to infinity (Figs. 46*a*, 46*b*).

Since in S a completely definite point $z = \omega(\xi + i\eta)$ of the z plane corresponds to every pair (ξ, η) for $\eta < 0$, the quantities ξ and η may be conceived as curvilinear coordinates in the z plane. The lines (ξ) and (η) form an orthogonal net of coordinate lines.

Let z be some point of S. Draw at z the tangents to the lines (ξ), (η) in the directions of increasing ξ and η. These tangents which will likewise be denoted by (ξ) and (η) will represent the axes of the curvilinear coordinates at the point z (Fig. 46*a*); all this is quite analogous to the procedure in § 49.

Let $\vec{A}$ be some vector starting from the point $z = (\xi + i\eta)$ and let A_x, A_y be its projections on the axes Ox, Oy and A_ξ, A_η its projections on the axes (ξ) and (η). As in § 49,

$$A_\xi + iA_\eta = e^{-i\alpha}(A_x + iA_y), \tag{92.2}$$

where α is the angle between the axes (ξ) and Ox, measured from Ox in the positive direction.

In order to determine $e^{i\alpha}$, the point z will be given a displacement dz in the direction (ξ); the corresponding point ζ will then undergo a displacement $d\xi > 0$ in the direction ξ of the ζ plane. Obviously

$$dz = |dz| e^{i\alpha} = |\omega'(\zeta)| e^{i\alpha} d\xi, \quad dz = \omega'(\zeta) d\xi,$$

whence

$$e^{i\alpha} = \frac{\omega'(\zeta)}{|\omega'(\zeta)|}, \quad e^{-i\alpha} = \frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}. \tag{92.3}$$

Thus one obtains

$$A_\xi + iA_\eta = \frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}(A_x + iA_y). \tag{92.2'}$$

Denote by v_ξ, v_η the components of displacement and by $\widehat{\xi\xi}$, $\widehat{\eta\eta}$, $\widehat{\xi\eta}$ the components of stress in the (ξ), (η) directions of the curvilinear coordinate system. By (92.2'),

$$v_\xi + iv_\eta = \frac{\overline{\omega'(\zeta)}}{|\omega'(\zeta)|}(u + iv), \tag{92.4}$$

where u, v are the components of displacement with respect to Ox, Oy. By § 8, the following relations hold between the stress components in

the two coordinate systems:

$$\widehat{\xi\xi} + \widehat{\eta\eta} = X_x + Y_y, \quad \widehat{\eta\eta} - \widehat{\xi\xi} + 2i\widehat{\xi\eta} = (Y_y - X_x + 2iX_y)e^{2i\alpha}, \tag{92.5}$$

where, by (92.3),

$$e^{2i\alpha} = \frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}}\cdot\frac{\omega'(\zeta)}{\omega'(\zeta)} = \frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}}. \tag{92.3'}$$

The expressions for the components of displacement and stress in terms of functions of the complex variable ζ may be found as in § 50; it will now be agreed to denote by

$$\varphi_1(z),\ \psi_1(z),\ \Phi_1(z),\ \Psi_1(z)$$

the functions

$$\varphi(z),\ \psi(z),\ \Phi(z),\ \Psi(z)$$

of Chapter 5 (and likewise of § 90). As in § 50, write

$$\begin{gathered}\varphi(\zeta) = \varphi_1[\omega(\zeta)], \quad \psi(\zeta) = \psi_1[\omega(\zeta)], \\ \Phi(\zeta) = \Phi_1[\omega(\zeta)] = \frac{\varphi'(\zeta)}{\omega'(\zeta)}, \quad \Psi(\zeta) = \Psi_1[\omega(\zeta)] = \frac{\psi'(\zeta)}{\omega'(\zeta)}.\end{gathered} \tag{92.6}$$

Applying now the formulae of § 32, expressing X_x, Y_y, X_y in terms of φ_1, ψ_1, Φ_1, Ψ_1, one finds by (92.5) and (92.3')

$$\widehat{\xi\xi} + \widehat{\eta\eta} = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}] = 4\Re\Phi(\zeta), \tag{92.7}$$

$$\eta\eta - \xi\xi + 2i\xi\eta = \frac{2}{\overline{\omega'(\zeta)}}\{\omega(\zeta)\,\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)\}. \tag{92.8}$$

Using (92.4) and the formula

$$2\mu(u + iv) = \varkappa\varphi(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi'(\zeta)} - \overline{\psi(\zeta)}, \tag{92.9}$$

the expression for $v_\xi + iv_\eta$ is also easily deduced. Finally, adding (92.7) and (92.8), one finds the useful relation

$$\eta\eta + i\xi\eta = \Phi(\zeta) + \overline{\Phi(\zeta)} + \frac{1}{\overline{\omega'(\zeta)}}\{\omega(\zeta)\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)\}. \tag{92.10}$$

§ 93. Solution of the first fundamental problem for the half-plane. Let the body S occupy the lower half-plane. By the conditions

of the problem

$$Y_y = N(t), \quad X_y = T(t) \text{ on } Ox, \tag{93.1}$$

where $N(t)$ and $T(t)$ are given functions of the abscissae t (which represent the normal and tangential stresses).

By (32.8)

$$Y_y - iX_y = \Phi(z) + \overline{\Phi(z)} + z\overline{\Phi'(z)} + \overline{\Psi(z)}. \tag{93.2}$$

Hence the boundary condition may be written

$$\Phi(t) + \Phi(t) + t\Phi'(t) + \Psi(t) = N - iT \tag{93.3}$$

or

$$\Phi(t) + \overline{\Phi(t)} + t\Phi'(t) + \Psi(t) = N + iT. \tag{93.4}$$

The condition (93.3) could, of course, have also been deduced from (41.9); however, it must not be overlooked that the quantity T of the present section is the quantity $(-T)$ of that formula. This follows from the fact that, when moving along the Ox axis in the positive direction, the region S lies on the right, and not on the left.

It will be assumed that N and T are continuous functions satisfying the condition (90.9), i.e., that

$$N = o\left(\frac{1}{t}\right), \quad T = o\left(\frac{1}{t}\right). \tag{93.5}$$

From the condition that $\Psi(t)$, defined by (93.4), is to be the boundary value of some function $\Psi(z)$, holomorphic in the lower half-plane and vanishing at infinity [as follows from (90.1)], one obtains by (76.21)

$$\frac{1}{2\pi i}\int \frac{N - iT}{t - z}\,dt - \frac{1}{2\pi i}\int \frac{\Phi(t)dt}{t - z} - \frac{1}{2\pi i}\int \frac{\overline{\Phi(t)}dt}{t - z} - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{t\overline{\Phi'(t)}}{t - z}\,dt = 0,$$

where z is any point of the lower half-plane. But $\Phi(t)$ is the boundary value of $\Phi(z)$, holomorphic in the lower half-plane and vanishing at infinity, and $\overline{\Phi(t)}$, $t\overline{\Phi'(t)}$ are the boundary values of $\bar{\Phi}(z)$, $z\bar{\Phi}'(z)$, holomorphic in the upper half-plane and likewise vanishing at infinity [cf. (90.1)], and hence, by (72.2′) and (72.2), the conclusion is drawn that

the second integral in the preceding formula is equal to $-\Phi(t)$, while the last two integrals vanish. Thus

$$\Phi(z) = -\frac{1}{2\pi i}\int^{+\infty} \frac{N - iT}{t}\, dt. \tag{93.6}$$

Having found $\Phi(z)$, the function $\Psi(z)$ may be determined from (72.2′), since the boundary value $\Psi(t)$ is given by (93.4). Thus one obtains, using the formulae of § 72,

$$\Psi(z) = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{N + iT}{t - z}\, dt - \Phi(z) - z\Phi'(z) =$$

$$= -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{T dt}{t - z} + \frac{z}{2\pi i}\int_{-\infty}^{+\infty} \frac{N - iT}{(t - z)^2}\, dt =$$

$$= -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{N + iT}{t - z}\, dt + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{N - iT}{(t - z)^2}\, t dt. \tag{93.7}$$

It is readily verified that, if the functions $N(t)$, $T(t)$ and their first derivatives $N'(t)$, $T'(t)$ satisfy the H condition at all finite points and if $tN(t)$, $tT(t)$, $t^2N'(t)$ and $t^2T'(t)$ satisfy the H condition in the neighbourhood of the point at infinity, then the above expressions for $\Phi(z)$ and $\Psi(z)$ satisfy all the required conditions. In particular, the functions $\Phi(z)$, $\Phi'(z)$, $\Psi(z)$ are continuous up to the boundary and have for large $|z|$ the form, determined by (90.1) and (90.5); in order to see this, it is sufficient to refer to the results at the end of § 71. Thus the problem is solved.

The above result agrees essentially with that obtained by G. V. Kolosov [1] by another method. The same problem was also solved later (and independently of G. V. Kolosov) by M. A. Sadovski [1, 2]. But neither of these authors presented a strict investigation of the solution.

NOTE. The right-hand sides of (93.6) and (93.7) obviously are holomorphic functions in the lower as well as in the upper half-planes, but, in general, they are not analytic on the common boundary Ox of the half-planes. It is clear, however, that, if any part of the boundary remains unloaded, the right-hand sides of (93.6) and (93.7) will also be analytic

on that part, and hence $\Phi(z)$, $\Psi(z)$ may be continued analytically through this part from the lower into the upper half-plane.

This property of the solution is easily proved directly without reference to (93.6) and (93.7). In fact, let

$$\Omega(z) = -\Phi(z) - z\Phi'(z) - \Psi(z); \tag{93.8}$$

since, by supposition, $\Phi(z)$ and $\Psi(z)$ are holomorphic in the lower half-plane, $\Omega(z)$ is likewise holomorphic there. Next consider the functions

$$\bar{\Phi}(z), \quad \bar{\Omega}(z) = -\bar{\Phi}(z) - z\bar{\Phi}'(z) - \bar{\Psi}(z)$$

which are holomorphic in the upper half-plane. By (93.3) and (93.4), one has on any unloaded part of the Ox axis

$$\Phi(t) = \bar{\Omega}(t), \quad \bar{\Phi}(t) = \Omega(t), \tag{93.9}$$

where $\Phi(t)$, $\Omega(t)$ are the boundary values, assumed by the corresponding functions for $z \to t$ from the lower half-plane, while $\bar{\Phi}(t)$, $\bar{\Omega}(t)$ are those assumed for $z \to t$ from the upper half-plane.

It follows from the first equality that $\bar{\Omega}(z)$, holomorphic in the upper half-plane, is the analytic continuation of $\Phi(z)$ from the lower into the upper half-plane, and hence the analytic continuity of $\Phi(z)$ is proved. Similarly, the second equality (93.9) leads to the conclusion that $\Omega(z)$ is analytically continued into the upper half-plane, where it takes the value $\bar{\Phi}(z)$. Hence it follows by (93.8) that $\Psi(z)$ may also be analytically continued, and the earlier proposition is proved.

In particular, it is now seen that, if only a finite segment of the boundary is loaded, the functions $\Phi(z)$ and $\Psi(z)$ may be expanded for sufficiently large $|z|$ in Laurent's series.

These results present simple means for studying the behaviour of $\Phi(z)$ and $\Psi(z)$ for large $|z|$, provided the behaviour of the stress components at infinity is known. It will be remembered that the behaviour of $\Phi(z)$ and $\Psi(z)$ at infinity had been postulated a priori in § 90 and that this step seemed to be artificial in character. However, it will be easy now to remove (or diminish) this artificiality. In fact, it may be readily proved that $\Phi(z)$ and $\Psi(z)$ *necessarily* have the form, given by (90.1), if only a finite part of the boundary is loaded and if the stresses are, for example, subject to the following condition: the quantities

$$X_x, \; Y_y, \; X_y, \quad y \cdot \frac{\partial(X_x + Y_y)}{\partial y}$$

tend (uniformly) to zero as z moves to infinity (remaining, of course, in the lower half-plane). No space will be devoted here to the proof which will easily be provided by the reader.

§93a Example. As an application of the above results consider the case when the segment

$$-a \leqslant t \leqslant a$$

of the Ox axis is subject to a uniform pressure p, while the remaining part of the boundary is free from external forces. Then $T = 0$ for all t, $N = -p$ for $-a \leq t \leq +a$, $N = 0$ for the remaining values of t, and (93.6), (93.7) give

$$\Phi(z) = \frac{p}{2\pi i}\int_{-a}^{+a}\frac{dt}{t-z} = -\frac{p}{2\pi i}\int_{-a}^{+a}\frac{dt}{z-t}, \quad \Psi(z) = -\frac{zp}{2\pi i}\int_{-a}^{+a}\frac{dt}{(t-z)^2},$$

whence

$$\Phi(z) = \frac{p}{2\pi i}\left[\log(z-t)\right]_{t=-a}^{t=+a} = \frac{p}{2\pi i}\log\frac{z-a}{z+a},$$
$$\Psi(z) = -\frac{paz}{\pi i(z^2-a^2)}. \tag{93.1a}$$

Note that the results of § 93 have been applied, although in the present case the given function $N(t)$ is discontinuous; the correctness of the final result may be verified directly.

In (93.1a) the term $\log (z-a)/(z+a)$ means the increase of the function $\log (z-t)$ for a continuous change of t from $-a$ to $+a$. For greater clarity, write $z-t = \rho e^{-i\theta}$, where $\rho = |z-t|$ and θ is the angle between the vector $z-t$ (starting from t and ending at z) and the axis Ox which will be assumed to lie between 0 and π and to be measured from the positive Ox axis in clockwise direction (Fig. 47). Then

$$\log(z-t) = \log\rho - i\theta, \quad \log\frac{z-a}{z+a} = \log\frac{\rho_1}{\rho_2} - i(\theta_1-\theta_2), \tag{93.2a}$$

where $\theta_1 - \theta_2$ is the angle subtended at z by the loaded segment of the Ox axis.

The stress components will now be calculated. One has

$$X_x + Y_y = 4\Re\Phi(z) = -\frac{2p}{\pi}(\theta_1 - \theta_2), \tag{93.3a}$$

$$X_y - X_x + 2iX_y = 2[\bar{z}\Phi'(z) + \Psi(z)] = \frac{2pa}{\pi i}\,\frac{\bar{z} - z}{z^2 - a^2} =$$

$$= -\frac{4pay}{\pi(z^2 - a^2)} = -\frac{4pay(\bar{z}^2 - a^2)}{\pi(z^2 - a^2)(\bar{z}^2 - a^2)}, \tag{93.4a}$$

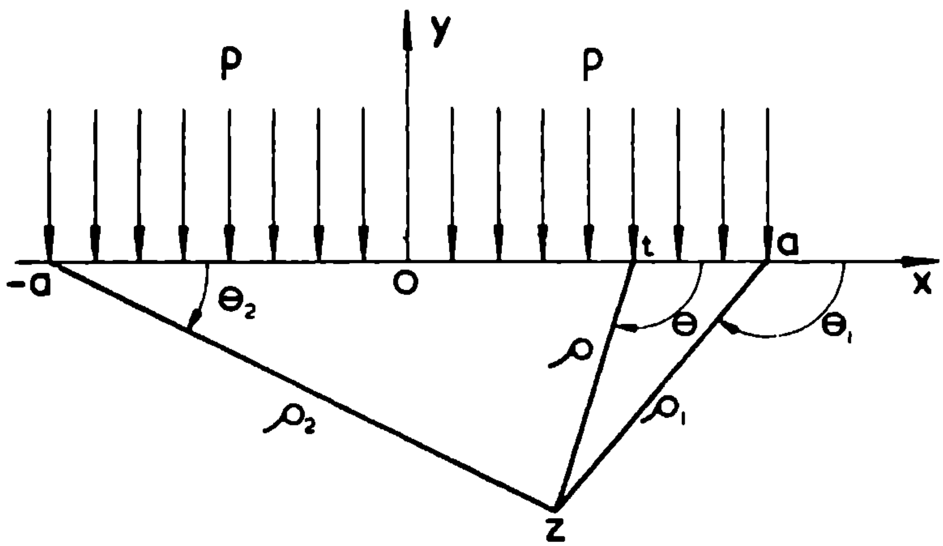


Fig. 47.

whence, finally,

$$X_x = -\frac{p}{\pi}(\theta_1 - \theta_2) + \frac{2pay(x^2 - y^2 - a^2)}{\pi[(x^2 + y^2 - a^2)^2 + 4a^2y^2]}$$

$$Y_y = -\frac{p}{\pi}(\theta_1 - \theta_2) - \frac{2pay(x^2 - y^2 - a^2)}{\pi[(x^2 + y^2 - a^2)^2 + 4a^2y^2]} \tag{93.5a}$$

$$X_y = \frac{4paxy^2}{\pi[(x^2 + y^2 - a^2)^2 + 4a^2y^2]}.$$

The solution of this problem was first given by J. H. Michell [3] and was later obtained by G. V. Kolosov [1, 2] by a different method. (However, both papers by Kolosov contain a misprint in the expression for X_y, where a^2 appears instead of a).

The law of the stress distribution becomes clearer, if one writes in

(93.4a) the term $z^2 - a^2$ as

$$z^2 - a^2 = \rho_1\rho_2\, e^{-i(\theta_1+\theta_2)};$$

then

$$Y_y - X_x + 2iX_y = -\frac{4pay}{\rho_1\rho_2}\, e^{i(\theta_1+\theta_2)} \tag{93.4'a}$$

which gives, in conjunction with (93.3a),

$$\begin{aligned} X_x &= -\frac{p}{\pi}(\theta_1 - \theta_2) + 2pa\frac{y\cos(\theta_1+\theta_2)}{\rho_1\rho_2}, \\ Y_y &= -\frac{p}{\pi}(\theta_1 - \theta_2) - 2pa\frac{y\cos(\theta_1+\theta_2)}{\rho_1\rho_2}, \\ X_y &= -2pa\frac{y\sin(\theta_1+\theta_2)}{\rho_1\rho_2}. \end{aligned} \tag{93.5'a}$$

These formula demonstrate that the stress components are continuous up to the boundary, provided the points $t = \pm a$ are excluded. At these points they cease to be continuous but *remain bounded* (as is seen by noting that $y = -\rho_1 \sin\theta_1 = -\rho_2 \sin\theta_2$). It is likewise clear that the boundary conditions are satisfied.

The components of displacement are also readily calculated and it is verified that they remain continuous up to the *entire* boundary (including the points $t = \pm a$), provided the point at infinity is excluded, since they increase with $|z| \to \infty$ like $\log|z|$.

The solution for a tangential stress, applied to a segment of the boundary, may be deduced just as simply.

§ 94. Solution of the second fundamental problem. In this case the boundary condition may be written

$$\varkappa\varphi(t) - t\,\overline{\varphi'(t)} - \overline{\psi(t)} = 2\mu(g_1 + ig_2) \tag{94.1}$$

or

$$\varkappa\overline{\varphi(t)} - \bar{t}\,\varphi'(t) - \psi(t) = 2\mu(g_1 - ig_2). \tag{94.2}$$

It will be assumed that the displacements remain bounded at infinity which, by what has been said in § 90, is equivalent to the condition $X = Y = 0$. (The more general case, considered in § 90, is easily reduced to the preceding one by means of a method analogous to that of § 78.)

The condition (94.1) may be replaced by one obtained by differentiating (94.1) with respect to t; in that case one has only to deal with $\Phi(z)$, $\Psi(z)$, and the difficulty arising from the presence of the logarithmic terms in $\varphi(z)$, $\psi(z)$ is removed. The problem will be solved by such methods in § 113, 2°.

Under this condition and those of § 90 the functions $\varphi(z)$, $\psi(z)$, $\varphi'(z) = \Phi(z)$, $\psi'(z) = \Psi(z)$, holomorphic in the lower half-plane, must satisfy (90.1′) and (90.2′) with $X = Y = 0$. Only the following of these conditions will be considered here:

$$\varphi(z) = o(1), \quad \psi(z) = c + o(1), \quad \varphi'(z) = o\left(\frac{1}{z}\right), \tag{94.3}$$

where c is some constant which is not given beforehand (so that the basis of the problem is somewhat more general in comparison with the conditions of § 90); the constant term in the expression for $\varphi(z)$ has been omitted, as usually, without affecting generality.

By (90.10) one has, in addition, to assume that for large $|t|$ the given functions satisfy the condition

$$g_1 + ig_2 = G + o(1), \tag{94.4}$$

where G is a constant which is, in general, complex. Further, it will now be assumed that $g_1 + ig_2$ satisfies the H condition on the boundary, including the point at infinity.

Expressing that the function $\psi(t)$, determined by (94.2), must be the boundary value of the function $\psi(z)$, holomorphic in the lower half-plane, one obtains by (76.21)

$$-\frac{\mu}{\pi i}\int_{-\infty}^{+\infty}\frac{g_1 + ig_2}{t - z}\,dt + \frac{\varkappa}{2\pi i}\int_{-\infty}^{+\infty}\frac{\varphi(t)dt}{t - z} - \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{t\overline{\varphi'(t)}dt}{t - z} = -\tfrac{1}{2}\bar{c},$$

where z is an arbitrary point of the lower half-plane, or, applying the formulae of § 72,

$$-\frac{\mu}{\pi i}\int_{-\infty}^{+\infty}\frac{g_1 + ig_2}{t - z}\,dt - \varkappa\varphi(z) = -\tfrac{1}{2}\bar{c},$$

where, in particular, use has been made of the fact that $t\overline{\varphi'(t)}$ is the boundary value of $z\bar{\varphi}'(z)$, holomorphic in the upper half-plane and vanishing at infinity. The value of $\bar{c}$ is obtained by letting $z \to \infty$ (in the lower half-plane); then, using the second formula (71.15), one finds

$$\tfrac{1}{2}\bar{c} = -\mu G.$$

Thus

$$\varkappa\varphi(z) = -\frac{\mu}{\pi i}\int_{-\infty}^{+\infty} \frac{g_1 + ig_2}{t - z} - \mu G. \tag{94.5}$$

The function $\psi(z)$ is now easily determined from its boundary value, given by (94.2); in fact, applying (72.2′), one obtains

$$\psi(z) = \frac{\mu}{\pi i}\int_{-\infty}^{+\infty} \frac{g_1 - ig_2}{t - z}\,dt - \frac{\varkappa}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\varphi(t)}\,dt}{t - z} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{t\varphi'(t)\,dt}{t - z} + \tfrac{1}{2}c$$

or, applying again the formulae of § 72 and substituting for c,

$$\psi(z) = \frac{\mu}{\pi i}\int_{-\infty}^{+\infty} \frac{g_1 - ig_2}{t - z}\,dt - z\varphi'(z) - \mu\bar{G}, \tag{94.6}$$

where, in particular, use has been made of the fact that $\overline{\varphi(t)}$ is the boundary value of $\bar{\varphi}(z)$, holomorphic in the upper half-plane and vanishing at infinity. It is easily seen, on the basis of the results of § 71, that these functions $\varphi(z)$, $\psi(z)$ satisfy all the conditions of the problem, including (94.3), if, for example, the expression $g_1 + ig_2$ and its derivative $g_1' + ig_2'$ with respect to t satisfy the H condition and if the expressions $t(g_1 + ig_2 - G)$, $t^2(g_1' + ig_2')$ satisfy that condition near the point at infinity. Thus the problem is solved.

If it is not only required to satisfy (94.3) but also the conditions (90.1′), (90.2′), it is sufficient to assume, in addition, that also $t^3(g_1'' + ig_2'')$ satisfies the H condition near the point at infinity.

The solution of this problem (by other means) was likewise given by M. A. Sadovski [1, 2] who also made a careful study of the character and of the conditions for the existence of the solution.

§ 95. Solution of the fundamental problems for regions, mapped on to the half-plane by means of rational functions. Case of a parabolic contour. When the given region S may be mapped on to the half-plane by a rational function $\omega(\zeta)$, the fundamental problem may be solved by elementary means, as in the analogous cases of § 84 et seq.

In view of the analogy with the earlier work, consideration will be

limited here to an explanation of the method of solution by the concrete example, where the boundary L is a parabola and S is the part of the plane, lying outside the parabola (i.e., not on the side of the focus).

Consider the transformation

$$z = \omega(\zeta) = i(\zeta - ia)^2, \quad (a > 0), \tag{95.1}$$

i.e.,

$$x = -2\xi(\eta - a), \quad y = \xi^2 - (\eta - a)^2. \tag{95.1'}$$

The real axis $\eta = 0$ of the ζ plane corresponds in the Oxy plane to a line with the parametric representation

$$x = 2a\xi, \quad y = \xi^2 - a^2,$$

i.e., to the line

$$x^2 = 4a^2(y + a^2); \tag{95.2}$$

this is the parabola L with parameter $2a^2$, its axis parallel to the axis Oy and its vertex at the point $(0, -a^2)$; the origin is the focus of the parabola.

When the point ζ moves along the ξ axis from the left to the right, the corresponding point z moves along the parabola likewise from the left to the right.

It is readily verified that (95.1) maps the region S, outside the parabola, on to the half-plane $\eta < 0$. The coordinate lines (ξ) and (η) are easily seen to be confocal parabolas; the axes of the parabolas (ξ) and (η) are orientated in opposite directions. Fig. 46*a* of § 92 shows several parabolae of the family (ξ) [i.e., $\eta =$ const.] and those parts of some parabolas $\xi =$ const. which are included in S.

The angle, subtended by the parabola at points inside the parabola (i.e., outside S), is seen to be (-2π), so that one has to take

$$\Pi' = -2\pi \tag{95.3}$$

in the formulae of § 91. The solution of the fundamental problems for S presents no difficulties. As an example, *the first fundamental problem* will now be solved (the second fundamental problem can be solved in an analogous manner).

Let σ denote points of the real axis of the ζ plane. Then, by (92.10), the boundary condition may be written

$$\Phi(\sigma) + \overline{\Phi(\sigma)} + \frac{\sigma + ia}{2}\Phi'(\sigma) - \frac{\sigma - ia}{\sigma + ia}\Psi(\sigma) = N + iT, \tag{95.4}$$

where N and T are the known boundary values of the normal and tangential stresses $\widehat{\eta\eta}$ and $\widehat{\xi\xi}$. The condition (95.4) will now be multiplied by $\sigma + ia$ which gives

$$(\sigma + ia)\Phi(\sigma) + (\sigma + ia)\overline{\Phi(\sigma)} + \frac{(\sigma + ia)^2}{2}\Phi'(\sigma) - (\sigma - ia)\Psi(\sigma) = F, \quad (95.5)$$

where

$$F = (N + iT)(\sigma + ia). \quad (95.6)$$

It has, of course, been assumed here that N and T are given in such a way that they do not violate the conditions imposed in § 90 with regard to the behaviour of the stresses at points, away from the origin.

The method of solution, to be used below, may, of course, be applied directly to (95.4); in the earlier editions of this book the problem under consideration was solved in that manner. The present method, however, leads more quickly to the solution.

The conjugate complex form of (95.5) is

$$(\sigma - ia)\Phi(\sigma) + (\sigma - ia)\overline{\Phi(\sigma)} + \frac{(\sigma - ia)^2}{2}\overline{\Phi'(\sigma)} -$$

$$- (\sigma + ia)\overline{\Psi(\sigma)} = \bar{F}. \quad (95.7)$$

The unknown functions $\Phi(\zeta)$, $\Psi(\zeta)$ which are holomorphic in the lower half-plane satisfy on the basis of (90.1) and of (95.1) the conditions

$$\Phi(\zeta) = O\left(\frac{1}{\zeta^2}\right), \quad \Psi(\zeta) = O\left(\frac{1}{\zeta^2}\right), \quad \Phi'(\zeta) = O\left(\frac{1}{\zeta^3}\right). \quad (95.8)$$

Expressing that the function $(\sigma - ia)\Psi(\sigma)$, determined by (95.5), is the boundary value of the function $(\zeta - ia)\Psi(\zeta)$, holomorphic in the lower half-plane and vanishing at infinity, one obtains, applying (76.21),

$$-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\bar{F}\,d\sigma}{\sigma - \zeta} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{(\sigma - ia)\,\Phi(\sigma)d\sigma}{\sigma - \zeta} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{(\sigma - ia)\,\overline{\Phi(\sigma)}d\sigma}{\sigma - \zeta} +$$

$$+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{(\sigma - ia)^2\,\overline{\Phi'(\sigma)}d\sigma}{2(\sigma - \zeta)} = 0,$$

where ζ is a point of the lower half-plane; noting that $(\sigma - ia)\Phi(\sigma)$ is the boundary value of $(\zeta - ia)\Phi(\zeta)$, holomorphic in the lower half-

plane and vanishing at infinity, and that $(\sigma - ia)\overline{\Phi(\sigma)}$ and $(\sigma - ia)^2\Phi'(\sigma)$ are the boundary values of $(\zeta - ia)\overline{\Phi}(\zeta)$ and $(\zeta - ia)^2\overline{\Phi}'(\zeta)$, holomorphic in the upper half-plane and vanishing at infinity, one finds, applying the formulae of § 72,

$$-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\overline{F}\,d\sigma}{\sigma - \zeta} - (\zeta - ia)\Phi(\zeta) = 0,$$

whence

$$\Phi(\zeta) = -\frac{1}{2\pi i(\zeta - ia)}\int_{-\infty}^{+\infty}\frac{\overline{F}\,d\sigma}{\sigma - \zeta}. \tag{95.9}$$

The function $(\zeta - ia)\Psi(\zeta)$ is now easily determined from its boundary value given by (95.5). One thus obtains

$$\Psi(\zeta) = \frac{1}{2\pi i(\zeta - ia)}\int_{-\infty}^{+\infty}\frac{F\,d\sigma}{\sigma - \zeta} + \frac{\zeta + ia}{\zeta - ia}\Phi(\zeta) + \frac{(\zeta + ia)^2}{2(\zeta - ia)}\Phi'(\zeta). \tag{95.10}$$

It is readily seen that the above solution satisfies the imposed conditions, if the given function F and its first derivative F' with respect to σ satisfy the H condition and if this condition is satisfied near the point at infinity by the functions σF and $\sigma^2 F'$.

Hence the problem is solved.

Chapter 17

SOME GENERAL METHODS OF SOLUTION OF BOUNDARY VALUE PROBLEMS. GENERALIZATIONS *)

One of the general methods of solution of the fundamental boundary value problems of the plane theory of elasticity for simply-connected regions has been studied in §§ 78, 79. The present chapter gives a short introduction to several other methods (also applicable to multiply connected regions) which are either generalizations of the methods of the earlier chapters of this Part or closely related to them.

Only one new method, due to D. I. Sherman (§§ 101, 102), for the solution of the first and second fundamental problems will be studied in detail and justified with complete proofs.

At the end of this chapter (§ 104), several other general problems of the theory of elasticity will be formulated to which analogous methods of solution may be applied.

§ 96. On the integral equations of S. G. Mikhlin. The method of reduction of the fundamental problems to integral equations which was studied in § 79 cannot be applied directly to multiply connected regions, since it relies on the conformal transformation of the region under consideration on to the circle and such a transformation (simple and invertible) is impossible, if the given region is multiply connected.

However, S. G. Mikhlin succeeded in modifying the above-mentioned method so that it becomes applicable also to multiply connected regions. The essentials of this modification will now be summarized. It is known from the theory of functions of a complex variable that the problem of conformal transformation of a region S, bounded by one simple contour L, on to the circle is equivalent to the determination of the so-called Green function for this region, i.e., of the real function $G(x, y)$ which is defined in the following manner:

1°. $G(x, y)$ is a regular harmonic function throughout S, except at a

*) This chapter is not necessary for the understanding of the later work.

given point (x_0, y_0) where it has a logarithmic singularity. Thus

$$G(x, y) = \log \frac{1}{r} + G_0(x, y),$$

where r is the distance between the points (x, y) and (x_0, y_0) and $G_0(x, y)$ is a regular harmonic function.

2°. The boundary value of $G(x, y)$ on L is zero.

If $H_0(x, y)$ is the harmonic function, conjugate to $G_0(x, y)$, then the analytic function of the complex variable

$$M(z) = \log \frac{1}{z - z_0} + G_0(x, y) + iH_0(x, y),$$

where $z_0 = x_0 + iy_0$, is called the complex Green function. As shown by the preceding formula, the function $M(z)$ is multi-valued because of the presence of the logarithmic term. Since the complex Green function depends on z as well as on z_0, it is more logical to denote it by $M(z, z_0)$ rather than by $M(z)$.

The fact that the problem of determination of the Green function is equivalent to the problem of conformal transformation of a given region on to the circle permits modification of the method of §§ 78, 79 so that the use of conformal mapping may be replaced by a study of the function $M(z, z_0)$.

On the other hand, the concept of the Green function, whether real or complex, may also be applied to multiply-connected regions, bounded by several contours. Hence the above-mentioned method may be generalized to the case of multiply-connected regions.

In this way S. G. Mikhlin reduced the first and second fundamental problems of the plane theory of elasticity for multiply connected regions to Fredholm integral equations which are somewhat more complicated (as was to be expected) than the equations of § 79 (which apply only to simply connected regions), but which are quite useful for general investigations. In particular, they have been used in a number of papers by S. G. Mikhlin [1—3, 7, 9] to prove the existence theorems. The reader's attention is drawn to these papers and likewise to the book [13] by the same author in which he gives a sufficiently complete study of the results.

Apart from the first and second fundamental problems for multiply connected regions, S. G. Mikhlin also solved by his method other boundary

problems which are of great interest; for example, the problem of elastic equilibrium of a body, composed in a definite manner of different homogeneous parts having different elastic constants (restricted, of course, to those bodies to which the solutions of plane elasticity may be applied); this problem is treated in S. G. Mikhlin [10] and several particular cases are considered by elementary means in his paper [8].

§ 97. On a general method of solution of problems for multiply connected regions. One general method of solution of boundary value problems, developed by D. I. Sherman [1, 5] and S. G. Mikhlin, deserves special consideration; this method permits the construction of the Fredholm equation for a given multiply connected region, if by some means the general solution of the corresponding problem has been deduced for simply connected regions each of which is bounded by one of the simple contours, constituting the boundary of the given multiply connected region. For this purpose these general solutions must be presented in a definite manner, e.g. in the form given by the solutions of the integral equations, stated in § 79.

Particularly simple and practically useful equations are obtained in the case where the above-mentioned, separate, simply connected regions are mapped on to the circle by rational functions for which the effective methods of solution, studied above, maybe applied. An example of such a case is the half-plane with elliptic holes, considered by D. I. Sherman [4].

In turning to the general case, it will be noted that the integral equations, obtained in the manner stated above, have the following, practically useful properties: If these equations are solved by the known algorithm of successive approximation (i.e., by expanding the solutions in so-called Neumann's series), then this algorithm coincides, in essence, with the algorithm, generalizing the algorithm of Schwarz for the problem of Dirichlet; with regard to this generalized algorithm of Schwarz comments have already been made in § 89.

A study of this method will also be found in S. G. Mikhlin's book [13]; Mikhlin's investigation differs somewhat from that by D. I. Sherman, since the latter starts from integral equations, obtained by the Author (§ 79), while Mikhlin starts from his own equations mentioned in the preceding section.

§ 98. The integral equations, proposed by the Author. The integral equations, deduced in § 79, are quite useful for general investigations and give effective, practically applicable results in a number of important particular cases, but they suffer from one essential disadvantage; namely, the transforming function $\omega(\zeta)$ is required for their construction. The same disadvantage attaches to S. G. Mikhlin's equations (§ 96), since the complex Green function $M(z, z_0)$ has to be determined.

Integral equations have long ago ceased to be useful only for general theoretical investigations; lately, rather effective methods have been developed for their numerical solution, in particular, in those cases, where they involve only simple (and not multiple) integrals, as is the case with those which are of interest here.

It is therefore very desirable for direct practical applications to have integral equations whose kernels are related directly and simply to the line elements constituting the boundary of the region and which do not involve elements the determination of which requires preliminary solution of auxiliary boundary problems as Dirichlet's problem (or its equivalent) for the determination of the functions $\omega(\zeta)$ or $M(z, z_0)$.

Equations of this type, used by G. Lauricella—D. I. Sherman, will be considered in detail in §§ 101, 102; these equations are, in the Author's opinion, the simplest and most suitable for the purpose of general investigations.

However, the Author proposes to devote some space here to equations, obtained in his papers [17, 18], since the trend of thought, leading to these equations, is closely related to that which led to the results of the preceding chapters of this Part, and since they are of interest in themselves.

These equations are very similar to those of G. Lauricella (cf. § 101), but nevertheless they differ significantly from Lauricella's. In the Author's opinion Lauricella's equations had (at least outwardly) a rather complicated form, so that at the time he did not notice the similarity and supposed that his equations were considerably simpler.

Moreover, the Author's equations were the subject of a number of investigations by other authors (in the first place D. I. Sherman) worthy of reference, since the methods of investigation, developed by them, may be successfully applied to the solution of analogous problems.

For the sake of clarity, a beginning will be made with the case of finite regions S, bounded by one simple smooth contour L; the positive direction on L will again be such that it leaves S on the left.

The first and second fundamental problems will be considered simultaneously. The boundary conditions for these problems may be written as

$$k\overline{\varphi(t)} + \bar{t}\varphi'(t) + \psi(t) = \overline{f(t)}, \tag{98.1}$$

where, in the notation of § 41, for *the first fundamental problem* $k = 1$,

$$f(t) = f_1(t) + if_2(t) = i\int^s (X_n + iY_n)ds + C, \tag{98.2}$$

while for *the second fundamental problem* $k = -\varkappa$,

$$f(t) = -2\mu(g_1 + ig_2); \tag{98.3}$$

$\varphi(t)$, $\varphi'(t)$, $\psi(t)$ are, of course, the corresponding boundary values. The arbitrary constant on the right-hand side of (98.2) may be fixed to suit convenience.

Conditions will now be stated such that the right-hand side of the equation

$$\psi(t) = \overline{f(t)} - k\overline{\varphi(t)} - \bar{t}\varphi'(t), \tag{98.4}$$

equivalent to (98.1), must be the boundary value of some function $\psi(z)$, holomorphic in S. It is known from § 73 that a necessary and sufficient condition for this to be so is given by

$$\frac{1}{2\pi i}\int_L \frac{\overline{f(t)} - k\overline{\varphi(t)} - \bar{t}\varphi'(t)}{t - z}\,dt = 0$$

for all z *outside* S, or

$$\frac{k}{2\pi i}\int_L \frac{\overline{\varphi(t)}dt}{t - z} + \frac{1}{2\pi i}\int_L \frac{\bar{t}\varphi'(t)dt}{t - z} = A(z) \tag{98.5}$$

for all z outside S, where

$$A(z) = \frac{1}{2\pi i}\int \frac{\overline{f(t)}dt}{t - z}. \tag{98.6}$$

In this way the functional equation has been deduced for the determination of $\varphi(z)$. Once one has succeeded in finding by some means the

function $\varphi(z)$, *holomorphic in* S and satisfying (98.5), the problem will be solved, since $\psi(t)$ can be determined from (98.4) by Cauchy's formula

$$\psi(z) = \frac{1}{2\pi i}\int_L \frac{\psi(t)dt}{t-z} = \frac{1}{2\pi i}\int_L \frac{\overline{f(t)}dt}{t-z} - \frac{k}{2\pi i}\int_L \frac{\overline{\varphi(t)}dt}{t-z} - \frac{1}{2\pi i}\int_L \frac{\bar{t}\varphi'(t)dt}{t-z} \quad (98.7)$$

(where, of course, z lies in S).

The functional equation (98.5) may be readily reduced to a Fredholm equation in the following manner. [It would be of interest to study (98.5) independently without reduction to a Fredholm equation. In all probability this would offer the opportunity of finding new classes of regions for which the fundamental problems may be solved effectively.] In (98.5), let z tend to some point t_0 of L (remaining, of course, outside S). Then, on the basis of the Plemelj formulae (cf. § 68), one obtains, assuming $\varphi(t)$, $\varphi'(t)$ and $f(t)$ to satisfy on L the H condition,

$$-\tfrac{1}{2}k\overline{\varphi(t_0)} + \frac{k}{2\pi i}\int_L \frac{\overline{\varphi(t)}dt}{t-t_0} - \tfrac{1}{2}\bar{t}_0\varphi'(t_0) + \frac{1}{2\pi i}\int_L \frac{\bar{t}\varphi'(t)dt}{t-t_0} = A(t_0), \quad (a)$$

where $A(t_0)$ is the boundary value of $A(z)$ as $z \to t_0$ *from outside* S, i.e.,

$$A(t_0) = -\tfrac{1}{2}\overline{f(t_0)} + \frac{1}{2\pi i}\int_L \frac{\overline{f(t)}dt}{t-t_0} = a(t_0) + ib(t_0); \quad (98.8)$$

$a(t_0)$, $b(t_0)$ denote here real functions which will be assumed known.

Equation (a) which is obviously not a Fredholm equation may be simplified as follows. Expressing that $\varphi(t)$ and $\varphi'(t)$ must be the boundary values of functions, holomorphic in S, one finds by (73.1′)

$$-\tfrac{1}{2}\varphi(t_0) + \frac{1}{2\pi i}\int_L \frac{\varphi(t)dt}{t-t_0} = 0,$$

$$-\tfrac{1}{2}\varphi'(t_0) + \frac{1}{2\pi i}\int_L \frac{\varphi'(t)dt}{t-t_0} = 0; \quad (b)$$

the first of these conditions becomes in its conjugate complex form

$$\tfrac{1}{2}\overline{\varphi(t_0)} - \frac{1}{2\pi i}\int \frac{\overline{\varphi(t)}d\bar{t}}{\bar{t}-\bar{t}_0} = 0. \quad (c)$$

Multiplying (*b*) and (*c*) by $-\bar{t}_0$ and k respectively and adding them to (*a*), one obtains

$$-k\overline{\varphi(t_0)} - \frac{k}{2\pi i}\int_L \overline{\varphi(t)}d\log\frac{\bar{t}-\bar{t}_0}{t-t_0} + \frac{1}{2\pi i}\int_L \varphi'(t)\frac{\bar{t}-\bar{t}_0}{t-t_0}\,dt = A(t_0),$$

and finally, integrating the second integral on the left-hand side by parts,

$$-k\overline{\varphi(t_0)} - \frac{k}{2\pi i}\int_L \overline{\varphi(t)}d\log\frac{\bar{t}-\bar{t}_0}{t-t_0} - \frac{1}{2\pi i}\int_L \varphi(t)d\frac{\bar{t}-\bar{t}_0}{t-t_0} = A(t_0). \quad (98.9)$$

This is the integral equation which was mentioned earlier and which was to be deduced.

It may still be written in a different way. In fact, if

$$t - t_0 = re^{i\vartheta}, \qquad (98.10)$$

where $r = |t - t_0|$ and $\vartheta = \vartheta(t_0, t)$ is the angle between the vector $\overrightarrow{t_0t}$ and the Ox axis measured in the positive direction, one has

$$\log\frac{\bar{t}-\bar{t}_0}{t-t_0} = -2i\vartheta, \quad \frac{\bar{t}-\bar{t}_0}{t-t_0} = e^{-2i\vartheta} = \cos 2\vartheta - i\sin 2\vartheta;$$

hence (98.9) becomes

$$k\overline{\varphi(t_0)} - \frac{1}{\pi}\int \{k\overline{\varphi(t)} + e^{-2i\vartheta}\varphi(t)\}d\vartheta = -A(t_0). \qquad (98.9')$$

By writing

$$\varphi(t) = p(t) + iq(t), \qquad (98.11)$$

where $p(t)$ and $q(t)$ are real functions, and by separating real and imaginary parts, (98.9′) may be represented in the form of the two real equations

$$\left.\begin{aligned} &kp(t_0) \quad \frac{1}{\pi}\int_L \{p(t)\,(k+\cos 2\vartheta) + q(t)\sin 2\vartheta\}d\vartheta = -a(t_0), \\ &kq(t_0) - \frac{1}{\pi}\int \{p(t)\sin 2\vartheta + q(t)(k-\cos 2\vartheta)\}d\vartheta = b(t_0). \end{aligned}\right\} \qquad (98.9'')$$

In these equations

$$d\vartheta = \frac{\partial\vartheta}{\partial s}\,ds,$$

where s is the arc coordinate of the contour, corresponding to the point t. It is easily seen that

$$\frac{\partial \vartheta}{\partial s} = \frac{\cos \alpha}{r},$$

where $\alpha = \alpha(t_0, t)$ is the angle between the outward normal at t and the vector $\overrightarrow{t_0 t}$.

In order to verify the last relation, it is sufficient to remember that by the Cauchy—Riemann equations

$$\frac{\partial \vartheta}{\partial s} = \frac{\partial \log r}{\partial n} = \frac{1}{r} \frac{\partial r}{\partial n},$$

because $\log r$ and ϑ are the real and imaginary parts of the function $\log (t - t_0)$ of the complex variable t (for fixed t_0); n denotes here the normal which points to the right as one moves in the positive direction of the tangent.

If it is assumed that the angle between the normal (or tangent) to L at the point t and some fixed direction (considering this angle as function of t or s) satisfies the H condition, then

$$\frac{\cos \alpha}{r} = \frac{K(t_0, t)}{r^{\mu}},$$

where μ is a constant such that $0 \leqslant \mu < 1$ and $K(t_0, t)$ is a function continuous on L (which even satisfies the H condition). (Cf., for example, the Author's book [25]).

Thus the system (98.9″) represents an ordinary system of Fredholm equations. Correspondingly the equations (98.9) or (98.9′), equivalent to (98.9″), may be called Fredholm equations.

As the study of these integral equations in the case of simply connected regions does not present any difficulties (cf. S. G. Mikhlin [13] for the case $k = 1$), the following results will be merely enunciated here and the reader will later be given references, where the corresponding proofs may be found. First of all, it will be noted that, as is easily seen, every (continuous) solution $\varphi(t)$ of (98.9′) will satisfy the H condition everywhere on L, on the basis of the conditions assumed above. But, in addition, the solution was to be such that the derivative $\varphi'(t)$ also satisfies this condition on L, because this had been assumed in the deduction of the equation. It is readily verified that fulfilment of the last condition is ensured, if it is assumed that the curvature of the line L satisfies at every

point the H condition and that the function $f(t)$, given on L, has a derivative with respect to t which satisfies the H condition.

Consider *the first fundamental problem*. In this case $k = 1$ and $f(t)$ is given by (98.2); in the latter formula the constant C may and will be assumed arbitrarily fixed. Since, by supposition, the function $f(t)$ is continuous on L, the condition that the resultant vector of the external forces is to vanish will be automatically satisfied; however, the condition of vanishing of the resultant moment is expressed by (cf. § 42)

$$\int (f_1 dx + f_2 dy) = 0. \tag{98.12}$$

It is easily verified that the homogeneous system, obtained from (98.9″) for $a(t) = b(t) = 0$, has the solution

$$p(t) + iq(t) = i\varepsilon t + \alpha + i\beta, \tag{98.13}$$

where ε, α, β are real constants; this follows from the fact that (cf. § 34) the state of stress and the constant C will not be changed by adding to $\varphi(z)$ an expression of the form $i\varepsilon z + \alpha + i\beta$. On the other hand, it may also be verified directly that $p(t)$, $q(t)$, as given by (98.13), satisfy (98.9″).

The formula (98.13) involves linearly three arbitrary real constants and it gives three linearly independent solutions of the homogeneous system.

Writing $t = \xi + i\eta$, one may take as these three solutions

1) $p = -\eta$, $q = \xi$, 2) $p = 1$, $q = 0$, 3) $p = 0$, $q = 1$.

It may be shown that the homogeneous system has no other linearly independent solutions. Hence, by the general theory of Fredholm equations, the system (98.9″) will only have solutions, if the right-hand sides of these equations satisfy three conditions of a well known form. However, a closer study shows that two of these conditions are automatically satisfied as a consequence of the fact that $a(t)$, $b(t)$ are not arbitrary, but such that $a(t) + ib(t)$ is the boundary value of a function, holomorphic outside S and vanishing at infinity; the third condition, as was to be expected, reduces to the condition (98.12).

Thus, if (98.12) is satisfied, the system (98.9″) or, what is the same thing, the equation (98.9′) has a solution which is determined apart from an expression of the form (98.13). In addition, it may be shown (and this

is not obvious beforehand) that every solution $\varphi(t)$ of (98.9′) will be the boundary value of a function, holomorphic in S; this function $\varphi(z)$ follows from $\varphi(t)$ by the help of Cauchy's formula, and $\psi(t)$ will then be determined by (98.7). The solution of the first fundamental problem has thus been obtained.

In the case of *the second fundamental problem*, where $k = -\varkappa$ and $f(t)$ is given by (98.3), quite analogous results may be found; the only difference is that the homogeneous system, corresponding to (98.9″), has now only the two linearly independent solutions

$$\varphi(t) = p(t) + iq(t) = \alpha + i\beta, \tag{98.14}$$

where α and β are arbitrary real constants; the system (98.9″), in spite of the presence of the solutions of the corresponding homogeneous system, is always soluble (as a consequence of the particular form of the right-hand sides) and its solution gives the solution of the original problem, as in the case of the first fundamental problem.

Hitherto, it has been assumed that the region S is finite and simply connected. Suppose now that S is bounded by several simple contours $L_1, L_2, \ldots, L_m, L_{m+1}$ the last of which contains all the preceding ones, as in § 35 (cf. Fig. 14); the contour L_{m+1} may be absent in which case S will be infinite (i.e., the infinite plane with holes). It will be assumed that the individual contours L_j satisfy in a certain way the conditions of smoothness, stated above. As always, let $L = L_1 + L_2 + \ldots + L_m + L_{m+1}$ denote the complete boundary of S; the positive direction of L will be chosen in such a way that it leaves S on the left.

The only difference from the case of finite, simply connected regions is that here the unknown functions $\varphi(z)$ and $\psi(z)$ may be (and, in general, will be) multi-valued. In fact, by (42.1),

$$\begin{aligned} \varphi(z) &= -\frac{1}{2\pi(1+\varkappa)} \sum_{j=1}^{m} (X_j + iY_j) \log (z - z_j) + \varphi_0(z), \\ \psi(z) &= \frac{\varkappa}{2\pi(1+\varkappa)} \sum_{j=1}^{m} (X_j - iY_j) \log (z - z_j) + \psi_0(z), \end{aligned} \tag{98.15}$$

where (X_j, Y_j) are the resultant vectors of the external forces, applied to the contours L_j, z_j are arbitrarily fixed points inside L_j $(j = 1, \ldots, m)$ and $\varphi_0(z)$, $\psi_0(z)$ are functions, holomorphic in S, if this region is finite (i.e., if L_{m+1} is present); if S is infinite (i.e., if L_{m+1} is absent), then (cf. § 36)

$$\varphi_0(z) = \Gamma z + \varphi^*(z), \quad \psi_0(z) = \Gamma' z + \psi^*(z), \tag{98.16}$$

where $\varphi^*(z)$, $\psi^*(z)$ are functions, holomorphic in S including the point at infinity; in the case of the first as well as of the second fundamental problem it will be assumed (§ 39) that the constants Γ, Γ' are known beforehand; further, in the case of the second fundamental problem for infinite regions, it will also be assumed that the quantities

$$X = \sum_{j=1}^{m} X_j, \quad Y = \sum_{j=1}^{m} Y_j,$$

i.e., the components of the resultant vector of the external forces, applied to the entire boundary L of S, are known.

For the sake of brevity, it will now be assumed that S is finite, i.e., that L_{m+1} is present; the case of infinite regions may be considered in quite an analogous manner.

A beginning will be made with *the first fundamental problem.* In this case the boundary condition may be written (as in the case of simply connected regions)

$$\overline{\varphi(t)} + \bar{t}\varphi'(t) + \psi(t) = \overline{f(t)}, \tag{98.17}$$

where now, instead of (98.2),

$$f(t) = i\int (X_n + iY_n)ds + C_j \text{ on } L_j \ (j = 1, 2, \ldots, m, m+1), \tag{98.18}$$

while the arc coordinate s is measured (in the positive direction) on each of the contours L_j from an arbitrarily fixed point of that contour, and C_j is a constant having, in general, different values on different L_j; these constants are not known beforehand, except for one, say, C_{m+1} which may be fixed arbitrarily, and it will be assumed here that $C_{m+1} = 0$.

Substituting from (98.15) into (98.17), one finds

$$\overline{\varphi_0(t)} + \bar{t}\varphi_0'(t) + \psi_0(t) = \overline{f_0(t)}, \tag{98.19}$$

where

$$f_0(t) = f(t) + \frac{1}{2\pi(1+\varkappa)} \sum_{j=1} \{X_j + iY_j\} \{\log (t - z_j) - \varkappa \log (\bar{t} - \bar{z}_j)\} + \\ + \frac{t}{2\pi(1+\varkappa)} \sum_{j=1}^{m} \frac{X_j - iY_j}{\bar{t} - \bar{z}_j} \tag{98.20}$$

On the left-hand side of the boundary condition (98.19) one has the

boundary values of holomorphic (i.e., single-valued, analytic) functions; the right-hand side is likewise a single-valued function (assuming, of course, the choice of definite branches on each contour L_j), because for a circuit in the positive direction (leaving S on the left) of the contour L_j ($j = 1, 2, \ldots, m$) the function $f(t)$ undergoes an increase $i(X_j + iY_j)$, while the second term on the right-hand side of (98.20) shows the same increase, but with opposite sign; similarly for L_{m+1}, under the condition (which is implied) that the resultant vector (X, Y) of all external forces, acting on L, is equal to zero.

Since in the case of the first fundamental problem the quantities X_j, Y_j are known beforehand, the function $f_0(t)$ in (98.19) is determined on every L_j ($j = 1, \ldots, m$) apart from the constants C_j, while it is completely known on L_{m+1} (because, by supposition, $C_{m+1} = 0$).

Applying to (98.19) the same reasoning as in the case of a single contour, one arrives at exactly the same equation (98.9) with $k = 1$ or at the equivalent equation (98.9') or, finally, at the system (98.9''); the only difference will be that $\varphi_0(t)$ takes now the place of $\varphi(t)$, while $f(t)$ is replaced by $f_0(t)$. In addition, the right-hand side now involves the initially unknown constants $C_1, C_2, \ldots, C_m$ which must be determined in the process of solving the problem.

In the case of *the second fundamental problem*, proceeding in an analogous manner, one finds, in the former notation, the boundary condition

$$-\varkappa\overline{\varphi_0(t)} + \bar{t}\varphi_0'(t) + \psi_0(t) = \overline{f_0(t)}, \tag{98.21}$$

where this time

$$f_0(t) = -2\mu(g_1 + ig_2) - \frac{\varkappa}{\pi(1+\varkappa)} \sum_{j=1}^{m} (X_j + iY_j) \log|t - z_j| + \\ + \frac{t}{2\pi(1+\varkappa)} \sum_{j=1}^{m} \frac{X_j - iY_j}{\bar{t} - \bar{z}_j}. \tag{98.22}$$

Thus one obtains the same equation (98.9) for $k = -\varkappa$ as in the case of a single contour, if one replaces $\varphi(t)$ by $\varphi_0(t)$ and $f(t)$ by $f_0(t)$. In the present case, the unknown constants X_j, Y_j appear on the right-hand side and they must be determined at the same time as the function $\varphi_0(t)$.

It may be shown that in the case of multiply connected regions the derived integral equations allow complete solution of the corresponding boundary problems.

A preliminary study of these integral equations was given in the Author's notes [17, 18], quoted earlier, in which, for the sake of definiteness, the first fundamental problem was considered; this study was based on the supposition that the existence theorems for multiply connected regions had already been proved by some other method.

Soon after, D. I. Sherman [2, 3, 6, 11] gave a very complete study of these equations in which he did not rely on other proofs of the existence theorems, but, on the contrary, proved these theorems directly by means of the equations under consideration.

D. I. Sherman also modified these equations so as to give them a form more convenient for studies of a general character and for applications. In particular, in his paper [11], he studied in detail the question of the distribution of the eigenvalues of the integral equations, obtained by a definite modification of the above equations, introducing some parameter λ, as is done in the general Fredholm theory. This investigation showed that for values of λ, corresponding to the first and second fundamental problems, the solutions of the relevant integral equations may be expanded in Neumann's series which, in general, can be obtained by the method of successive approximation.

By means of the method of this section D. I. Sherman [6] also solved one particular case of the mixed fundamental problem when the external stress is given on one of the contours, bounding the region, while the displacements are given on the others.

Further, D. I. Sherman [8] solved by a method, analogous to the preceding one, the first and second fundamental problems for bodies, consisting of different homogeneous parts; as indicated in § 96, the same problem was solved somewhat earlier by S. G. Mikhlin by use of another method.

In later papers D. I. Sherman gave new solutions of the above as well as of some other boundary problems by means of a method which is a generalization of that of the present section; this work will be discussed below.

Finally, one more interesting problem will be mentioned which was solved by G. N. Savin [7] by a method, analogous to that above; this problem deals with the equilibrium of an elastic plane with an infinite number of identical, equally spaced holes which are subject to the same external forces. A study of this solution may likewise be found in S. G. Mikhlin [13].

§ 99. Application to contours with corners. The form of the equation (98.9') or of the system (98.9'') suggests that, if the integrals occurring there be considered as Stieltjes integrals, these equations may be applied to regions, bounded by contours of a much more general form than has been assumed for their deduction.

The investigations of L. G. Magnaradze [1—3], based on known results by J. Radon and partly by T. Carleman, show that this is actually the case and that the above-mentioned equations, interpreted in a suitable generalized sense, apply, for example, to the case where the contours, bounding the region, have corners other than cusps. There may even be infinitely many such corners; it is sufficient if the boundary of the region consists of contours having so-called "bounded rotation" (according to J. Radon).

It may be noted that L. G. Magnaradze [4] succeeded in extending these results also to one very general class of three-dimensional bodies the surfaces of which may have polygonal edges (even an infinite, but denumerable number of them); in this case one has, of course, to apply the corresponding integral equations for three-dimensional bodies.

§ 100. On the numerical solution of the integral equations of the plane theory of elasticity. Equation (98.9') or the equivalent system (98.9'') may, thanks to their simplicity, be used successfully for the numerical solution of the corresponding boundary problems of the plane theory of elasticity. One of the methods of numerical solution is outlined in the Author's note [21] and it was studied in greater detail by A. Ya. Gorgidze and A. K. Rukhadze [1] who applied this method to several examples and gave also estimates of accuracy.

This method seems to give satisfactory results also in cases when the boundaries have corners.

§ 101. The integral equations of D. I. Sherman-G. Lauricella. Recently D. I. Sherman [15—17] succeeded in deducing integral equations for the solution of the first and second as well as of the mixed fundamental boundary value problems of the plane theory of elasticity which deserve greater attention. Apparently the most natural way of arriving at these equations is the following which is based on one simple general idea, analogous to that used by I. Fredholm [1] for the deduction of the integral equations for the second fundamental problem in the three-dimensional case.

D. I. Sherman begins directly from the formulae (101.3) and (101.4) without indicating the means by which these were obtained (and he considers separately the cases $k = -\varkappa$ and $k = 1$).

I. Fredholm's idea consists, in principle, of the following. If one takes instead of the body under consideration the half-space and writes down the known formulae solving the corresponding boundary problem in closed form, using definite integrals over the plane boundary of the half-space, then these formulae, when applied to the given body (taking now the integrals over the surface of that body instead of over the plane), do not, of course, solve the boundary problem in closed form; they lead, however, to integral equations which under certain conditions will be Fredholm equations.

At first suppose that the region S under consideration is finite and bounded by one simple contour which satisfies the same conditions as in § 98. The boundary conditions of the first and second fundamental problems will now be written, using the notation of § 98,

$$k\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t), \tag{101.1}$$

remembering that $k = 1$ and $k = -\varkappa$ for the first and second problem respectively.

Under the supposition that S is the upper half-plane, L is the real axis and $\varphi(z)$, $\psi(z)$, $z\varphi'(z)$ vanish at infinity, the solution of the boundary problem (101.1) is given by

$$\varphi(z) = \frac{1}{2\pi i k}\int_L \frac{f(t)\,dt}{t - z}, \qquad \psi(z) = -\frac{1}{2\pi i}\int_L \frac{\overline{f(t)}\,dt}{t - z} - z\varphi'(z).$$

The solution for $k = -\varkappa$ can be found in § 94, where in the present case $G = 0$ and the difference in sign arises from the fact that the problem has been solved there for the lower half-plane. The solution for $k = 1$ is obtained in an analogous manner. It may also be deduced from the solution of the first fundamental problem for the half-plane, found in § 93.

Substituting in the second formula above for $\varphi'(z)$ and introducing the notation

$$\omega(t) = \frac{1}{k}f(t), \tag{101.2}$$

one finds

$$\varphi(z) = \frac{1}{2\pi i}\int \frac{\omega(t)\,dt}{t - z}, \tag{101.3}$$

$$\psi(z) = \frac{k}{2\pi i}\int_L \frac{\overline{\omega(t)}dt}{t-z} + \frac{1}{2\pi i}\int_L \frac{\omega(t)\,d\bar{t}}{t-z} - \frac{1}{2\pi i}\int_L \frac{\bar{t}\omega(t)dt}{(t-z)^2}. \quad (101.4)$$

In the case when L is the real axis (as it has been assumed for the present), $\bar{t} = t$, $d\bar{t} = dt$; the reason why $d\bar{t}$ and $\bar{t}$ have been written instead of dt and t in the second and third integrals of (101.4) respectively will become clear later on.

Integrating the last integral of (101.4) by parts, this formula may be rewritten

$$\psi(z) = \frac{k}{2\pi i}\int_L \frac{\overline{\omega(t)}dt}{t-z} - \frac{1}{2\pi i}\int_L \frac{\bar{t}\omega'(t)dt.}{t-z}. \quad (101.4')$$

Now the case when S is not the half-plane will be considered and an attempt will be made to find the solution of the boundary problem (101.1) in the form (101.3), (101.4), where $\omega(t)$ now *denotes some function of points of the contour L which is initially unknown* and has to be determined. It will be assumed that $\omega(t)$ has a derivative $\omega'(t)$ which satisfies the H condition.

Using the Plemelj formulae for boundary values of Cauchy integrals and substituting into (101.1), the boundary values of the functions $\varphi(z)$, $\psi(z)$, determined by (101.3), (101.4'), and likewise the function

$$\varphi'(z) = \frac{1}{2\pi i}\int_L \frac{\omega(t)dt}{(t-z)^2} = \frac{1}{2\pi i}\int_L \frac{\omega'(t)dt}{t-z}$$

[where the latter expression is obtained by an integration by parts], one finds the integral equation

$$k\omega(t_0) + \frac{k}{2\pi i}\int_L \omega(t)d\log\frac{t-t_0}{\bar{t}-\bar{t}_0} - \frac{1}{2\pi i}\int_L \overline{\omega(t)}d\frac{t-t_0}{\bar{t}-\bar{t}_0} = f(t_0). \quad (101.5)$$

This is the integral equation, obtained by D. I. Sherman in the quoted papers [15, 16]. It is seen to be very similar to the equation (98.9) which, for the purpose of comparison, will now be written in its conjugate complex form

$$k\varphi(t_0) - \frac{k}{2\pi i}\int_L \varphi(t)d\log\frac{t-t_0}{\bar{t}-\bar{t}_0} - \frac{1}{2\pi i}\int_L \overline{\varphi(t)}d\frac{t-t_0}{\bar{t}-\bar{t}_0} = -\overline{A(t_0)}. \quad (a)$$

However, it differs essentially from (101.5) by the sign of the first in-

tegral, by its right-hand side and, what is more important, by the character of the conditions, imposed on the unknown functions. In fact, the unknown function of (101.5) is subject to no other condition except one referring to its continuity, while the unknown function $\varphi(t)$ of equation (*a*) must be the boundary value of a function, holomorphic in S. This last condition, as already stated in § 98, is automatically satisfied in the case of finite simply-connected regions which will now be considered; but in the general case it plays an essential part.

Equation (101.5) will now be considered. As in § 98, let $t - t_0 = re^{i\vartheta}$; (101.5) then becomes

$$k\omega(t_0) + \frac{1}{\pi}\int_L \{ k\omega(t) - e^{2i\vartheta}\,\overline{\omega(t)}\}d\vartheta = f(t_0). \tag{101.5'}$$

Further, writing

$$\omega(t) = p(t) + iq(t), \quad f(t) = f_1(t) + if_2(t), \tag{101.6}$$

one obtains the system of two Fredholm equations

$$\begin{aligned} kp(t_0) + \frac{1}{\pi}\int_L \{p(t)(k - \cos 2\vartheta) - q(t)\sin 2\vartheta\}d\vartheta &= f_1(t_0), \\ kq(t_0) - \frac{1}{\pi}\int_L \{p(t)\sin 2\vartheta - q(t)(k + \cos 2\vartheta)\}d\vartheta &= f_2(t_0). \end{aligned} \tag{101.5''}$$

The following should be noted with regard to the system (101.5″). For $k = 1$, i.e., for the first fundamental problem, this system reduces to that, deduced by G. Lauricella [3] for the solution of the fundamental biharmonic problem which, as has been pointed out earlier, is equivalent (with certain reservations in the case of multiply connected regions) to the first fundamental problem of plane elasticity. For $k = -\varkappa$, i.e., for the second fundamental problem, the system (101.5″) corresponds to the system, likewise deduced by G. Lauricella [1, 2] for the second fundamental problem in the three-dimensional case.

However, Lauricella does not use Cauchy integrals and he presents the connection between the functions, which directly occur in the corresponding problems (the biharmonic function U in the fundamental biharmonic problem, the displacement components in the second fundamental problem), and the auxiliary functions p, q of points of the boundary L in a (apparently) very complicated form. Lauricella's integral equations

likewise are not as simple as (101.5″). This last circumstance is, of course, of no significant importance, but the formulae (101.3) and (101.4), expressing the relations between the functions $\varphi(z)$, $\psi(z)$ and $\omega(t) = p(t) + iq(t)$, are of great value and so is the form (101.5) of the integral equation which clearly demonstrates the connection with Cauchy integrals. In fact, the discovery of this relationship considerably simplifies the analysis, in particular in the case of multiply connected regions (with regard to which more will be said later), and, in addition, offers the opportunity of deducing (relatively) simple solutions of a number of other important boundary value problems. Therefore it seems only just to call (101.5) or (101.5′) the equations of D. I. Sherman—G. Lauricella.

In the case of multiply connected regions, it is advisable, according to D. I. Sherman, to somewhat modify (101.3), (101.4) and the integral equations which follow from them, thus leading to (relatively) very simple results; this question will be treated in greater detail in the next section.

§ 102. Solution of the first and second fundamental problems by the method of D. I. Sherman *). Let the region S be bounded by several, simple, non-intersecting contours $L_1, L_2, \ldots, L_m, L_{m+1}$, the last of which contains all the others, and let $L = L_1 + L_2 + \ldots + L_{m+1}$ denote the complete boundary of S. In addition, it will be assumed that each of the contours L has a curvature, satisfying the H condition. The finite regions, bounded by the L_j $(j = 1, 2, \ldots, m)$, will be denoted by S_j, and the infinite region, bounded by L_{m+1}, by S_{m+1}.

The first fundamental problem will be solved first. Without affecting generality, it may be assumed that the unknown functions $\varphi(z)$ and $\psi(z)$ are single-valued, since multi-valued terms (which are known beforehand) may be removed from them and placed on the right-hand side of the equation, expressing the boundary condition [Cf. (98.15), (98.19), (98.20)].

The boundary condition is

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) + C_j \text{ on } L_j (j = 1, 2, \ldots, m+1), \quad (102.1)$$

where $f(t)$ is a given function (which is single-valued and continuous on every L_j) and $C_1, C_2, \ldots, C_m, C_{m+1}$ are initially unknown constants

*) D. I. Sherman [15, 16]. These papers have been reproduced here with only insignificant modifications.

only one of which may be fixed arbitrarily; it will be assumed that

$$C_{m+1} = 0.$$

Following D. I. Sherman, the solution will be written in the form

$$\varphi(z) = \frac{1}{2\pi i}\int_L \frac{\omega(t)dt}{t - z} + \sum_{j=1}^{m} \frac{b_j}{z - z_j}, \tag{102.2}$$

$$\psi(z) = \frac{1}{2\pi i}\int_L \frac{\overline{\omega(t)}dt}{t - z} + \frac{1}{2\pi i}\int_L \frac{\omega(t)d\bar{t}}{t - z} - \frac{1}{2\pi i}\int_L \frac{\bar{t}\omega(t)dt}{(t - z)^2} + \sum_{j=1}^{m} \frac{b_j}{z - z_j}, \tag{102.3}$$

where $\omega(t)$ is a function of points of L, subject to definition, z_j are arbitrarily fixed points of S_j, $j = 1, \ldots, m$ (so that they lie outside S) and b_j are *real* constants, related to $\omega(t)$ in the following manner:

$$b_j = i\int_{L_j} \{\omega(t)d\bar{t} - \overline{\omega(t)}dt\}, \quad j = 1, 2, \ldots, m. \tag{102.4}$$

The introduction of b_j leads to the modification of the integral equations which was mentioned at the end of § 101. In the case of simply connected regions ($m = 0$), the formulae (102.2), (102.3) become (101.3) and (101.4) for $k = 1$.

Substituting in (102.1) the boundary values of the functions $\varphi(z)$, $\varphi'(z)$, $\psi(z)$ as determined by (102.2) and (102.3), one obtains as in § 101

$$\omega(t_0) + \frac{1}{2\pi i}\int \omega(t)d\log\frac{t - t_0}{\bar{t} - \bar{t}_0} - \frac{1}{2\pi i}\int \overline{\omega(t)}d\,\frac{t - t_0}{\bar{t} - \bar{t}_0} +$$

$$+ \sum_{j=1}^{m}\left\{\frac{b_j}{t_0 - z_j} + \frac{\bar{b}_j}{\bar{t}_0 - \bar{z}_j}\left(1 - \frac{t_0}{\bar{t}_0 - \bar{z}_j}\right)\right\} - C_k = f(t_0) \text{ on } L_k,$$

$$k = 1, 2, \ldots, m + 1. \tag{102.5}$$

It will be expedient to further modify this equation by adding to the left-hand side the term

$$\frac{b_{m+1}}{t_0} + \frac{\bar{b}_{m+1}}{\bar{t}_0}\left(1 - \frac{t}{\bar{t}_0}\right), \tag{a}$$

where b_{m+1} is a *purely imaginary* constant, related to $\omega(t)$ by the formula [cf. note following (102.4)]

$$b_{m+1} = \frac{1}{2\pi i}\int \left\{\frac{\omega(t)}{t^2}\,dt + \frac{\omega(t)}{\bar{t}^2}\,d\bar{t}\right\}; \tag{102.6}$$

it will be assumed that the origin of coordinates lies in S.

Thus one obtains the equation

$$\omega(t_0) + \frac{1}{2\pi i}\int_L \omega(t)\log\frac{t-t_0}{\bar t-\bar t_0} - \frac{1}{2\pi i}\int_L \overline{\omega(t)}\,d\,\frac{t-t_0}{\bar t-\bar t_0} +$$
$$+\sum_{j=1}^{m+1}\left\{\frac{b_j}{t_0-z_j}+\frac{\bar b_j}{\bar t_0-\bar z_j}\left(1-\frac{t_0}{\bar t_0-\bar z_j}\right)\right\} - C_k = f(t_0) \text{ on } L_k,$$
$$k = 1, 2, \ldots, m+1, \qquad (102.5')$$

where $z_{m+1} = 0$.

In addition, the unknown constants C_k will be related to the unknown function $\omega(t)$ by the formulae

$$C_k = -\int_{L_k} \omega(t)ds,\ k = 1, 2, \ldots, m, \qquad (102.7)$$

where ds is the element of arc of L_k.

If one now replaces the constants b_j, C_j on the left-hand side of (102.5′) by the expressions (102.4), (102.6) and (102.7), then (102.5′) becomes an integral equation which *involves no other unknowns* but $\omega(t)$. Separating real and imaginary parts, as was done in § 101, one obtains a system of two Fredholm equations, but since this system serves no purpose in what follows it will not be written down here.

The integral equation (102.5′) will be called the equation of D. I. Sherman. In the case of simply connected regions, it differs from (101.5) only by the term

$$\frac{b_1}{t_0}+\frac{\bar b_1}{\bar t_0}\left(1-\frac{t_0}{\bar t_0}\right).$$

It will now be shown that, if (102.5′) has a solution, then necessarily $b_{m+1} = 0$, provided the resultant moment of the external forces is zero; the vanishing of the resultant vector of the external forces is ensured by the single-valuedness and continuity of the function $f(t)$. The condition for the resultant moment may obviously be written [cf. (42.5)]

$$\Re\int_L f(t)dt = 0, \qquad (102.8)$$

while (102.1) may be presented in the form

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} + b_{m+1}\left\{\frac{1}{t}-\frac{1}{\bar t}+\frac{t}{\bar t^2}\right\} - C_j = f(t) \text{ on } L_j, \quad (102.1')$$

if one interprets $\varphi(t)$, $\varphi'(t)$ and $\psi(t)$ as the boundary values of the expressions (102.2) and (102.3). Multiplying both sides of (102.1′) by $d\bar{t}$ and integrating over L, one finds after an integration by parts

$$\int_L \{\varphi(t)d\bar{t} - \overline{\varphi(t)}dt\} + b_{m+1} \int_L \left\{ \frac{d\bar{t}}{t} + \frac{dt}{\bar{t}} \right\} + 2\pi i b_{m+1} = \int_L f(t)dt.$$

Since the last term on the left-hand side of this equation is real and all the other terms are purely imaginary, one must have $b_{m+1} = 0$, as was to be proved.

Thus, in order to satisfy (102.8), every solution $\omega(t)$ of (102.5′) is at the same time a solution of the original equation (102.5), and hence a solution of the boundary problem (102.1), and the constants C_j will be determined by the formulae (102.7).

It will now be proved that the equation (102.5′) always has a solution. [Equation (102.5) would only be soluble under the condition (102.8)]. For this purpose the homogeneous equation, obtained from (102.5′) for $f(t) = 0$, will be considered and it will be shown that it has no non-zero solutions. Let $\omega_0(t)$ be any solution of this equation and $\varphi_0(z)$, $\psi_0(z)$, C_j^0 be the corresponding values of $\varphi(z)$, $\psi(z)$ and the constants C_j, determined by (102.2), (102.3), (102.4) and (102.7) for $\omega(t) = \omega_0(t)$; in particular, by (102.2) and (102.3),

$$\varphi_0(z) = \frac{1}{2\pi i} \int_L \frac{\omega_0(t)dt}{t - z} + \sum_{j=1}^{m} b_j^0 \log (z - z_j) \tag{102.9}$$

$$\psi_0(z) = \frac{1}{2\pi i} \int_L \frac{\overline{\omega_0(t)}dt}{t - z} - \frac{1}{2\pi i} \int_L \frac{\bar{t}\omega_0'(t)dt}{t - z} + \sum_{j=1}^{m} \frac{\bar{b}_j^0}{z - z_j} \tag{102.10}$$

where b_j^0 are the constants, given by (102.4) for $\omega(t) = \omega_0(t)$, and the expression for $\psi_0(z)$ has been transformed by means of an integration by parts. The functions $\varphi_0(z)$, $\psi_0(z)$ satisfy the boundary condition

$$\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} - C_j^0 = 0 \text{ on } L_j, j = 1, \ldots, m + 1, C_{m+1}^0 = 0, \tag{102.11}$$

as may be seen from (102.1′), taking into consideration that in the present case $f(t) = 0$ and also $b_{m+1} = 0$, since obviously (102.8) will be satisfied. Hence $\varphi_0(t)$, $\psi_0(t)$ solve the first fundamental problem in the absence of external forces, and therefore, by the uniqueness theorem,

$$\varphi_0(z) = i\varepsilon z + c, \tag{102.12}$$

where ε is a real and c a complex constant; thus, by (102.11), using the fact that $C_{m+1}^0 = 0$,

$$\psi_0(z) = -\bar{c}, \tag{102.13}$$

and, obviously,

$$C_j^0 = 0, \quad j = 1, 2, \ldots, m+1. \tag{102.14}$$

It follows from (102.9), (102.10), (102.12) and (102.13) that

$$i\varepsilon z + c = \frac{1}{2\pi i}\int_L \frac{\omega_0(t)dt}{t - z} + \sum_{j=1}^m \frac{b_j^0}{z - z_j}, \tag{102.9'}$$

$$-\bar{c} = \frac{1}{2\pi i}\int_L \frac{\overline{\omega_0(t)}dt}{t - z} - \frac{1}{2\pi i}\int \frac{\bar{t}\omega_0'(t)dt}{t - z} + \sum_{j=1}^m \frac{b_j^0}{z - z_j}. \tag{102.10'}$$

Introduce now the notation

$$i\varphi^*(t) = \omega_0(t) + \sum_{j=1}^m \frac{b_j^0}{t - z_j} - i\varepsilon t - c, \tag{102.15}$$

$$i\psi^*(t) = \overline{\omega_0(t)} - \bar{t}\omega_0'(t) + \sum_{j=1}^m \frac{b_j^0}{t - z_j} + \bar{c}. \tag{102.16}$$

The equations (102.9′), (102.10′) may then be written

$$\frac{1}{2\pi i}\int_L \frac{\varphi^*(t)dt}{t - z} = 0, \quad \frac{1}{2\pi i}\int_L \frac{\psi^*(t)dt}{t - z} = 0 \text{ for all } z \text{ in } S,$$

and hence (cf. § 74) $\varphi^*(t)$, $\psi^*(t)$ are the boundary values of the functions $\varphi^*(z)$, $\psi^*(z)$, holomorphic in the regions $S_1, S_2, \ldots, S_{m+1}$, and

$$\varphi^*(\infty) = \psi^*(\infty) = 0.$$

It will now be recalled that in the present case $b_{m+1} = 0$, where b_{m+1} is given by (102.6) for $\omega(t) = \omega_0(t)$. Substituting in (102.6) with $b_{m+1} = 0$ for $\omega(t)$ the expression $\omega_0(t)$, obtained from (102.15), and taking into consideration the previously stated property of $\varphi^*(t)$, it is easily seen that $\varepsilon = 0$.

Further, eliminating $\omega_0(t)$ from (102.15) and (102.16), one finds

$$\overline{\varphi^*(t)} + \bar{t}\varphi^{*\prime}(t) + \psi^*(t) = i\sum_{j=1}^m b_j^0 \left\{\frac{1}{\bar{t} - \bar{z}_j} - \frac{1}{t - z_j} + \frac{\bar{t}}{(t - z_j)^2}\right\} - 2i\bar{c}$$

on L.

Multiplying both sides of this equation by dt and integrating over the contours $L_k (k = 1, 2, \ldots, m)$, one obtains

$$\int_{L_k} \{\overline{\varphi^*(t)}dt - \varphi^*(t)d\bar{t}\} = i \sum_{j=1}^{m} b_j^0 \int_{L_k} \left\{\frac{dt}{\bar{t} - \bar{z}_j} + \frac{d\bar{t}}{t - z_j}\right\} - 2\pi b_k^0,$$

and hence, since the b_j^0 are real,

$$b_k^0 = 0, \quad k = 1, 2, \ldots, m. \tag{102.17}$$

Therefore

$$\varphi^*(t) + \bar{t}\varphi^{*\prime}(t) + \psi^*(t) = -2i\bar{c} \text{ on } L_k,\ k = 1, 2, \ldots, m + 1.$$

Consequently $\varphi^*(z)$, $\psi^*(z)$ solve the first fundamental problem for the regions S_k, $k = 1, 2, \ldots, m + 1$, in the absence of external forces. Applying the uniqueness theorem to the region S_{m+1} and using the condition $\varphi^*(\infty) = \psi^*(\infty) = 0$, one finds $\varphi^*(z) = \psi^*(z) = 0$ in S_{m+1}, and hence $c = 0$. Further, the uniqueness theorem applied to the regions $S_k (k = 1, \ldots, m)$ gives (remembering that $c = 0$)

$$\varphi^*(z) = i\varepsilon_k z + c_k, \quad \psi^*(z) = -\bar{c}_k \text{ on } L_k,\ k = 1, 2, \ldots, m,$$

whence, by (102.15)—(102.17),

$$\omega_0(t) = -\varepsilon_k t + ic_k \text{ on } L_k, \quad k = 1, 2, \ldots, m;$$

in addition, since $\varphi^*(z) = \psi^*(z) = 0$ in S_{m+1},

$$\omega_0(t) = 0 \text{ on } L_{m+1}.$$

Finally, using successively the equations (102.4), (102.17), (102.7) and (102.14), it is easily verified that $\varepsilon_k = c_k = 0$ for all k, and hence $\omega_0(t) = 0$ everywhere on L.

Thus the homogeneous equation, corresponding to (102.5′), has no solutions, different from zero. Consequently the equation (102.5′) has one and only one solution $\omega(t)$. Substituting this solution $\omega(t)$ in (102.2) and (102.3), one obtains the solution of the original problem, provided the condition (102.8), expressing zero resultant moment of the external forces, is satisfied (the vanishing of the resultant vector of the external forces being ensured by the continuity of $f(t)$ on L). Thus the problem is solved.

If (102.8) is not satisfied, $\varphi(z)$ and $\psi(z)$ will not satisfy the boundary condition (102.1), since in that case $b_{m+1} \neq 0$ and the solution of (102.5′) will not be a solution of (102.5).

Next consider *the second fundamental problem*. In this case the boundary condition may be written

$$\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} = g(t), \tag{102.18}$$

where, as before,

$$g(t) = 2\mu(g_1 + ig_2). \tag{102.19}$$

Taking into consideration (101.3) and (101.4) (for $k = -\varkappa$) and likewise the form of the functions $\varphi(z)$, $\psi(z)$, as determined by (35.10) and (35.11), the solution of this problem will be sought in the form

$$\varphi(z) = \frac{1}{2\pi i}\int\limits_L \frac{\omega(t)dt}{t-z} + \sum_{j=1}^{m} A_j \log(z - z_j), \tag{102.20}$$

$$\psi(z) = -\frac{\varkappa}{2\pi i}\int\limits_L \frac{\overline{\omega(t)}dt}{t-z} + \frac{1}{2\pi i}\int\limits_L \frac{\omega(t)d\bar{t}}{t-z} - \frac{1}{2\pi i}\int\limits_L \frac{\bar{t}\omega(t)dt}{(t-z)^2} - \\ - \sum_{j=1}^{m} \varkappa\bar{A}_j \log(z - z_j), \tag{102.21}$$

where the A_j are constants. These constants will be related to the unknown function $\omega(t)$ by the formulae

$$A_j = \int\limits_L \omega(t)ds. \tag{102.22}$$

It is easily seen that the displacements, corresponding to the functions $\varphi(z)$, $\psi(z)$, are single-valued in S.

As in the preceding problem, one obtains for the determination of $\omega(t)$ the integral equation

$$\varkappa\omega(t_0) + \frac{\varkappa}{2\pi i}\int\limits_L \omega(t)d\log\frac{t-t_0}{\bar{t}-\bar{t}_0} + \frac{1}{2\pi i}\int\limits_L \overline{\omega(t)}d\frac{t-t_0}{\bar{t}-\bar{t}_0} + \\ + \sum_{j=1}^{m}\varkappa\{\log(t_0 - z_j) + \log(\bar{t}_0 - \bar{z}_j)\}\int\limits_{L_j}\omega(t)ds = g(t_0) \text{ on } L, \tag{102.23}$$

where $\log(t_0 - z_j) + \log(\bar{t}_0 - \bar{z}_j)$ must be conceived as a single-valued function which is equal to $2\log|t_0 - z_j|$.

The integral equation (102.23) is found to be always soluble. In order to see this, consider the homogeneous equation, obtained from it for

$g(t) = 0$. Let $\omega_0(t)$ be any solution of this homogeneous equation and $\varphi_0(z)$, $\psi_0(z)$ the corresponding expressions for the functions $\varphi(z)$, $\psi(z)$. Then

$$\varkappa\varphi_0(t) + t\overline{\varphi_0'(t)} - \overline{\psi_0(t)} = 0 \text{ on } L.$$

Hence, by the uniqueness theorem,

$$\varphi_0(z) = c, \quad \psi_0(z) = \varkappa\bar{c},$$

where c is a constant. As a consequence of the single-valuedness of $\varphi_0(z)$, $\psi_0(z)$ which are simply constants, one obtains from (102.20) or (102.21) that

$$A_j^0 = 0, \quad j = 1, 2, \ldots, m, \tag{102.24}$$

where the A_j^0 are determined by (102.22) for $\omega(t) = \omega_0(t)$.

Further, it follows from (102.20) and (102.21) that

$$\frac{1}{2\pi i}\int_L \frac{\omega_0(t)dt}{t-z} = c,$$

$$-\frac{\varkappa}{2\pi i}\int_L \frac{\overline{\omega_0(t)}dt}{t-z} - \frac{1}{2\pi i}\int_L \frac{\bar{t}\omega_0'(t)dt}{t-z} = \varkappa\bar{c},$$

whence it is easily concluded (cf. the case of the first fundamental problem) that the functions $\varphi^*(t)$, $\psi^*(t)$ determined by

$$i\varphi^*(t) = \omega_0(t) - c, \quad -i\psi^*(t) = \varkappa\omega_0(t) + \bar{t}\omega_0'(t) + \varkappa\bar{c} \tag{102.25}$$

are the boundary values of some functions $\varphi^*(z)$, $\psi^*(z)$, holomorphic in the regions $S_1, S_2, \ldots, S_{m+1}$, while $\varphi^*(\infty) = \psi^*(\infty) = 0$.

Eliminating $\omega_0(t)$ from (102.25), one obtains

$$\varkappa\overline{\varphi^*(t)} - \bar{t}\varphi^{*\prime}(t) - \psi^*(t) = -2i\varkappa\bar{c} \text{ on } L_k,\ k = 1, 2, \ldots, m+1. \tag{102.26}$$

Applying the uniqueness theorem for the second fundamental problem to each of the regions S_k, one finds

$$\varphi^*(z) = c_k,\ \psi^*(z) = \varkappa\bar{c}_k + 2i\varkappa\bar{c} \text{ in } S_k,\ k = 1, 2, \ldots, m+1,$$

and, since one has in S_{m+1}: $\psi^*(\infty) = \varphi(^*\infty) = 0$, obviously $c_{m+1} = 0$, $c = 0$.

The functions $\varphi^*(z) = 0$, $\psi^*(z) = 2i\varkappa\bar{c}$ clearly solve the second fundamental problem for S_k with the boundary condition (102.26); by the uniqueness theorem,

the most general solution is obtained by adding to $\varphi^*(z)$ some constant c_k and to $\psi^*(z)$ the constant $\varkappa \bar{c}_k$ [cf. (34.13) and the remarks following it].

One thus has

$$\varphi^*(z) = c_k,\ \psi^*(z) = \varkappa \bar{c}_k \text{ in } S_k (k = 1, \ldots, m),$$
$$\varphi^*(z) = \psi^*(z) = 0 \text{ in } S_{m+1}.$$

But then, by (102.25),

$$\omega_0(t) = ic_k \text{ on } L_k, \quad k = 1, \ldots, m, \quad \omega_0(t) = 0 \text{ on } L_{m+1}.$$

It follows from this by (102.24) and (102.22) that all $c_k = 0$, and consequently $\omega_0(t) = 0$.

Thus the homogeneous equation, corresponding to (102.23), has no solutions different from zero, and therefore the equation (102.23) has always one and only one solution. Hence the problem is solved.

With obvious insignificant modifications, the above results likewise apply to the case when the contour L_{m+1} is absent and hence S is the infinite plane with holes.

§ 103. On the solution of the mixed fundamental problem and of certain other boundary problems by means of D. I. Sherman's method. The method of the preceding section may be successfully extended to the solution of certain other important boundary value problems.

In the first place *the mixed fundamental problem* will be mentioned. This problem was solved for regions of the same shape as in the preceding section by D. I. Sherman [17] who used in this case the same representation for the functions $\varphi(z)$, $\psi(z)$ as in the case of the second fundamental problem, i.e., (102.20), (102,21). However, this time that representation does not lead immediately to Fredholm equations, but to so-called singular equations of a rather simple form. These last equations, on their part, are easily reduced to Fredholm equations and the problem may be solved by a method, analogous to that of § 102. (Cf. D. I. Sherman[17]; in § 79, another method of solution for the case of simply connected regions, likewise due to D. I. Sherman, has been mentioned).

Using a method, analogous to the preceding one, D. I. Sherman [20] gave a new and simpler solution than that by S. G. Mikhlin [10, 8] of the first fundamental problem for bodies consisting in a definite manner of homogeneous parts having different elastic constants; this problem

(and likewise the corresponding second fundamental problem) has already been mentioned in §§ 96, 98.

Finally, D. I. Sherman [22] gave (by means of a method, analogous to the above) the general solution of the following problem. Let S be a region of the same shape as in § 102, and let it be required to find the elastic equilibrium of a (homogeneous) body, occupying S, if the normal component of displacement v_n and the tangential component of the external stresses T on the boundary L of S are given. For $T = 0$, this problem reduces to that of the frictionless contact of the body under consideration with rigid profiles at its boundary L.

In the following Part, this last problem will be solved for the case when the region S is simply connected and mapped on to the circle by means of rational functions, as has already been mentioned in § 88, 2°.

§ 104. Generalization to anisotropic bodies. The methods of the present Part may be successfully generalized to the case of homogeneous anisotropic bodies. As shown by S. G. Lekhnitzki, complex representation of the solution may also be given in this case, although it will, of course, be more complicated than for isotropic bodies. By means of such a representation and of a suitable generalization of the above methods a number of general as well as particular problems have been solved. The scope of this book does not permit a study of these questions. Therefore only reference will be made to the literature on this subject which is already fairly extensive.

The interesting papers by S. G. Lekhnitzki will not be quoted here in detail, since they are studied in his recently published book [1]. Among publications of a theoretical character, giving the general solution of several fundamental boundary value problems, the following should be mentioned: S. G. Mikhlin [11], G. N. Savin [3, 4], D. I. Sherman [9, 19], I. N. Vekua [2].

The solution of many particular, but practically important problems was given in the above-mentioned book by S. G. Lekhnitzki; this book does not only summarize the author's work, but also that of other investigators. For this reason no detailed reference will be made here to work, giving solutions of a particular character, as this can be found in that book and in G. N. Savin [5, 6].

§ 105. On other applications of the general representation of solutions. The methods of solution, studied in Parts IV and V (as well

as in Part VI), of the boundary problems of plane elasticity are based on the general representation of the solutions of the corresponding differential equations by means of functions of a complex variable. Such general representations of the solutions of partial differential equations by means of "arbitrary" functions acquired exaggerated importance at the outset of the development of mathematical physics, similar to that given to the integration of ordinary differential equations by means of quadrature. But it soon became clear that the determination of a "general solution" by no means exhausted a problem and that for the solution of the corresponding boundary problems such general solutions were often next to useless.

This fact caused the usual reaction in such cases and led to other extreme points of view which have been dominant until quite recent times, i.e., that no benefit whatsoever may be derived from "general solutions".

However, in actual fact, this is not so. The general solutions, if they can be found and if they are used efficiently, are often extraordinarily useful, particulary in practical problems. In a number of such cases they permit the construction of a complete theory of a given problem in a manner which is simpler and more thorough than would have been possible by other, hitherto known methods; the theory of plane elasticity may serve here as an example.

In contrast to this, the hitherto known, general solutions of the equations of the three-dimensional theory of elasticity do not permit the construction of a complete general theory; however, they have been found to be useful for the solution of a number of problems of a special character and have served as means for the solution of several general problems.

Therefore it has been found very desirable to extend methods, analogous to those studied above, to other sections of the theory of elasticity as well as to a wider range of problems. There exist already results in this direction which deserve more attention and further development.

Since there is no space to dwell at length on this range of problems, reference will be made to the work of I. N. Vekua, where the method of complex representation of solutions is generalized to a wider class of differential equations of the elliptic type, to which the equations of plane elasticity belong (in the static case); a full study of this work is given in I. N. Vekua's book [1] and therefore only those of his papers [4, 5] will be quoted here which refer directly to the theory of elasticity.

The method of general representation has also successfully been applied to several problems of elastic vibrations, but again no more can be said about this here.

Finally, only a reference will be made to the general solutions of the equations of the three-dimensional theory of elasticity, stated by Boussinesq, B. G. Galerkin, P. F. Papkovich and others (see also the earlier remarks referring to this problem). Some information on this question may be found in the text books by L. S. Leibenson [1], P. F. Papkovich [1] and A. E. H. Love [1].

PART VI

SOLUTION OF THE BOUNDARY PROBLEMS OF THE PLANE THEORY OF ELASTICITY BY REDUCTION TO THE PROBLEM OF LINEAR RELATIONSHIP

Many important problems of the theory of elasticity, including the problems considered in Chapters 15 and 16, may be solved very simply by reduction to a single boundary problem of complex function theory which the Author calls *the problem of linear relationship of the boundary values*, or, briefly, *the problem of linear relationship*. The formulation of this problem and its solution for several particular cases (which will be required later on) is given in Chapter 18.

This problem has been called by many authors the Riemann problem. It would have been more correct to call it the Hilbert problem, as has been done in the Author's book [25]. However, the Author proposes now to use the above term as an alternative.

CHAPTER 18

THE PROBLEM OF LINEAR RELATIONSHIP

§ 106. Sectionally holomorphic functions. As in § 65, let L be the union of a finite number of simple, non-intersecting arcs and contours in the complex z plane; these arcs and contours will always be assumed to be smooth. As in § 65, L will be called a simple smooth line and it will be assumed that it has a definite positive direction (i.e., on each arc or contour which is a component of L). The ends of the arcs (if such exist) will form part of L and will be called ends of the line L.

These closed arcs (i.e., including their end points) will often be denoted by ab or, if there are several of them, by $a_k b_k$, $k = 1, 2, \ldots$, where the symbols are to indicate that the positive direction is from a to b or from a_k to b_k.

As in § 65, a distinction will be made between "left" and "right" neighbourhoods of the points of L, other than its ends.

Denote by S' that part of the plane which contains all points not belonging to L; in other words, S' is the z plane cut along L. If L consists only of arcs, then S' is a connected region, while, if L includes contours, S' consists of several connected regions, bounded by these contours.

Let $F(z)$ be some function, given in S' (but not on L) and satisfying the following conditions:

1°. The function $F(z)$ is holomorphic everywhere in S'.

2°. It is continuous from the left and from the right at all points of L, other than the ends a_k, b_k.

3°. Near the ends a_k, b_k

$$|F(z)| < \frac{A}{|z - c|^{\mu}}, \quad 0 \leqslant \mu < 1, \tag{106.1}$$

where c is anyone of the ends a_k, b_k, A is a positive constant and μ is likewise a constant, subject to the stated condition.

Such a function $F(z)$ will be called *sectionally holomorphic in the entire plane* or, more simply, *sectionally holomorphic*. The line L will be called the *line of discontinuity* of $F(z)$ or the *boundary*.

As in § 65, denote by $F^+(t)$ and $F^-(t)$ the boundary values of the function $F(z)$ at the point t of L from the left and right respectively.

On occasion, functions will be considered which satisfy the above conditions everywhere, except at a finite number of points $z_1, z_2, \ldots$ not belonging to L, where $F(z)$ may have poles (without any other stipulations). In such cases the function $F(z)$ will be said to be sectionally holomorphic everywhere, except at the points $z_1, z_2, \ldots$. In particular, functions will often be considered which are sectionally holomorphic everywhere, except at the point at infinity, where they have a pole, i.e., functions which have for sufficiently large $|z|$ a series expansion of the form

$$F(z) = C_m z^m + C_{m-1} z^{m-1} + \ldots + C_0 + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \ldots \qquad (106.2)$$

Finally, the following condition will be introduced: when it is said that $F(z)$ vanishes at some point t_0 of L which is not an end, this will imply that $F^+(t_0) = F^-(t_0) = 0$; if t_0 is an end, then the vanishing of $F(z)$ at t_0 will imply that $F(z) \to 0$ as $z \to t_0$.

NOTE. The definition of the concept of sectionally holomorphic functions may, of course, be extended to the case, where the function is not given in the entire cut plane S', but only in some part of it. For example, let S_0 be some connected region bounded by one or several contours, the union of which will be denoted by L_0, and let the union L of contours and arcs, considered above, be entirely contained in S_0.

If the function $F(z)$, given in S_0 (except at points of L), satisfies the conditions 1°—3° and if, in addition, it takes definite boundary values on the boundary L_0 of S_0, then it may be called a function, sectionally holomorphic in S_0. Such a function may be extended into a function, sectionally holomorphic in the entire plane, by putting, for example, $F(z) = 0$ outside S_0.

In the sequel, unless stated otherwise, sectionally holomorphic functions will be assumed to be given in the entire plane (except on the line of discontinuity).

§ 107. The problem of linear relationship (the Hilbert problem).

Let L be a given smooth line which satisfies the conditions of the preceding section. The following problem will be considered:

To find the sectionally holomorphic function $F(z)$ with the line of dis-

continuity L the boundary values of which from the left and from the right satisfy the condition

$$F^+(t) = G(t)F^-(t) + f(t) \text{ on } L \tag{107.1}$$

(except at the ends), *where $G(t)$ and $f(t)$ are functions, given on L and $G(t) \neq 0$ everywhere on L.*

In addition, it will be assumed that the functions $G(t)$ and $f(t)$ satisfy the H condition.

Since the concept of boundary values from the left and from the right is not defined for the ends of the line L, the reservation has been introduced that (107.1) is to be satisfied on L, except at the ends. In the sequel, this stipulation will be omitted, although it will always be implied.

The above problem will be called *the problem of linear relationship of the boundary values* or simply *the problem of linear relationship* or *the Hilbert problem* [because the boundary values are connected (related) by a linear expression (with, in general, variable coefficients)].

If $f(t) = 0$ everywhere on L, the problem will be called homogeneous. The homogeneous problem was first considered by Hilbert for the case where L is a simple contour; the non-homogeneous problem (for the same case) was proposed by I. I. Privalov (under somewhat more general assumptions). However, a complete, but very simple solution has only been found recently. This solution and its literature is studied in the Author's book [25].

Only the particular and very simple case when $G(t)$ is a constant will be studied here, because it is the case required in the later sections. For the sake of clarity, the cases when $G(t) = 1$ and $G(t) = g$, where g is an arbitrary constant different from unity, will be considered separately.

§ 108. Determination of a sectionally holomorphic function for a given discontinuity. The simplest case of the problem of § 107 occurs when $G(t) = 1$. Then the problem reduces to the determination of a sectionally holomorphic function $F(z)$ for the given discontinuity $f(t)$ so that

$$F^+(t) - F^-(t) = f(t) \text{ on } L. \tag{108.1}$$

The solution of this problem may be written down immediately. In fact, consider the Cauchy integral

$$F_0(z) = \frac{1}{2\pi i}\int \frac{f(t)dt}{t - z}.$$

On the basis of the statements of § 68, $F_0(z)$ is a sectionally holomorphic function which vanishes at infinity and for which by (68.4)

$$F_0^+(t) - F_0^-(t) = f(t) \text{ on } L \text{ (except at the ends).} \tag{a}$$

By Note 4 of § 68 the function $F_0(z)$ satisfies near any end c of L the condition (106.1), viz.,

$$|F_0(z)| < \frac{A}{|z-c|^\mu},$$

even for arbitrarily small, positive μ.

Hence $F_0(z)$ is one of the solutions of the problem. Next consider the difference $F(z) - F_0(z) = F_*(z)$, where $F(z)$ is an unknown solution. By (108.1) and (a)

$$F_*^+(t) - F_*^-(t) = 0 \text{ on } L.$$

Thus, on the basis of a known property of functions of a complex variable (§ 37, 2°), the values of $F_*(z)$ on the left and on the right of L continue each other analytically. Therefore, if one prescribes for the function $F_*(z)$ suitable values on L, this function will be holomorphic in the entire plane, except possibly in the neighbourhoods of the ends a_k, b_k of L. However, since in the neighbourhood of any end c, by (106.1),

$$|F_*(z)| < \frac{A}{|z-c|^\mu}, \quad 0 \leqslant \mu < 1, \tag{b}$$

the point c is a removable singularity and it may be assumed that $F_*(z)$ is holomorphic in the entire plane.

By (b) the product $(z-c)F_*(z)$ remains bounded near c; hence this product function has a removable singularity at that point (cf. for example, I. I. Privalov [1]). Therefore $(z-c)F_*(z)$ may be assumed to be holomorphic near c, i.e., $(z-c)F_*(z) = F_{**}(z)$, where $F_{**}(z)$ is a holomorphic function. Thus $F_*(z)$ can only have a first order pole at c; but by (b) there is no such pole, because $(z-c)F_*(z) \to 0$ as $z \to c$.

Consequently, by the Liouville theorem, $F_*(z) = C = \text{const.}$ in the entire plane and the general solution of the problem is given by

$$F(z) = F_0(z) + C$$

or

$$F(z) = \frac{1}{2\pi i}\int \frac{f(t)dt}{t-z} + C, \tag{108.2}$$

where C is an arbitrary constant. If it is required that $F(\infty) = 0$, then one has to take $C = 0$.

The solution of a somewhat more general problem will now be found. In fact, it will be assumed that the unknown function $F(z)$ is sectionally holomorphic everywhere, except at the point at infinity where it may have a pole of order not greater than m, i.e., it must have there the form (106.2). It is then easily seen (by application of the generalized Liouville theorem) that

$$F(z) = \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} + P_m(z), \tag{108.3}$$

where $P_m(z)$ is a polynomial of degree not higher than m, i.e.,

$$P_m(z) = C_m z^m + C_{m-1}z^{m-1} + \ldots + C_0; \tag{108.4}$$

$C_0, C_1, \ldots, C_m$ are here arbitrary constants.

The generalized Liouville theorem consists of the following: If a function $F(z)$ is holomorphic in the whole plane, except at the point $z = \infty$, and if for large $|z|$

$$F(z) = O(z^m),$$

where m is a positive integer, then $F(z)$ is a polynomial of degree not higher than m.

Finally, if the solution is permitted to have poles of order not greater than $m_1, m_2, \ldots, m_l, m$ at the given points $z_1, z_2, \ldots, z_l, \infty$, then

$$F(z) = \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} + R(z), \tag{108.5}$$

where $R(z)$ is an arbitrary rational function with poles of the type indicated, i.e.,

$$R(z) = \sum_{j=1}^{l} \frac{C_{j1}}{z-z_j} + \frac{C_{j2}}{(z-z_j)^2} + \ldots + \frac{C_{jm_j}}{(z-z_j)^{m_j}} + \\ + C_0 + C_1 z + \ldots + C_m z^m, \tag{108.6}$$

where the C_{jk}, C_k are arbitrary constants.

NOTE. It follows from the above statements that every sectionally holomorphic function $F(z)$ may be represented in the form of a Cauchy integral

$$F(z) = \frac{1}{2\pi i}\int \frac{f(t)dt}{t-z} + C,$$

where $f(t)$ denotes the discontinuity of $F(z)$ on the line L, i.e.,

$$f(t) = F^+(t) - F^-(t),$$

and C is a constant.

Further, if $F(z)$ is a sectionally holomorphic function in some region S_0 which does not coincide with the whole plane (as in the Note at the end of § 106), then this function may always be represented in the form of the sum of a function holomorphic in S_0 and a Cauchy integral, i.e.,

$$F(z) = \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} + F^*(z), \tag{108.7}$$

where L is the line of the discontinuity $f(t) = F^+(t) - F^-(t)$ inside S_0 and $F^*(z)$ is a function, holomorphic in S_0. The expression (108.7) holds true everywhere in S_0, except at points of L where $F(z)$ is, in general, not defined.

The truth of (108.7) follows from the fact that the difference

$$F(z) - \frac{1}{2\pi i}\int_L \frac{f(t)dt}{t-z} = F^*(z)$$

is a function, holomorphic in S_0 except at points of L, where obviously

$$F^{*+}(t) - F^{*-}(t) = 0 \text{ on } L,$$

and hence $F^*(z)$ is holomorphic everywhere in S_0, provided it is given suitable values on L.

The function $F^*(z)$ may likewise be represented by a Cauchy integral taken over the boundary L_0 of the region S_0.

§ 109. Application. One interesting application of the formula (108.7) will be stated here which is due to D. I. Sherman [14]. Let there be given an elastic body, such as a plate with several holes, and let solid discs *of the same material be* inserted into these holes; however, let the contours of these discs differ slightly, in the unstressed state, from those of the holes. It will be supposed that the boundaries of the inserted discs and of the corresponding holes are brought into contact without any gaps and that they are welded together (or restrained by frictional forces from sliding relatively to each other).

Denote the body, obtained in this manner, by S_0 and its boundary by L_0. It will be assumed that L_0 is a simple contour.

The results of this section will also remain true (with obvious insignificant modifications), if it is assumed that L_0 consists of several contours; this corresponds to the case, where not all holes are filled by discs and where some of the inserted discs have holes.

Further, denote by L the union of the contours of the holes into which discs have been inserted.

As before, let $\varphi(z)$ and $\psi(z)$ be the functions which determine the elastic equilibrium of the body S_0. These functions are defined in each of the regions into which the region S_0 is divided by L and they are holomorphic there; however, they suffer discontinuous changes for a passage through L.

This is obvious for regions, occupied by discs, because these functions are assumed to be single-valued. However, single-valuedness of $\varphi(z)$, $\psi(z)$ in the regions, occupied by the material surrounding the discs, follows from the fact that the resultant vector (and also the resultant moment) of the forces, applied to the edges of the original holes by the discs, obviously vanish.

It will be assumed that the external forces X_n, Y_n, acting on the boundary of the body S_0, are given and that, in addition, the discontinuities in the displacements for a passage through L are determined by

$$u^+ - u^- = g_1(t), \quad v^+ - v^- = g_2(t) \text{ on } L, \tag{109.1}$$

where the functions $g_1(t)$ and $g_2(t)$ are likewise given; they depend on the shapes of the holes and inserted discs before deformation and on the method by which the edges of the discs and of the surrounding plate were brought into contact before welding occurred.

Under these conditions one has the following boundary conditions:

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) \text{ on } L_0, \tag{109.2}$$

$$\varphi^+(t) + t\overline{\varphi'^+(t)} + \overline{\psi^+(t)} = \varphi^-(t) + t\overline{\varphi'^-(t)} + \overline{\psi^-(t)} \text{ on } L, \tag{109.3}$$

$$\varkappa\varphi^+(t) - t\overline{\varphi'^+(t)} - \overline{\psi^+(t)} = \varkappa\varphi^-(t) - t\overline{\varphi'^-(t)} - \overline{\psi^-(t)} + 2\mu g(t) \text{ on } L, \tag{109.4}$$

where

$$f(t) = i\int_0^s (X_n + iY_n)ds \text{ on } L_0, \; g(t) = g_1(t) + ig_2(t) \text{ on } L \tag{109.5}$$

are known functions.

The condition (109.2) expresses that the external stresses acting

on the boundary L_0 of the body S_0, are given. The condition (109.3) indicates that the stresses, acting from either side on the element of the line of division, balance each other. Finally, (109.4) shows that the discontinuities in the displacements of the line of division are known.

As a matter of fact, (109.2) must only be satisfied exactly, apart from an arbitrary constant; similarly, (109.3) must be fulfilled, apart from arbitrary constants on each of the contours, constituting L. However, it is easily seen that the last constants may be included with the unknown functions.

Adding (109.3) and (109.4), one finds

$$\varphi^+(t) - \varphi^-(t) = \frac{2\mu g(t)}{\varkappa + 1} \text{ on } L. \tag{109.6}$$

Further, taking the conjugate complex form of (109.3) and using (109.6), one obtains

$$\psi^+(t) - \psi^-(t) = \frac{2\mu h(t)}{\varkappa + 1} \text{ on } L, \tag{109.7}$$

where

$$h(t) = -\overline{g(t)} - \bar{t}g'(t), \quad g'(t) = \frac{dg(t)}{dt}, \tag{109.8}$$

i.e., $h(t)$ is a function, known on L.

On the basis of the statements in the Note at the end of § 108, one deduces that

$$\begin{aligned} \varphi(z) &= \varphi_0(z) + \frac{\mu}{\pi i(\varkappa + 1)} \int_L \frac{g(t)dt}{t - z}, \\ \psi(z) &= \psi_0(z) + \frac{\mu}{\pi i(\varkappa + 1)} \int_L \frac{h(t)dt}{t - z} \end{aligned} \tag{109.9}$$

where $\varphi_0(z)$, $\psi_0(z)$ are functions, holomorphic in S_0.

For the sake of brevity, let

$$\varphi_*(z) = \frac{\mu}{\pi i(\varkappa + 1)} \int_L \frac{g(t)dt}{t - z}, \quad \psi_*(z) = \frac{\mu}{\pi i(\varkappa + 1)} \int_L \frac{h(t)dt}{t - z}, \tag{109.10}$$

and (109.9) becomes

$$\varphi(z) = \varphi_0(z) + \varphi_*(z), \quad \psi(z) = \psi_0(z) + \psi_*(z); \tag{109.11}$$

the holomorphic functions $\varphi_0(z)$, $\psi_0(z)$ are subject to definition, while $\varphi_*(z)$, $\psi_*(z)$ are known, sectionally holomorphic functions, determined by (109.10).

Substituting (109.11) in the boundary condition (109.2), one obtains

$$\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = f_0(t) \text{ on } L_0, \tag{109.12}$$

where

$$f_0(t) = f(t) - \varphi_*(t) - t\overline{\varphi_*'(t)} - \overline{\psi_*(t)} \tag{109.13}$$

is a function, known on L_0.

One has thus arrived at the usual first fundamental problem for the body S_0. After having determined $\varphi_0(z)$, $\psi_0(z)$, the functions $\varphi(z)$, $\psi(z)$ may be found from (109.9) or (109.11).

Consequently, it is seen that *the problem under consideration reduces directly to the customary first fundamental problem for the same region* S_0.

If, instead of the stresses, displacements are given on L_0, the problem will reduce in the same manner to the customary second fundamental problem.

If the discs and the surrounding body have different elastic properties, then the above is no longer true; more will be said about the solution of this case later on.

§ 109a. Example. Consider the simplest case of a circular ring with outer radius 1 and inner radius r into which a circular disc of radius $r + \varepsilon$ has been inserted, where ε is a known quantity. Then S_0 is the unit circle, L_0 the circumference of this circle and L a circle with radius $r < 1$.

If it is assumed that the origin lies at the centre and that the positive direction on L (as well as on L_0) is counter-clockwise, it is easily seen that, with $t = \rho e^{i\vartheta}$, one has

$$g(t) = -\varepsilon(\cos\vartheta + i\sin\vartheta) = -\varepsilon e^{i\vartheta} = -\frac{\varepsilon t}{r} \text{ on } L$$

and

$$h(t) = -\overline{g(t)} - \bar{t}g'(t) = \frac{2\varepsilon\bar{t}}{r} = \frac{2\varepsilon r}{t} \text{ on } L.$$

Hence (109.10) gives

$$\varphi_*(z) = \begin{cases} \dfrac{2\mu\varepsilon}{r(\varkappa+1)}\, z & \text{for } |z| < r \\ 0 & \text{for } |z| > r \end{cases} \qquad \psi_*(z) = \begin{cases} 0 & \text{for } |z| < r \\ -\dfrac{4\mu\varepsilon r}{\varkappa+1}\dfrac{1}{z} & \text{for } |z| > r \end{cases},$$

and the functions $\varphi_0(z)$, $\psi_0(z)$ are determined by the boundary condition

$$\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = f_0(t) \text{ on } L_0,$$

where

$$f_0(t) = f(t) + \frac{4\mu\varepsilon r}{\varkappa + 1} t.$$

The last expression follows from (109.13), since $\varphi_*(t) = \varphi_*'(t) = 0$ on L_0, because $\varphi_*(z) = 0$ for $|z| > r$ and

$$\psi_*(t) = -\frac{4\mu\varepsilon r}{\varkappa + 1} \cdot \frac{1}{\bar{t}} = -\frac{4\mu\varepsilon r}{\varkappa + 1} t \quad \text{on} \quad L_0.$$

Thus, in order to solve the original problem, one has to find the solution of the customary first fundamental problem for the circle, after adding to the stresses, actually acting on L_0 and characterized by $f(t)$, the fictitious stresses corresponding to the second term in the expression for $f_0(t)$, obtained above .These fictitious stresses are easily seen to correspond to a distribution of uniform normal tension of magnitude

$$\frac{4\mu r}{\varkappa + 1} \varepsilon.$$

Thus the solution of the problem may be written down directly, using the formulae of § 80.

§ 110. Solution of the problem: $F^+ = gF^- + f$.

Consider the case $G(t) = g$, where $g \neq 1$ is a given, in general complex constant. The boundary condition in this case will have the form

$$F^+(t) - gF^-(t) = f(t) \text{ on } L, \text{ except at the ends.} \qquad (110.1)$$

It will now be assumed that L consists of n simple smooth arcs L_k $(k = 1, 2, \ldots, n)$ which have no points in common; these arcs, as indicated earlier, will be denoted by a_kb_k, where a_k, b_k are the ends of L_k and the positive direction is from a_k to b_k. (Fig. 48).

The case, where L consists only of contours, is easily seen to reduce directly to the problem of § 109. For example, if L consists of one simple contour which divides the plane into two regions S^+ and S^-, adjoining L on the left and right respectively, one may consider, instead of $F(z)$, the function $F_*(z)$, defined as follows: $F_*(z) = F(z)$ in S^+, $F_*(z) = gF(z)$ in S^-; then (110.1) takes the form $F_*^+(t) - F_*^-(t) = f(t)$. One may proceed in an analogous manner, when L consists of several contours.

In the case where L consists of contours and arcs, the problem is likewise easily solved.

First, a solution will be studied which may have a pole of arbitrary order at infinity, and a beginning will be made with the homogeneous problem

$$F^{+}(t) - gF^{-}(t) = 0. \tag{110.1'}$$

A particular solution $X_0(z)$ of this problem will be sought in the form

$$X_0(z) = \prod_{j=1}^{n} (z - a_j)^{-\gamma}(z - b_j)^{\gamma-1}, \tag{110.2}$$

where $\gamma = \alpha + i\beta$ is a constant.

The function $X_0(z)$ is holomorphic in S', i.e., in the plane cut along L, if a definite branch of this function is selected, e.g. the branch for which

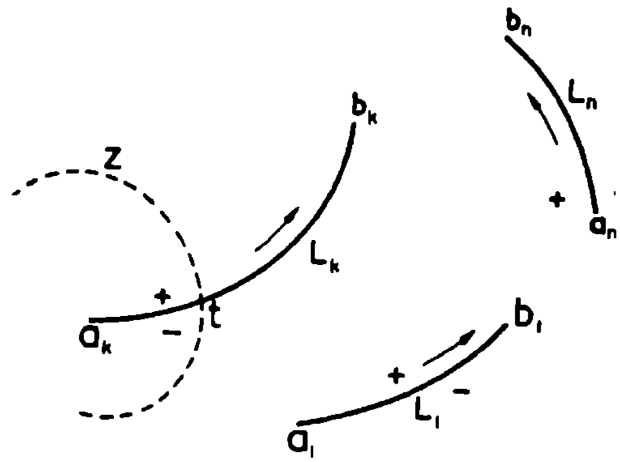


Fig. 48.

$\lim_{z\to\infty} [z^n X_0(z)] = 1$; or, in other words, the branch which has for large $|z|$ the form

$$X_0(z) = \frac{1}{z^n} + \frac{\alpha_{-n+1}}{z^{n-1}} + \ldots; \tag{110.3}$$

in the sequel, unless stated otherwise, this branch will always be implied.

It is readily verified by an investigation of the variation in the argument of $z - a_k$ or $z - b_k$, when z describes a closed path beginning at a point t of the arc $a_k b_k$ and leading, without intersecting L, from the left side of $a_k b_k$ around the end a_k to the right side of the arc (as in Fig. 48) or around the end b_k (not shown in Fig. 48), that

$$X_0^{-}(t) = e^{-2\pi i\gamma} X_0^{+}(t), \quad \text{i.e.,} \quad X_0^{+}(t) = e^{2\pi i\gamma} X_0^{-}(t). \tag{110.4}$$

By definition, the power with complex exponent

$$(z - a_k)^{-\gamma} = e^{-\gamma \log(z-a_k)} = e^{-\gamma[\log|z-a_k| + i\vartheta]} = e^{-\gamma \log|z-a_k|}\, e^{-i\gamma\vartheta},$$

where $\log|z - a_k|$ is real, so that

$$(z - a_k)^{-\gamma} = |z - a_k|^{-\gamma} e^{-i\gamma\vartheta},$$

where $\vartheta = \arg(z - a_k)$ and $|z - a_k|^{-\gamma}$ is the uniquely defined value $e^{-\gamma \log|z-a_k|}$.

When z goes from the left side of L_k around a_k to the right side, as shown in Fig. 48, then ϑ increases by $(+2\pi)$, and hence $(-i\gamma\vartheta)$ by $(-2\pi i\gamma)$, and therefore $(z - a_k)^{-\gamma}$ must be multiplied by $e^{-2\pi i\gamma}$.

When z goes from the left side of L_k around b_k to the right side, then $\vartheta = \arg(z - b_k)$ increases by (-2π) and

$$(z - b_k)^{\gamma-1} = |z - b_k|^{\gamma-1} \quad e^{i(\gamma-1)\vartheta}$$

must be multiplied by $e^{-2\pi i(\gamma-1)} = e^{-2\pi i\gamma}$, as in the first case, since $e^{2\pi i} = 1$.

Hence $X_0(z)$ will satisfy the boundary condition (110.1'), provided $e^{2\pi i\gamma} = g$, i.e.,

$$\gamma = \alpha + i\beta \quad \frac{\log g}{2\pi i} \quad \frac{\log|g|}{2\pi i} + \frac{\theta}{2\pi}, \tag{110.5}$$

where θ denotes the argument of the constant g. This argument is determined, apart from an additive term $2k\pi i$, where k is an integer; however, θ will always be chosen in such a manner that

$$0 \leqslant \theta < 2\pi, \tag{110.6}$$

by which condition θ is completely determined. In particular, if g is a real, positive number, then $\theta = 0$, while, if g is a real, negative number, $\theta = \pi$.

It will now be investigated as to whether the inequality

$$X_0(z) < \frac{A}{|z - c|} \quad 0 \quad 1 \tag{110.7}$$

is fulfilled, where c is any end a_k, b_k; this condition must, by supposition, be satisfied by any sectionally holomorphic function. By (110.6),

$$0 \leqslant \alpha < 1. \tag{110.8}$$

If g is not a real positive number, then $\alpha \neq 0$, $1 - \alpha < 1$. Hence, excluding the case when g is real and positive, the inequality (110.7) will be fulfilled by taking $\mu = \alpha$ for $c = a_k$, $\mu = 1 - \alpha$ for $c = b_k$.

One has

$$(z - a_k)^{-\gamma} = (z - a_k)^{-(\alpha+i\beta)} = e^{-(\alpha+i\beta)\log(z-a_k)} = e^{-(\alpha+i\beta)(\log r + i\vartheta)},$$

where $r = |z - a_k|$, $\vartheta = \arg(z - a_k)$. Hence

$$(z - a_k)^{-\gamma} = e^{-\alpha \log r} . \Theta = \frac{\Theta}{r^{\alpha}} = \frac{\Theta}{|z - a_k|^{\alpha}},$$

where $\Theta = e^{\beta\vartheta - i(\alpha\vartheta - \beta \log r)}$, so that $|\Theta| = {}^{\beta\vartheta}$. When z is in the neighbourhood of the point a_k in the plane, cut along L, then ϑ lies between finite limits (because z cannot cross L) and therefore $|\Theta|$ is bounded and, in addition, $|\Theta| > a$, where a is a positive constant. Similar reasoning applies to the neighbourhood of the point b_k.

Thus, a particular solution $X_0(z)$ of the homogeneous problem has been found (for $\alpha \neq 0$); it is given by (110.2) with γ determined by (110.5). This particular solution does not vanish anywhere in the finite part of the plane and it is unbounded like $|z - a_k|^{-\alpha}$ and $|z - b_k|^{\alpha - 1}$ near the ends a_k and b_k respectively.

The most general solution of the homogeneous problem will now be found which has a pole at infinity. For this purpose it will be noted that $X_0(z)$, being a solution of the homogeneous problem, satisfies the condition

$$X_0^+(t) = g X_0^-(t) \text{ on } L, \tag{110.9}$$

whence

$$g = \frac{X_0^+(t)}{X_0^-(t)} \text{ on } L. \tag{110.9'}$$

Replacing in (110.1′) g by its value (110.9′), one obtains

$$\frac{F^+(t)}{X_0^+(t)} - \frac{F^-(t)}{X_0^-(t)} = 0 \text{ on } L,$$

or

$$F_*^+(t) - F_*^-(t) = 0 \text{ on } L,$$

where $F_*(z)$ denotes the sectionally holomorphic function $F(z)/X_0(z)$. It follows from the preceding relation that $F_*(z)$ is holomorphic in the entire plane, except at the point $z = \infty$, provided it is given suitable values on L (cf. § 109). Further, since $F_*(z)$ can only have a pole at infinity, it must, by the generalized Liouville theorem, be a polynomial.

Thus, the most general solution of the homogeneous problem is given by

$$F(z) = X_0(z) P(z), \tag{110.10}$$

where $P(z)$ is an arbitrary polynomial.

If it is desired to obtain a solution which is also holomorphic at

infinity, it must be assumed that the degree of the polynomial $P(z)$ does not exceed n; this follows from the behaviour of $X_0(z)$ at infinity, as determined by (110.3). If one requires that $F(\infty) = 0$, then the degree of $P(z)$ may not be higher than $n - 1$.

In general, the solution (110.10) will not be bounded near the ends. However, if it is desired to find the solution which is bounded near the given ends $c_1, c_2, \ldots, c_p$, the polynomial $P(z)$ must be chosen in such a manner that it vanishes at these points, i.e., $P(z) = (z - c_1)(z - c_2) \ldots (z - z_p)Q(z)$, where $Q(z)$ is a polynomial. In that case the solution $F(z)$ will not only be bounded near the ends c_k, but it will vanish there. (It is seen that a solution which is bounded near certain ends, but does not vanish there, does not exist, assuming, of course, all the time that $\alpha \neq 0$.) Writing

$$X_p(z) = X_0(z)(z - c_1)(z - c_2) \ldots (z - c_p), \tag{110.11}$$

all solutions bounded near the ends $c_1, c_2, \ldots, c_p$ may be represented in the form

$$F(z) = X_p(z) . Q(z), \tag{110.12}$$

where $Q(z)$ is an arbitrary polynomial.

Naturally, $X_p(z)$ is itself a particular solution of the homogeneous problem, similar to $X_0(z)$. However, it is bounded near the given ends and it vanishes at these ends in such a way that

$$X_p(z) = | z - c_j |^{\mu} . \Theta, \quad 0 < \mu < 1, \tag{110.13}$$

Θ being a bounded quantity; in fact, $| \Theta | > a = \text{const} > 0$. [cf. remarks following (110.8)].

Among the solutions $X_p(z)$ the two following will be specially noted:

$$X(z) = X_0(z) \prod_{j=1} (z - a_j)(z - b_j) = \prod_{j=1} (z - a_j)^{1-\gamma} (z - b_j)^{\gamma}, \tag{110.14}$$

which is *bounded near all ends* (where it actually vanishes), and

$$X_*(z) = X_0(z) \prod_{j=1} (z - b_j) = \prod_{j=1} \; z - b_j \tag{110.15}$$

which is *bounded near the ends* b_j, $j = 1, 2, \ldots, n$ (where it vanishes). For large $| z |$ these solutions have the forms

$$X(z) = z^n + \beta_{n-1} z^{n-1} + \ldots + \beta_0 + \frac{\beta_{-1}}{z} + \ldots, \tag{110.16}$$

$$X_*(z) \quad 1 + \frac{\Upsilon_{-1}}{\cdot} + \frac{\Upsilon_{-2}}{\circ} + \qquad (110.17)$$

Next consider the non-homogeneous problem. Using (110.9′), the boundary condition (110.1) may be written

$$\frac{F^+(t)}{X_0^+(t)} - \frac{F^-(t)}{X_0^-(t)} = \frac{f(t)}{X_0^+(t)}$$

or

$$F_*^+(t) - F_*^-(t) = f_*(t),$$

where $F_*(z) = F(z)/X_0(z)$, $f_*(t) = f(t)/X_0^+(t)$.

Using the results of § 109, one finds

$$F(z) = \frac{X_0(z)}{2\pi i} \int \frac{f(t)dt}{X_0^+(t-z)} + X_0(z)P(z), \qquad (110.18)$$

where $P(z)$ is an arbitrary polynomial. This is the general solution of the problem, admitting a pole at infinity.

If it is desired to obtain the solution, holomorphic at $z = \infty$, it must be assumed, in view of (110.3), that $P(z)$ is a polynomial of degree not higher than n:

$$P(z) = C_0 z^n + C_1 z^{n-1} + \ldots + C_{n-1} z + C_n, \qquad (110.19)$$

where $C_0, C_1, \ldots, C_n$ are arbitrary constants. If, in addition, it is required that $F(\infty) = 0$, one has to assume $C_0 = 0$.

In general, the solution $F(z)$ will not be bounded at the ends a_k, b_k. However, by a suitable choice of the polynomial $P(z)$, it may be arranged that it is bounded at certain ends $c_1, c_2, \ldots, c_p$. It is simple to prove this directly by constructing the general solution, having that property. For this it will be sufficient to repeat the reasoning leading to (110.18), but using, instead of the particular solution $X_0(z)$, the solution $X_p(z)$ as determined by (110.11).

Also in that case one is led to the condition $F_*^+(t) - F_*^-(t) = f_*(t)$, where this time

$$F_*(z) = \frac{F(z)}{X_p(z)}, \quad f_*(t) = \frac{f(t)}{X_p{}^+(t)}$$

however, $X_p(z)$ vanishes now at the ends $c_1, \ldots, c_p$; but since, by supposition, the unknown function is bounded near these ends, $|F_*(z)| < A/|z - c_j|^\mu$, $0 \le \mu < 1$, and the former reasoning applies, provided one does not consider the fact that,

in general, $f_*(t)$ is not bounded near the ends c_j. It is readily shown by a study of the behaviour of $f_*(t)$ near these ends [cf. N. I. Muskhelishvili [25]) that this circumstance is of no importance. As a matter of fact, it has been shown earlier that, if a solution of the required type exists, it is given by (110.20) and that the first term on the right-hand side of (110.20) actually remains bounded near the ends $c_1, c_2, \ldots, c_p$.

Thus the general solution, bounded near the ends $c_1, c_2, \ldots, c_p$, is given by

$$F(z) = \frac{X_p(z)}{2\pi i}\int_L \frac{f(t)dt}{X_p^+(z)(t-z)} + X_p(z)P(z), \tag{110.20}$$

where $P(z)$ is an arbitrary polynomial.

It follows from (110.11) and (110.3) that for large $|z|$

$$X_p(z) = z^{p-n} + \delta_{p-n-1}z^{p-n-1} + \ldots; \tag{110.21}$$

therefore $X_p(z)$ is holomorphic at infinity only for $p \leqslant n$.

If $p \leqslant n+1$, the first term on the right-hand side of (110.20) remains bounded as $z \to \infty$ and, in order to obtain a solution which is also holomorphic for $z = \infty$, one must assume that $P(z)$ is a polynomial of degree not higher than $n-p$; for $p = n+1$, one has to assume $P(z) = 0$.

However, if $p > n+1$, a solution which is holomorphic at infinity will only exist, provided certain conditions, to be stated now, are satisfied. Since for large $|z|$

$$\frac{1}{t-z} = -\frac{1}{z} - \frac{t}{z^2} - \frac{t^2}{z^3} - \ldots,$$

one has the expansion, likewise valid for large $|z|$,

$$\frac{1}{2\pi i}\int_L \frac{f(t)dt}{X_p^+(t)(t-z)} = -\frac{A_1}{z} - \frac{A_2}{z^2} - \ldots, \tag{110.22}$$

where

$$A_k = \frac{1}{2\pi i}\int_L \frac{t^{k-1}f(t)dt}{X_p^+(t)}, \quad k = 1, 2, \ldots. \tag{110.23}$$

Hence, if it is required that the solution (110.20) should be holomorphic at infinity for $p > n+1$, then one has to put $P(z) = 0$ and, in addition, $f(t)$ must be subject to the conditions $A_k = 0$, $k = 1, 2, \ldots,$

$p - n - 1$, i.e.,

$$\frac{1}{2\pi i}\int_L \frac{t^{k-1}f(t)dt}{X_p^+(t)} = 0, \quad k = 1, 2, \ldots, p - n - 1. \qquad (110.24)$$

Thus (110.24) must be satisfied, in order that for $p > n + 1$ a solution, holomorphic at infinity and bounded at the ends $c_1, c_2, \ldots, c_p$, may exist.

Further, if it be required that $F(\infty) = 0$, then in the preceding formula $k = 1, 2, \ldots, p - n$.

Hitherto the case where $\alpha = 0$ has been excluded, i.e., the case where g is a real, positive number. If $\alpha = 0$, one may use the particular solution $X_*(z)$, determined by (110.15). It is readily seen that this solution remains bounded near all ends and does not vanish anywhere when $\gamma = i\beta$. Applying the same reasoning as before, one obtains the general solution of the problem

$$F(z) = \frac{X_*(z)}{2\pi i}\int_L \frac{f(t)dt}{X_*^+(t)(t-z)} + X_*(z)P(z), \qquad (110.25)$$

where $P(z)$ is an arbitrary polynomial; in the present case the solution is seen to be necessarily bounded near all ends. If it is desired to find a solution, holomorphic for $z = \infty$, one has to put $P(z) = C = \text{const.}$, as may be seen from (110.17); if, in addition, $F(\infty) = 0$ is required, then $P(z) = 0$.

The formula (110.25) is, of course, also applicable for $\alpha \neq 0$, when it becomes a particular case of (110.20) and gives all solutions, bounded at the ends $b_1, b_2, \ldots, b_n$.

Finally, the general solution of the problem will be found under the supposition that it may have poles of order not greater than $m_1, m_2, \ldots, m_l, m$ at the given points $z_1, z_2, \ldots, z_l, \infty$. Reasoning as before and taking into consideration (108.5), one obtains for the general solution of the required form, applicable for $\alpha \neq 0$,

$$F(z) = \frac{X_0(z)}{2\pi i}\int_L \frac{f(t)dt}{X_0^+(t)(t-z)} + X_0(z)R(z), \qquad (110.26)$$

where $R(z)$ is a rational function of the form [cf. (109.6)]

$$R(z) = \sum_{j=1}^{l}\left\{\frac{C_{j1}}{z-z_j} + \frac{C_{j2}}{(z-z_j)^2} + \ldots + \frac{C_{jm_j}}{(z-z_j)^{m_j}}\right\} + P(z), \quad (110.27)$$

$P(z)$ being an arbitrary polynomial of degree not higher than $m + n$.

The polynomial $P(z)$ must be such that the pole of $P(z)X_0(z)$ at infinity is not of greater order than m; however, it is known that for large $|z|$: $X_0(z) = O(1/z^n.)$

When $\alpha = 0$, an analogous formula results which is obtained by substituting $X_*(z)$ for $X_0(z)$; in that case the degree of the polynomial $P(z)$ must not exceed m, because for large $|z|$

$$X_*(z) = 1 + \frac{\gamma_{-1}}{z} + \dots .$$

In conclusion, note the particular case of the problem under consideration when $g = -1$; the boundary condition takes then the form

$$F^+(t) + F^-(t) = f(t), \tag{110.28}$$

and one has

$$\gamma = \alpha + i\beta = \frac{\log(-1)}{2\pi i} = \frac{1}{2}. \tag{110.29}$$

Hence, by (110.2),

$$X_0(z) = \prod_{j=1}^{n} (z - a_j)^{-\frac{1}{2}} (z - b_j)^{-\frac{1}{2}} = \\ = \frac{1}{\sqrt{(z-a_1)(z-b_1)\dots(z-a_n)(z-b_n)}} \tag{110.30}$$

and, by (110.14) and (110.15),

$$X(z) = \sqrt{(z-a_1)(z-b_1)\dots(z-a_n)(z-b_n)}, \tag{110.31}$$

$$X_*(z) = \frac{\sqrt{(z-b_1)\dots(z-b_n)}}{\sqrt{(z-a_1)\dots(z-a_n)}}. \tag{110.32}$$

The most general solution of the problem, admitting a pole at infinity, is given by (110.18) with $X_0(z)$ replaced by the value (110.30); using the fact that now $X_0(z) = 1/X(z)$, one has for the general solution

$$F(z) = \frac{1}{2\pi i\, X(z)} \int_L \frac{X^+(t)f(t)dt}{t-z} + \frac{P(z)}{X(z)}, \tag{110.33}$$

where $X(z)$ is determined by (110.31) and $P(z)$ is an arbitrary polynomial. Assuming the degree of the polynomial not to exceed n, one obtains the solution which remains holomorphic at infinity.

The solution, bounded near all ends, is obtained from a formula which follows from the general formula (110.20) for $p = 2n$ and $X_p(z)$ deter-

mined by (110.31):

$$F(z) = \frac{X(z)}{2\pi i}\int_L \frac{f(t)dt}{X^+(t)(t-z)} + X(z)P(z). \tag{110.34}$$

If the solution is also to be holomorphic at infinity, one must assume $P(z) = 0$ and, in addition, subject the function $f(t)$ to the following conditions, which follow from (110.24):

$$\frac{1}{2\pi i}\int \frac{t^{k-1}f(t)dt}{X^+(t)} \quad 0, \quad k = 1, 2, \ldots, n \quad 1. \tag{110.35}$$

If also $F(\infty) = 0$, then $k = 1, 2, \ldots, n$ in (110.35).

NOTE. 1. In many cases the integrals, occurring in the formulae of the present section, may be easily evaluated in finite form. For example, this is true in the case, which often occurs in practice, when $f(t)$ is a polynomial

$$f(t) = A_m t^m + A_{m-1}t^{m-1} + \ldots + A_0. \tag{110.36}$$

In fact, consider the integral

$$I(z) = \int_L \frac{f(t)dt}{X_p^+(t)(t-z)} \tag{110.37}$$

which occurs in (110.20); in particular, for $p = 0$, one obtains the integral of (110.18).

Simultaneously with the integral I consider another integral

$$\Omega(z) = \frac{1}{2\pi i}\int_\Lambda \frac{f(\zeta)d\zeta}{X_p(\zeta)(\zeta - z)}. \tag{110.38}$$

where Λ is the union of the n contours $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$, surrounding the arcs $L_1, L_2, \ldots, L_n$ in clockwise direction, as shown in Fig. 49, and assume that in this case z remains outside these contours.

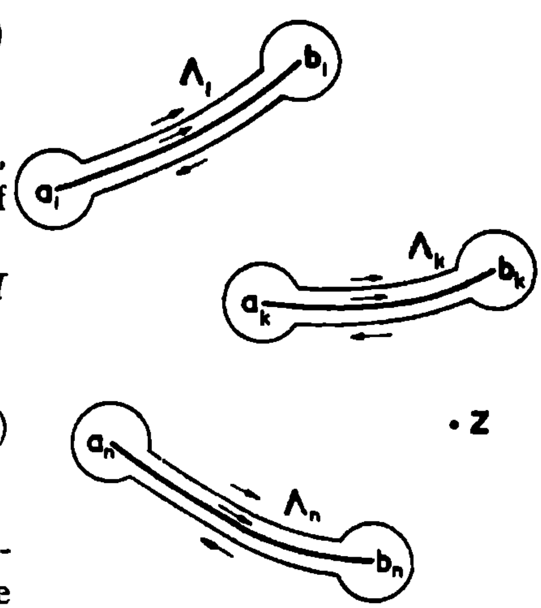


Fig. 49.

Using (110.21), it is concluded that for large $|\zeta|$

$$\frac{f(\zeta)}{X_p(\zeta)} = \alpha_q\zeta^q + \alpha_{q-1}\zeta^{q-1} + \ldots + \alpha_0 + \frac{\alpha_{-1}}{\zeta} + \frac{\alpha_{-2}}{\zeta^2} + \ldots, \quad (110.39)$$

where $q = n - p + m$ and the coefficients $\alpha_q, \alpha_{q-1}, \ldots, \alpha_0$ (the others not being required) are easily determined by elementary means.

In fact, $1/X_p(\zeta)$ is the product of binomial terms of the form $(\zeta - c)^\lambda$ which may be expanded as follows

$$(\zeta - c)^\lambda = \zeta^\lambda\left(1 - \frac{c}{\zeta}\right)^\lambda = \zeta^\lambda\left(1 - \lambda\frac{c}{\zeta} + \frac{\lambda(\lambda-1)}{1.2}\frac{c^2}{\zeta^2} - \ldots\right).$$

By (110.21), the sum of the exponents λ equals $n - p$, i.e. an integer (or zero).

Hence, by (70.3′), one obtains

$$\Omega(z) = \frac{f(z)}{X_p(z)} \quad \alpha_q z^q \qquad\qquad \alpha_0$$

(if $q < 0$, the polynomial on the right-hand side will vanish).

In (70.3′) the integral is taken over one contour only, but obviously this is of no importance. The difference in sign arises from the fact that the positive direction of the path of integration in the present case is opposite to that in § 70.

On the other hand, letting the contours Λ_k shrink into the arcs L_k and noting that $X_p(\zeta)$ in (110.38) will then tend to $X_p^+(t)$ or $X_p^-(t)$, depending on the position of ζ on Λ_k, it is readily seen that

$$\int_{\Lambda_k} \frac{f(\zeta)d\zeta}{X_p(\zeta)\,(\zeta - z)} \qquad \int_{a_kb_k} \frac{f(\zeta)d\zeta}{X_p^+(t)\,(t - z)} + \int_{b_ka_k} \frac{f(\zeta)d\zeta}{X_p^-(\zeta)(\zeta - z)}$$

or, since one has $X_p^-(t) = 1/g\,X_p^+(t)$ on L_k and must change the sign of the integral when replacing the path b_ka_k by the path a_kb_k,

$$\int_{\Lambda_k} \frac{f(\zeta)d\zeta}{X_p(\zeta)\,(\zeta - z)} = (1 - g)\int \frac{f(t)dt}{X_p^+(t)\,t - z)};$$

hence it is obvious that

$$\int \frac{f(\zeta)d\zeta}{X_p(\zeta)\,(\zeta - z)} \qquad (1 - g)\int \frac{f(t)dt}{X_p^+(t)\,(t - z)}$$

Consequently

$$I(z) = \int_L \frac{f(t)dt}{X_p^+(t)\,(t-z)} = \frac{2\pi i}{1-g}\,\Omega(z) =$$

$$= \frac{2\pi i}{1-g}\left\{\frac{f(z)}{X_p(z)} - \alpha_q z^q - \ldots - \alpha_0\right\}. \quad (110.40)$$

The fact that the function $1/X_p(z)$ may be unbounded near the ends is easily seen not to influence the above reasoning, since near any end c

$$\frac{1}{|X_p(\zeta)|} < \frac{\text{const,}}{|\zeta - c|^{\mu}}, \quad \mu < 1,$$

and hence integrals, taken over a small circle surrounding the end, tend to zero (cf. Fig. 49).

Integrals of the form

$$\int_L \frac{t^{m-1}f(t)dt}{X_p^+(t)}, \quad (110.41)$$

where m is an integer (positive, negative or zero), may likewise be evaluated in finite form. In fact, proceeding as in the preceding case, it is found that

$$(1-g)\int_L \frac{t^{m-1}f(t)dt}{X_p^+(t)} = \int_\Lambda \frac{\zeta^{m-1}f(\zeta)d\zeta}{X_p(\zeta)},$$

where Λ is the same as before. On the other hand, if α_{-m} denotes the coefficient of z^{-m} in (110.39), then, by the residue theorem,

$$\int_\Lambda \frac{\zeta^{m-1}f(\zeta)d\zeta}{X_p(\zeta)} = -2\pi i\alpha_{-m}$$

and, consequently,

$$\int_L \frac{t^{m-1}f(t)dt}{X_p^+(t)} = -\frac{2\pi i\alpha_{-m}}{1-g}. \quad (110.42)$$

Analogous results may be obtained when $f(t)$ is a *rational function*, and not only a polynomial.

It must be borne in mind that, if $f(t)$ is represented by different polynomials (or different rational functions) on the different arcs constituting L, then the calculation of the preceding integrals may, in general, not be as easily performed.

If L only consists of one arc and if $f(t)$ is not a polynomial, then in the majority of cases which are of practical interest one may replace $f(t)$ with sufficient accuracy by a polynomial with a small number of terms, or by a rational function.

NOTE. 2. When solving the boundary problems above, the selected particular solutions $X_p(z)$ of the homogeneous case have been used in a definite manner. However, it is obvious that nothing will be changed, if $X_p(z)$ is replaced by $CX_p(z)$, where C is an arbitrary constant which is not zero. It is only important in (110.20) and those formulae, connected with it, that $X_p^+(t)$ on L is the value taken by the selected functions $X_p(z)$ on L from the left.

For example, let $g = -1$ and L be a segment ab of the real axis. In agreement with the above condition, that branch of the function

$$X(z) = \sqrt{(z-a)(z-b)}$$

in (110.33) must be taken which for large $|z|$ is given by

$$\sqrt{(z-a)(z-b)} = z\left(1-\frac{a}{z}\right)^{\frac{1}{2}}\left(1-\frac{b}{z}\right)^{\frac{1}{2}} = z - \frac{a+b}{2} - \frac{(a-b)^2}{8z} + \dots$$

This function takes on the segment ab purely imaginary values. But sometimes it is more convenient to deal with a function, having real values on this segment. Such a function is, for example,

$$\sqrt{(z-a)(b-z)} = \pm i\sqrt{(z-a)(z-b)} = \pm iX(z).$$

If one takes the lower sign, i.e.,

$$\sqrt{(z-a)(b-z)} = -iX(z),$$

one obtains a function which is easily seen to take *positive* values on the left side of the segment ab of the Ox axis.

§ 111. Case of discontinuous coefficients. It is also not difficult to find the solution in the case, where the coefficient $G(t)$ in the boundary condition

$$F^+(t) - G(t)F^-(t) = f(t) \text{ on } L \tag{111.1}$$

remains constant on different parts of L, but changes discontinuously for transition from one part to another; in this connection the term "part" refers to sections, into which the line L may be divided by a finite number of points c_j' on it. However, in this case one has to admit that the function

$F(z)$ may be unbounded near the points of discontinuity c_j' as well as near the ends of the line L, where it must satisfy the same condition as at the ends; in general the points c_j' will be analogous to ends.

It will be left to the reader to deduce the solution in the general case; only the following particular case will be studied here, which is the only one required in the sequel.

Let L be a simple contour and let this contour consist of the n arcs $L_1 = a_1b_1$, $L_2 = a_2b_2$, ..., $L_n = a_nb_n$ without common ends (Fig. 50); the notation has been chosen such that during a circuit of L in the positive direction the points $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ are encountered in the stated order.

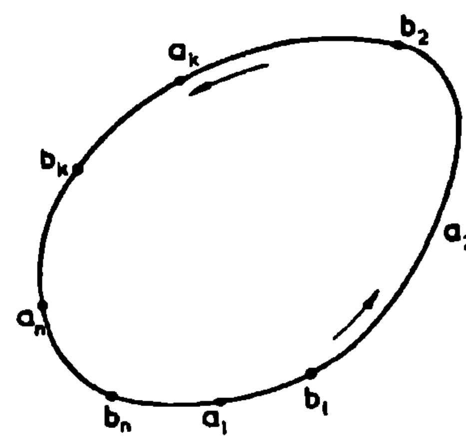


Fig. 50.

Denote by L' the union of the arcs L_k, $k = 1, 2, \ldots, n$, and by L'' the remaining part of L, i.e., the union of the arcs $b_1a_2, b_2a_3, \ldots b_na_1$, and assume that

$$G(t) = g \text{ on } L', \; G(t) = 1 \text{ on } L'', \qquad (111.2)$$

where $g \neq 1$ is, in general, a complex constant. Thus the boundary condition (111.1) takes the form

$$F^+(t) - gF^-(t) = f(t) \text{ on } L', \; F^+(t) - F^-(t) = f(t) \text{ on } L''. \quad (111.3)$$

It will be assumed that the given function $f(t)$, satisfying the H condition on L' and L'' separately, may change discontinuously for passage through the points a_j, b_j.

The homogeneous problem will be considered first:

$$F^+(t) - gF^-(t) = 0 \text{ on } L', \quad F^+(t) - F^-(t) = 0 \text{ on } L''. \quad (111.3')$$

The second of these equations shows that the values of the unknown function $F(z)$ inside and outside L continue each other analytically through the part L'' of L, or, in other words, that L'' is not effectively a line of discontinuity.

Thus one arrives at the same homogeneous problem

$$F^+(t) - gF^-(t) = 0 \text{ on } L', \qquad (111.3'')$$

where $F(z)$ is a function, holomorphic in the plane cut along L', as in the preceding section. Hence, for example, the function $X_0(z)$, deter-

mined by (110.2) and (110.5), is, for $\alpha \neq 0$, a particular solution of (111.3′), as may easily be verified directly; for $\alpha = 0$, one has to take $X_*(z)$, determined by (110.15), instead of $X_0(z)$.

Now consider the non-homogeneous problem (111.3). Using the solution $X_0(z)$ and noting that, by (110.9′) [remembering that here L' takes the place of L in § 110], one finds that

$$g = \frac{X_0^+(t)}{X_0^-(t)} \text{ on } L' \tag{111.4}$$

and that

$$\frac{X_0^+(t)}{X_0^-(t)} = 1 \text{ on } L'', \tag{111.5}$$

because the function $X_0(z)$ is holomorphic everywhere, except at points of L'; consequently (111.3) may be written as one single formula

$$\frac{F^+(t)}{X_0^+(t)} - \frac{F^-(t)}{X_0^+(t)} = \frac{f(t)}{X_0^+(t)} \text{ on } L,$$

whence, as in § 108,

$$F(z) = \frac{X_0(z)}{2\pi i} \int_L \frac{f(t)dt}{X_0^+(t)\,(t-z)} + X_0(z)P(z), \tag{111.6}$$

where $P(z)$ is an arbitrary polynomial and $F(z)$ may have a pole at infinity.

The formula (111.6) does not appear to differ from (110.18); however, this is due to the notation: in (110.18), the integral is taken, in the notation of this section, along L', and not along L. The formula (111.6) will actually agree with (110.18), if $f(t) = 0$ on L'', and this is quite natural, since in that case one is dealing effectively with the same problem.

If the function is to be holomorphic at $z = \infty$, it must be assumed that the degree of $P(z)$ does not exceed n; if $F(z)$ is to vanish at infinity, the degree of $P(z)$ must not exceed $n - 1$.

If g is a real, positive number, then one has to take in (111.6) $X_*(z)$ instead of $X_0(z)$.

If $F(z)$ may have poles at given points, not on L, the formula (111.6) must be replaced by another one, analogous to (110.26).

Finally, it may be noted that the preceding results are easily extended to the case, where the line L extends to infinity, e.g., where it is an infinite straight line; this extension is so obvious that no more need be said here.

SOLUTION OF THE FUNDAMENTAL PROBLEMS FOR THE HALF-PLANE AND FOR THE PLANE WITH STRAIGHT CUTS

The results, studied in Chapter 18, offer the possibility of solving in a very simple manner the principal boundary problems for those cases, where the region under consideration is the half-plane or the plane with straight cuts (along one and the same straight line), including the first and second fundamental problems for the half-plane which have already been considered in the preceding Part. Some of these problems, and likewise the important problem of contact of two elastic half-planes, will be considered in this chapter.

Fig. 51.

§ 112. Transformation of the general formulae for the half-plane *). It will be assumed that the elastic body occupies the lower half-plane $y < 0$ which will be denoted by S^-, so that the region S^- is to the right, if one moves in the positive direction along the Ox axis. The upper half-plane will be denoted by S^+ and the Ox axis by L. (Fig. 51).

First, general formulae will be recalled which will constantly be used:

$$X_x + Y_y = 2[\Phi(z) + \overline{\Phi(z)}], \tag{112.1}$$

$$Y_y - X_x + 2iX_y = 2[\bar{z}\Phi'(z) + \Psi(z)], \tag{112.2}$$

$$2\mu(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \tag{112.3}$$

where $\Phi(z) = \varphi'(z)$, $\Psi(z) = \psi'(z)$ are functions, holomorphic in S^-. As in Chapter 16, it will be assumed that the resultant vector (X, Y) of the external forces, acting on the boundary, is finite and that *the stresses*

*) N.I. Muskhelishvili [22].

and rotation vanish at infinity; hence one has for large $|z|$, as in § 90,

$$\Phi(z) = -\frac{X+iY}{2\pi z} + o\left(\frac{1}{z}\right), \quad \Phi'(z) = \frac{X+iY}{2\pi z^2} + o\left(\frac{1}{z^2}\right), \tag{112.4}$$

$$\Psi(z) = \frac{X-iY}{2\pi z} + o\left(\frac{1}{z}\right), \tag{112.4'}$$

$$\begin{aligned} \varphi(z) &= -\frac{X+iY}{2\pi}\log z + o(1) + \text{const.}, \\ \psi(z) &= \frac{X-iY}{2\pi}\log z + o(1) + \text{const.} \end{aligned} \tag{112.5}$$

The following formula follows from (112.1) and (112.2):

$$Y_y - iX_y = \Phi(z) + \overline{\Phi(z)} + z\overline{\Phi'(z)} + \overline{\Psi(z)}, \tag{112.6}$$

while the formula

$$2\mu(u' + iv') = \varkappa\Phi(z) - \overline{\Phi(z)} - z\overline{\Phi'(z)} - \overline{\Psi(z)} \tag{112.7}$$

is obtained from (112.3) by differentiation with respect to x, so that

$$u' = \frac{\partial u}{\partial x}, \quad v' = \frac{\partial v}{\partial x}, \tag{112.8}$$

where this notation will be used in the sequel.

These formulae will now be transformed by extending the definition of the function $\Phi(z)$ to the upper half-plane. This extension may be achieved in many ways, because $\Phi(z)$ is, in general, not defined in the upper half-plane. However, practically useful formulae are obtained by executing the process by definite methods, one of which has been stated earlier, while another analogous one will be indicated in § 113. In fact, the function $\Phi(z)$ has been defined in the upper half-plane in such a way that *its values there continue $\Phi(z)$ analytically from the lower half-plane through the unloaded parts of the boundary* (if such exist).

A way of doing this is easily deduced from (112.6) [cf. Note in § 93]. Actually, this formula shows that on the unloaded parts of the boundary, where obviously $Y_y^- = X_y^- = 0$,

$$\Phi^-(t) - \Phi^+(t) = 0, \tag{a}$$

provided that in the upper half-plane the function $\Phi(z)$ is defined in

the following manner:

$$\Phi(z) = -\overline{\Phi}(z) - z\overline{\Phi}'(z) - \overline{\Psi}(z) \text{ (for } z \text{ in } S^+). \tag{112.9}$$

Here use has been made of the notation, introduced earlier; in fact, the signs (+) and (—) indicate the boundary values from the left and from the right of L (i.e., in the present case, from the upper and lower half-planes), while $\overline{F}(z)$ denotes the function obtained from $F(z)$ in the following manner (§ 76):

$$\overline{F}(z) = \overline{F(\bar{z})}; \tag{112.10}$$

if $F(z)$ is holomorphic in S^-[or S^+], then $\overline{F}(z)$ will be holomorphic in S^+[S^-].

It will also be remembered [(76.6), (76.6′)] that, if $F(z)$ is defined, say, in the lower half-plane and if $F^-(t)$ exists at a point t of the real axis, then $\overline{F}^+(t)$ exists and

$$\overline{F^-(t)} = \overline{F}^+(t); \tag{112.10'}$$

similarly, interchanging the parts played by the upper and lower half-planes,

$$\overline{F^+(t)} = \overline{F}^-(t). \tag{112.10''}$$

It is clear from the above that the right-hand side of (112.9) represents a function, holomorphic in the upper half-plane S^+, and that on the unloaded parts of the boundary the condition (*a*) holds true.

Replacing in (112.9) the variable z by $\bar{z}$, assuming that z lies in S^- (and hence $\bar{z}$ in S^+) and taking conjugate complex values on both sides of (112.9), one finds

$$\overline{\Phi}(z) = -\Phi(z) - z\Phi'(z) - \Psi(z),$$

whence

$$\Psi(z) = -\Phi(z) - \overline{\Phi}(z) - z\Phi'(z); \tag{112.11}$$

this formula expresses $\Psi(z)$ for z in S^- in terms of $\Phi(z)$ which has been extended into the upper half-plane.

Without the extension of $\Phi(z)$, the formula (112.11) does not make sense, because it involves $\overline{\Phi}(z)$ which is, by definition, equal to $\overline{\Phi(\bar{z})}$; however, in order that $\Phi(\bar{z})$ for z in the lower half plane (i.e., for $\bar{z}$ in the upper half-plane) have a meaning, the function $\Phi(z)$ must be defined in the upper half-plane.

Introducing the value (112.11) for $\Psi(z)$ in (112.2), one obtains the following expressions for the stress components in terms of the *single function* $\Phi(z)$, defined in the upper as well as in the lower half-plane:

$$X_x + Y_y = 2[\Phi(z) + \overline{\Phi}(\bar{z})], \tag{112.12}$$

$$Y_y - X_x + 2iX_y = 2[(\bar{z} - z)\Phi'(z) - \Phi(z) - \overline{\Phi}(\bar{z})], \tag{112.13}$$

whence

$$Y_y - iX_y = \Phi(z) - \Phi(\bar{z}) + (z - \bar{z})\overline{\Phi}'(\bar{z}), \tag{112.14}$$

where the last formula could also have been obtained directly from (112.6). Further, it follows from (112.7) that

$$2\mu(u' + iv') = \varkappa\Phi(z) + \Phi(\bar{z}) - (z - \bar{z})\overline{\Phi}'(\bar{z}). \tag{112.15}$$

An expression for $2\mu(u + iv)$ is likewise easily deduced, if one also extends $\varphi(z)$ into the upper half-plane under the condition $\varphi'(z) = \Phi(z)$ in the upper half-plane as well. Noting that, by (112.9), in the upper half-plane $\Phi(z) = -[z\overline{\varphi}'(z) + \overline{\psi}(z)]'$, one obtains

$$\varphi(z) = -z\overline{\varphi}'(z) - \overline{\psi}(z) + \text{const.} \quad (\text{for } z \text{ in } S^+), \tag{112.16}$$

and hence, as before,

$$\psi(z) = -\overline{\varphi}(z) - z\varphi'(z) + \text{const.} \quad (\text{for } z \text{ in } S^-). \tag{112.17}$$

With this value for $\psi(z)$ the formula (112.3) becomes

$$2\mu(u + iv) = \varkappa\varphi(z) + \varphi(\bar{z}) - (z - \bar{z})\overline{\varphi}'(\bar{z}) + \text{const.}; \tag{112.18}$$

it is seen that for a given function $\Phi(z)$ the displacements are defined apart from a rigid body translation, as was to be expected.

If the rotation does not vanish at infinity, the arbitrary constant which influences the rigid body rotation enters into the function $\Phi(z)$ and in this way into $\varphi(z)$ on the right-hand side of (112.18). However, under the present condition by which the rotation is to vanish at infinity, the function $\Phi(z)$ is uniquely determined for any given state of stress of the body.

It will be noted that it follows from the definition of the function $\Phi(z)$ in the upper half-plane and from the conditions (112.4) and (112.4′) which, of course, refer to the lower half-plane that *the conditions* (112.4) *also hold for the upper half-plane.*

In the sequel, it will be assumed that $\Phi(z)$ is continuous from the left and from the right at all points t of the real axis, except possibly at a

finite number of points $t_1, t_2, \ldots, t_k$ which will always be specially mentioned (so that no such points will exist, unless stated otherwise) and that

$$\lim_{y \to 0} y\Phi'(z) = \lim_{y \to 0} y\Phi'(t + iy) = 0 \tag{112.19}$$

for any point t of the real axis, except possibly for $t_1, t_2, \ldots, t_k$. These restrictions are to ensure that the stresses X_x, Y_y, X_y tend to definite limits for $z \to t$; the exclusion of the points t_j has been introduced, in order not to lose solutions which are practically important.

Finally, it will be assumed that near the points t_j, mentioned above,

$$|\Phi(z)| < \frac{A}{|z - t_j|^{\alpha}}, \quad 0 \leqslant \alpha < 1. \tag{112.20}$$

§ 113. Solution of the first and second fundamental problems for the half-plane. The solutions of the first and second fundamental problems were given in §§ 93, 94; new solutions will be deduced here on the basis of the formulae of § 112, thus exhibiting their simplest application.

1°. First fundamental problem. In this problem the external stresses are given, i.e., the pressure $P(t) = -Y_y^-$ [denoted in § 93 by $-N(t)$] and the tangential stress $T(t) = X_y^-$, applied to the entire boundary Ox which will again be denoted by L. It will be assumed that $P(t)$ and $T(t)$ satisfy the H condition on L, including the point at infinity, and that they vanish for $t = \infty$.

By (112.14), the boundary condition takes the form

$$\Phi^+(t) - \Phi^-(t) = P(t) + iT(t), \tag{113.1}$$

because for z, tending to t from the lower half-plane, the functions $\Phi(z) \to \Phi^-(t)$, $\Phi(\bar{z}) \to \Phi^+(t)$, while $(z - \bar{z})\bar{\Phi}'(\bar{z}) = 2iy\bar{\Phi}'(\bar{z})$ tends to zero by (112.19).

Strictly speaking, (112.19) only ensures the limit $y\Phi'(z) \to 0$, when $z \to t$ along the normal to the boundary. However, it is readily verified by means of a simple estimate of the derivative of a Cauchy integral, stated in the Author's book [25], that the final result gives a regular solution of the problem.

A solution of the boundary problem (113.1), vanishing at infinity, may be written down on the basis of the results of § 108, where only the case of a finite line L has been considered; however, the applicability

of the relevant formulae to the present case is obvious. In fact,

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{P(t) + iN(t)}{t - z}\, dt; \tag{113.2}$$

it is easily decided what additional conditions must be imposed on $P(t)$ and $T(t)$ so that $\Phi(z)$ satisfies (112.4), and this will be left to the reader.

The problem is thus solved, because $\Phi(z)$ determines the components of stress and displacement by the formulae of § 112.

2°. Second fundamental problem. In this problem the boundary values of the displacement components u, v are given, i.e., $u^- = g_1(t)$, $v^- = g_2(t)$ on L. It will be assumed that the given functions $g_1(t)$ and $g_2(t)$ have derivatives $g_1'(t)$, $g_2'(t)$, satisfying the H condition on L including the point at infinity, and that they vanish for $t = \infty$.

The boundary condition

$$u^- + iv^- = g_1(t) + ig_2(t) \text{ on } L \tag{113.3}$$

gives, after differentiation with respect to t,

$$(u^-)' + i(v^-)' = g_1'(t) + ig_2'(t)$$

or, assuming that the boundary values of the partial derivatives u', v' are equal to the derivatives $(u^-)'$, $(v^-)'$ of the boundary values,

$$u'^- + iv'^- = g_1'(t) + ig_2'(t). \tag{113.4}$$

By (112.15), this condition takes the form

$$\Phi^+(t) + \varkappa\Phi^-(t) = 2\mu[g_1'(t) + ig_2'(t)] \text{ on } L. \tag{113.5}$$

The validity of the assumption $u'^- = (u^-)'$, $v'^- = (v^-)'$ under known conditions referring to the given functions $g_1(t)$, $g_2(t)$ may be verified after the solution has actually been constrtucted.

For the time being, denote by $\Omega(z)$ the following sectionally holomorphic function: $\Omega(z) = \Phi(z)$ in S^+, $\Omega(z) = -\varkappa\Phi(z)$ in S^-. Then (113.5) takes the form

$$\Omega^+(t) - \Omega^-(t) = 2\mu[g_1'(t) + ig_2'(t)] \text{ on } L, \tag{113.6}$$

and hence, as in the preceding case,

$$\Omega(z) = \frac{\mu}{\pi i}\int_L \frac{g_1'(t) + ig_2'(t)}{t - z}\, dt, \tag{113.7}$$

so that, finally,

$$\Phi(z) = \begin{cases} \Omega(z) & \text{for } y > 0, \\ -\dfrac{1}{\varkappa}\,\Omega(z) & \text{for } y < 0, \end{cases} \tag{113.8}$$

where $\Omega(z)$ is determined by (113.7). The investigation of the behaviour of this solution at infinity will be left to the reader.

The sectionally holomorphic function $\Omega(z)$, introduced in the above manner, may again be denoted by $\Phi(z)$; one then obtains an extension of the original function $\Phi(z)$ into the upper half-plane, different from that stated in the preceding section. This new method of extension is characterized by the fact that the values of $\Phi(z)$ in the upper half-plane analytically continue the values in the lower half-plane through those parts of the boundary (if such exist) where $u' = v' = 0$

The solution is also easily deduced when one begins with (112.18); however, the above solution is more convenient, because it does not require an additional investigation, arising from the fact that $\varphi(z)$ is not holomorphic at infinity, but behaves there like $\log z$, unless a further condition, restricting generality, is imposed, as was done in § 94.

§ 114. Solution of the mixed fundamental problem.

If there is only one segment of the real axis on which displacements are given, i.e., if in the notation of the main part of this section $n = 1$, the problem is fairly simple; however, no effective solution of this problem was obtained in the Author's paper [20], published in 1935. Two years later, V. M. Abramov [1] gave a more effective solution of this case, using Mellin's integrals. The solution of this section for arbitrary n which was first presented by the Author in his paper [22] is incomparably simpler. A somewhat more complicated solution which is in essence closely related to that studied here was deduced soon afterwards by N. I. Glagolev [1, 2] who was not acquainted with the Author's work [22].

Let $L' = L_1 + L_2 + \ldots + L_n$ be the union of a finite number of segments $a_k b_k$ of the real axis Ox, numbered in such a way that they are encountered in the order $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ when moving along the real axis in the positive direction. Let the components of displacement be given on L' and those of the external forces on the remaining part L'' of the real axis. As before, it will be assumed that the elastic body occupies the lower half-plane S^-; its boundary, i.e., the real axis, will be denoted by $L = L' + L''$.

Since one knows how to solve the first fundamental problem, the mixed fundamental problem under consideration may obviously always be reduced to the case where the components Y_y, X_y of the external

stresses, given on L'', are zero. Hence it will be assumed that the part L'' of the boundary is free from external forces, i.e., that

$$Y_y^- = X_y^- = 0 \text{ on } L''. \tag{114.1}$$

(The solution of the general case may also be deduced directly; cf. the Note at the end of this section).

In view of the considerable practical importance of the present problem, the ordinary mixed problem in the form, formulated above and to be called Problem A, will be studied side by side with a somewhat modified problem, to be called Problem B.

In both these problems the boundary condition on L' has the form

$$u^- + iv^- = g(t) + c(t) \text{ on } L', \tag{114.2}$$

where $g(t) = g_1(t) + ig_2(t)$ is a function given on L'.

In Problem A, it will be assumed that $c(t) = c = \text{const.}$ on L' and that, in addition, the resultant vector (X, Y) of the external forces, applied to L', is known. Without affecting generality, one may, for example, put $c = 0$, because its value only influences the rigid body translation.

In Problem B, it will be assumed that $c(t) = c_k$ on L_k, where the c_k are constants which are intially unknown and which are, in general, different on different segments. In this case it will be assumed that the resultant vectors (X_k, Y_k) of the external forces, applied to the individual segments L_k, are known. Without affecting generality, one of the constants c_k may be fixed arbitrarily, i.e., one may, for example, put $c_1 = 0$. For $n = 1$, the problems A and B coincide.

The physical meaning of these problems will now be explained. One may imagine that rigid stamps with given profiles are placed on the segments $L_k = a_k b_k$, that the points of the segments L_k of the boundary of the elastic body are brought in a definite manner into contact with the points of the profiles of the rigid stamps and that the same points remain (or are welded) together. Further, suppose that given forces be applied to the stamps and that *the stamps can only move vertically*. The problem of finding the equilibrium of the elastic half-plane under these conditions is Problem A, provided the stamps are rigidly interconnected; if the stamps can move vertically, independently of each other, one arrives at Problem B.

For greater understanding, consider the following particular case. Let there be only one stamp the profile of the base of which before contact with the elastic half-plane is given by $y = f(x)$, $a < x < b$. Further,

assume that the stamp is pressed into the half-plane by a given force, perpendicular to the boundary, and that the friction between the stamp and the elastic body is so great that no slip can occur. Then, assuming that the segment $L' = ab$ of the boundary of the body has come into contact with the stamp, one obtains the boundary condition (114.2) in which $g(t) = g_1(t) + ig_2(t)$, where $g_1(t) = f(t)$, $g_2(t) = 0$ and $c(t) = c$ is a constant which may be put equal to zero. The case of several stamps, whether they are interconnected or not, is quite analogous.

These problems represent idealizations of the problems of the pressure of foundations on the ground in the presence of sufficiently large friction (or, more correctly, cohesion), since slip and after-effects are excluded.

The above problem may be generalized in the way that the stamps are not only permitted to move vertically, but also to rotate; this case will again be referred to later on.

It is easily shown that neither of the problems A and B can have more than one solution, neglecting rigid body motion of the entire system (elastic half-plane and stamps). In fact, the proof of § 40 (cf. also end of § 90) may be repeated almost word for word, if it is noted that in the present case the integral of the expression $(X_n u + Y_n v)ds$ along the boundary of the region for the difference of two possible solutions is zero. Actually, one has on the part L'' of the boundary: $X_n = Y_n = 0$. Further, for Problem A, it may be assumed that the difference of the solutions $u = v = 0$ on the boundary; however, for Problem B (where for the difference of the solutions $u = \text{const.}$, $v = \text{const.}$ on L_k), all the integrals

$$\int_{L_k} (uX_n + vY_n)ds = u\int_{L_k} X_n dt + v\int_{L_k} Y_n dt = uX_k + vY_k$$

vanish, since the resultant vectors (X_k, Y_k), applied to L_k, are zero for the difference of two solutions.

In the sequel, the uniquess theorem will also be applied to cases, where the solution under consideration may cease to be regular near the points a_k, b_k. In such cases, uniqueness is obviously maintained, if the integrals of the expression $(X_n u + Y_n v)ds$, formed for the difference of solutions, when taken in S along infinitely small semi-circles about the points a_k, b_k as centres, tend to zero together with the radii of the semi-circles. In all the cases considered below this condition will be fulfilled, as the reader may easily verify in each individual case.

It will be assumed below that the function $\Phi(z)$ satisfies the conditions,

stated in § 112, and that the ends a_j, b_j of the segments L_j play here the parts of the points t_j. In addition, the known function $g(t)$ is to have a first derivative $g'(t)$, satisfying the H condition on L'.

In both problems A and B the boundary condition (114.2) gives $(u^-)' + (iv^-)' = g'(t)$ on L', whence by (112.15) [cf. remarks following (113.5)]

$$\Phi^+(t) + \varkappa\Phi^-(t) = 2\mu g'(t) \text{ on } L'. \tag{114.3}$$

However, the condition (114.1) is equivalent to the requirement that $\Phi^+(t) - \Phi^-(t) = 0$ on L'', i.e., that $\Phi(z)$ should be holomorphic in the entire plane, cut along L', and this will now be assumed.

The condition (114.3) differs essentially from (113.5) in that now L' is only *part* of the Ox axis.

A solution of the boundary problem (114.3), vanishing at infinity, may immediately be written down on the basis of (110.18). In the present case, by (110.5),

$$\gamma = \frac{\log(-\varkappa)}{2\pi i} = \frac{\log \varkappa}{2\pi i} + \frac{1}{2}$$

or

$$\gamma = \tfrac{1}{2} - i\beta,$$

where β is the real quantity

$$\beta = \frac{\log \varkappa}{2\pi} \tag{114.4}$$

which was denoted by $(-\beta)$ in § 110.

Therefore, by (110.2),

$$X_0(z) = \prod_{k=1}^{n} (z - a_k)^{-\frac{1}{2}+i\beta} (z - b_k)^{-\frac{1}{2}-i\beta}.$$

In future, $X_0(z)$ will be denoted by $X(z)$ and $X^+(t)$ by $X(t)$. Thus

$$X(z) = \prod_{k=1}^{n} (z - a_k)^{-\frac{1}{2}+i\beta} (z - b_k)^{-\frac{1}{2}-i\beta} \tag{114.5}$$

and

$$X(t) = X^+(t) = \prod_{k=1}^{n} (t - a_k)^{-\frac{1}{2}+i\beta} (t - b_k)^{-\frac{1}{2}-i\beta}, \tag{114.6}$$

where (114.6) refers to the value, taken by $X(z)$ on the upper side of the Ox axis (i.e., from the left of Ox); if the point t lies outside L', i.e.,

on L'', the values from the left and from the right coincide: $X^-(t) = X^+(t) = X(t)$; however, if t lies on L', one has, by the definition of the function $X(z)$, $X^+(t) + \varkappa X^-(t) = 0$, whence

$$X^-(t) = -\frac{1}{\varkappa} X^+(t) = -\frac{1}{\varkappa} X(t). \tag{114.7}$$

Applying now (110.18), one obtains

$$\Phi(z) = \frac{\mu X(z)}{\pi i} \int\limits_{L'} \frac{g'(t)dt}{X(t)\,(t - z)} + X(z)P_{n-1}(z), \tag{114.8}$$

where $P_{n-1}(z)$ is the polynomial of degree not higher than $n - 1$

$$P_{n-1}(z) = C_0 z^{n-1} + C_1 z^{n-2} + \ldots + C_{n-1}, \tag{114.9}$$

because, by supposition, the function $\Phi(z)$ is to vanish at infinity.

The coefficients $C_0, C_1, \ldots, C_{n-1}$ of the polynomial $P_{n-1}(z)$ have still to be determined from the supplementary conditions of problems A and B.

Problem B will be considered first; in this case the known resultant vectors (X_k, Y_k) serve as supplementary conditions. In order to express these, the normal pressure $P = -Y_y^-$ and the tangential stress $T = X_y^-$, acting on the boundary of the half-plane at the stamps, i.e., on L', will be calculated.

By (112.14), one has for the point t_0 on L'

$$P(t_0) + iT(t_0) = \Phi^+(t_0) - \Phi^-(t_0) \tag{114.10}$$

or, by (114.3),

$$P(t_0) + iT(t_0) = \frac{\varkappa + 1}{\varkappa} \Phi^+(t_0) - \frac{2\mu}{\varkappa} g'(t_0) \text{ on } L'. \tag{114.11}$$

Applying to the right-hand side of (114.8) the Plemelj formulae, one easily finds

$$\Phi^+(t_0) = \mu g'(t_0) + \frac{\mu X(t_0)}{\pi i} \int\limits_{L'} \frac{g'(t)dt}{X(t)\,(t - t_0)} + X(t_0)P_{n-1}(t_0);$$

substituting this value of $\Phi^+(t_0)$ in (114.11) and writing

$$\Phi_0(t_0) = \frac{\mu X(t_0)}{\pi i} \int\limits_{L'} \frac{g'(t)dt}{X(t)\,(t - t_0)}, \tag{114.12}$$

one obtains

$$P(t_0) + iT(t_0) = \frac{\mu(\varkappa - 1)}{\varkappa} g'(t_0) + \frac{\varkappa + 1}{\varkappa} \Phi_0(t_0) + \\ + \frac{\varkappa + 1}{\varkappa} X(t_0) P_{n-1}(t_0) \text{ on } L'. \quad (114.13)$$

By expressing that

$$\int_{L_k} [P(t_0) + iT(t_0)]dt_0 = -Y_k + iX_k, \; k = 1, 2, \ldots, n, \quad (114.14)$$

one deduces a system of n linear equations for the determination of the n constants C_j; this system has a unique solution, as may be seen from the uniqueness of the solution of the original problem.

Next consider Problem A. Since the solution found above satisfies the condition $u'^- + iv'^- = u^{-\prime} + iv^{-\prime} = g'(t)$ on L' (as is easily verified by substitution), one has on the segments L_k

$$u^- + iv^- = g(t) + c_k,$$

where the c_k are constants. One has now to formulate the condition

$$c_1 = c_2 = \ldots = c_n; \quad (114.15)$$

having succeeded in satisfying this condition, one can also fulfill the condition $c_1 = c_2 = \ldots = c_n = 0$ by adding an arbitrary constant to the right-hand side of (112.18).

In order to express the condition (114.15), the value of $u'^- + iv'^-$ will be determined on the unloaded part L'' of the boundary. By (112.15), one has for a point t_0 of L''

$$2\mu(u'^- + iv'^-) = (\varkappa + 1)\Phi(t_0) = (\varkappa + 1)\Phi_0(t_0) + (\varkappa + 1)X(t_0)P_{n-1}(t_0), \quad (114.16)$$

where $\Phi_0(t_0)$ is given by (114.12) and t_0 is now on Ox outside L' (i.e., on L''). Obviously (114.15) reduces to the following conditions:

$$g(a_{k+1}) - g(b_k) = \int_{b_k}^{a_{k+1}} (u'^- + iv'^-)dt_0, \; k = 1, 2, \ldots, n-1. \quad (114.17)$$

Substituting in (114.17) for $u'^- + iv'^-$ from (114.16), one obtains a system of $n - 1$ linear equations for the determination of the C_j. One more equation is given by the condition that the resultant vector of

the external forces (X, Y) applied to L' is known. This last condition is most simply expressed, using the first of the formulae (112.4) which gives

$$\lim_{z\to\infty} z\Phi(z) = -\frac{X+iY}{2\pi}. \tag{114.18}$$

Applying (114.18) to (114.8), one deduces directly that the coefficient C_0 of z^{n-1} in the polynomial $P_{n-1}(z)$ is given by

$$C_0 = -\frac{X+iY}{2\pi}. \tag{114.19}$$

Thus only the coefficients $C_1, C_2, \ldots, C_{n-1}$ have still to be determined from the above system of $n-1$ linear equations. This system will always have a unique solution, as in the preceding case.

In the particular case $g'(t) = 0$ (stamps with straight profiles, parallel to the axis Ox), the formulae (114.8) and (114.13) become very simple, since the integrals vanish.

Hitherto it has been assumed that the stamps may only move vertically (i.e., at right angles to the boundary). The case will now be considered, where the stamps may rotate (of course, in their own plane).

In Problem A (the stamps being rigidly interconnected), let ε denote the angle of rotation of the system of stamps, measured counter-clockwise. Then in the boundary condition (114.2) the function $g(t)$ must be replaced by $g(t) + i\varepsilon t$ and hence in all the subsequent formulae $g'(t)$ by $g'(t) + i\varepsilon$. Correspondingly an additional term, involving ε as a multiplier, will appear in the expression for $\Phi(z)$. (This additional term may be calculated in finite form; cf. example 2° of § 114*a*).

The additional displacements (u_0, v_0) of a point t of the boundary which are caused by the rotation of the stamps are given by $u_0 = 0$, $v_0 = \varepsilon t$, because, in general, the displacements, arising from a rigid body rotation by an angle ε about the origin, are $u_0 = -\varepsilon y$, $v_0 = \varepsilon x$, while on the boundary $y = 0$, $x = t$.

The quantity ε may not be given directly; for example, one may be given instead the resultant moment M about the origin of the external forces which act on the stamps.

These external forces do not, of course, coincide with those, applied by the profiles of the stamps to the sides of the elastic body; these last forces must balance the external forces, applied to the stamps. Obviously, the resultant vector (X, Y) and the resultant moment M of the external forces is equal to the resultant vector and moment of the forces, applied to the boundary of the elastic body by the faces of the stamps.

One thus has for the determination of ε the additional relation

$$M = -\int_L t_0 P(t_0) dt_0. \tag{114.20}$$

In Problem B, the angles ε_k $(k = 1, 2, \ldots, n)$ of the rotations of the various stamps may differ; if they are not given directly, but if, say, the moments M_k of the external forces, acting on the individual stamps, are known, then one will have the n additional conditions for the determination of the ε_k

$$M_k = -\int_{L_k} t_0 P(t_0) dt_0. \tag{114.21}$$

It is easily shown that these conditions completely determine the solution apart from a vertical rigid body displacement of the entire system; the proof is quite analogous to that stated above for the case where the stamps may only move vertically.

NOTE. If the part L'' of the boundary is loaded by given external stresses, the boundary condition takes the form

$$\Phi^+(t) - k\Phi^-(t) = f(t) \text{ on } L, \tag{114.22}$$

where

$$k = -\varkappa \text{ on } L', \quad k = 1 \text{ on } L'', \tag{114.23}$$

and $f(t)$ is a given function:

$$f(t) = 2\mu g'(t) \text{ on } L', \quad f(t) = P(t) + iT(t) \text{ on } L''. \tag{114.24}$$

One is thus led to the problem, considered in § 111. Applying (111.6), one obtains in the notation of the present section

$$\Phi(z) = \frac{X(z)}{2\pi i} \int_L \frac{f(t)dt}{X(t)\,(t-z)} + X(z)P_{n-1}(z), \tag{114.25}$$

where the integral must now be taken over the entire boundary. The coefficients of the polynomial $P_{n-1}(z)$ may be determined as before.

§ 114a. Examples.

1°. Stamp with straight, horizontal base.

Consider the case of *one* stamp $(n = 1)$ with a straight line profile,

parallel to the axis Ox, which may only move vertically so that

$$g'(t) = 0 \text{ on } L'; \tag{114.1a}$$

as stated earlier, this problem was solved by V. M. Abramov [1], using a quite different method. In addition, it will be assumed that the external forces, acting on the stamp, have a resultant in the downward direction, so that

$$X = 0, \quad Y = -P_0, \tag{114.2a}$$

where P_0 is a given positive constant.

The segment L' of the boundary which is in contact with the stamp will be assumed to lie symmetrically with regard to the origin; its length will be denoted by $2l$, so that for points t of L': $-l \leqslant t \leqslant l$.

Thus, in this notation and that of the preceding section,

$$X(z) = (z + l)^{-\frac{1}{2}+i\beta}(z - l)^{-\frac{1}{2}-i\beta}, \tag{114.3a}$$

and (114.8) gives

$$\Phi(z) = C_0 X(z)$$

or, using (114.19),

$$\Phi(z) = \frac{iP_0}{2\pi} X(z) = \frac{iP_0}{2\pi}(z + l)^{-\frac{1}{2}+i\beta}(z - l)^{-\frac{1}{2}-i\beta}, \tag{114.4a}$$

and the problem is solved.

The pressure $P(t)$ and the tangential stress $T(t)$, acting on the body underneath the stamp, are given by (114.11) which becomes

$$P(t) + iT(t) = \frac{\varkappa + 1}{\varkappa}\Phi^+(t).$$

Hence, substituting from (114.4a), one obtains

$$P(t) + iT(t) = \frac{iP_0}{2\pi}\frac{\varkappa + 1}{\varkappa}X(t) =$$

$$= \frac{iP_0}{2\pi}\frac{\varkappa + 1}{\varkappa}(t + l)^{-\frac{1}{2}+i\beta}(t - l)^{-\frac{1}{2}-i\beta}, \quad (-l \leqslant t \leqslant l), \tag{114.5a}$$

where $X(t)$ stands for $X^+(t)$, i.e., the value taken by $X(z)$ on the left side of the segment $(-l, +l)$, and it should be remembered that the branch of $X(z)$ is determined by the condition $\lim_{z\to\infty} zX(z) = 1$.

It is easily verified that under these conditions

$$X(t) = \frac{1}{\sqrt{t^2 - l^2}} \left[\frac{t+l}{t-l}\right]^{i\beta} = \frac{1}{i\sqrt{t^2 - l^2}} e^{i\beta\left[\log \frac{l+t}{l-t} - \pi i\right]} =$$

$$= \frac{e^{\pi\beta}}{i\sqrt{l^2 - t^2}} e^{i\beta \log \frac{l+t}{l-t}}, \quad -l < t < l,$$

where the root $\sqrt{l^2 - t^2}$ is positive and the logarithm real. Since

$$\beta = \frac{\log \varkappa}{2\pi}, \quad e^{\pi\beta} = \sqrt{\varkappa},$$

one may still write

$$X(t) = \frac{\sqrt{\varkappa}}{i\sqrt{l^2 - t^2}} e^{i\beta \log \frac{l+t}{l-t}} =$$

$$= \frac{\sqrt{\varkappa}}{i\sqrt{l^2 - t^2}} \left\{\cos\left[\beta \log \frac{l+t}{l-t}\right] + i \sin\left[\beta \log \frac{l+t}{l-t}\right]\right\} \qquad (114.6a)$$

$(-l \leqslant t \leqslant l)$. Substituting this expression in (114.5*a*) and separating real and imaginary parts, one obtains

$$P(t) = \frac{P_0}{\pi\sqrt{l^2 - t^2}} \frac{1+\varkappa}{\sqrt{\varkappa}} \cos\left[\frac{\log \varkappa}{2\pi} \log \frac{l+t}{l-t}\right], \qquad (114.7a)$$

$$T(t) = \frac{P_0}{\pi\sqrt{l^2 - t^2}} \frac{1+\varkappa}{\sqrt{\varkappa}} \sin\left[\frac{\log \varkappa}{2\pi} \log \frac{l+t}{l-t}\right]. \qquad (114.8a)$$

These formulae agree with those, obtained by V. M. Abramov [1].

It follows from (114.7*a*) that $P(t)$ changes its sign an infinite number of times as t approaches the values $-l$ and $+l$, so that effectively tensile forces, instead of compressive forces, act on certain parts of the boundary underneath the stamp. However, it is easily seen that these parts lie in very small neighbourhoods of the ends of the segment $-l$, $+l$. In fact, the point t at which $P(t)$, positive for $t = 0$, vanishes for the first time, when t approaches one of the values $\pm l$, is determined by the equation

$$\beta \log \frac{l+t}{l-t} = \pm \frac{\pi}{2}$$

or

$$\log \frac{l+t}{l-t} = \pm \frac{\pi^2}{\log \varkappa},$$

whence

$$t = \pm\, l \tanh \frac{\pi^2}{2 \log \varkappa}. \tag{114.9a}$$

However, for all actual bodies, $1 < \varkappa < 3$, since

$$\varkappa = \frac{\lambda + 3\mu}{\lambda + \mu} \quad \lambda > 0, \quad \mu > 0;$$

therefore the smallest possible value of $|\,t\,|$ is obtained by putting in (114.9a) $\varkappa = 3$ which gives

$$t = \pm\, 0.9997l.$$

Thus a change in sign of $P(t)$ only occurs in those places near which the solution obtained does not, in general, describe the actual state of the body, because, obviously, Hooke's law does not apply for the stresses which must occur according to the formulae above.

2°. Stamp with straight, inclined base.

Consider the case of the same stamp as in the preceding example,

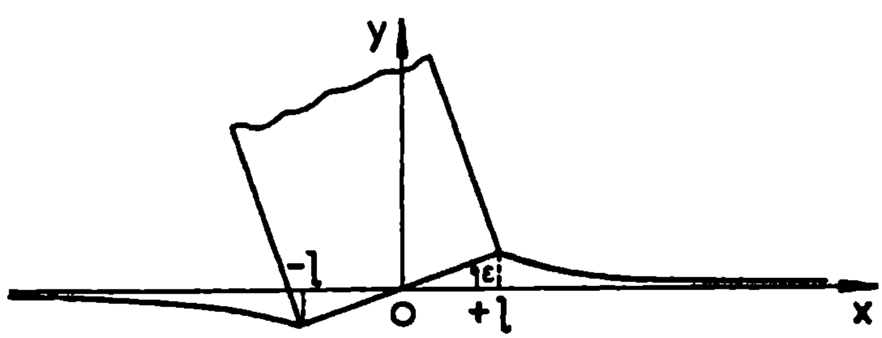


Fig. 52.

but assume now that the resultant vector of the external forces, acting on the stamp, is zero, while the base of the stamp forms an angle ε with the Ox axis, the angle being measured in the positive direction (Fig. 52).

Thus, in the present case,

$$g(t) = g_1 + ig_2 = i\varepsilon t, \quad g'(t) = i\varepsilon \tag{114.10a}$$

and

$$X = Y = 0. \tag{114.11a}$$

Hence one has, by (114.8), in the notation of the preceding example

$$\Phi(z) = \frac{\varepsilon\mu X(z)}{\pi} \int\limits^{l} \frac{dt}{X(t)\,(t - z)}, \tag{114.12a}$$

where $X(z)$ is given by (114.3a) and $X(t) = X^+(t)$.

On the basis of Note 1 at the end of § 110, the integral may be evaluated in closed form. In the present case, for large $|z|$,

$$\frac{1}{X(z)} = (z + l)^{\frac{1}{2} - i\beta} (z - l)^{\frac{1}{2} + i\beta} = z\left(1 + \frac{l}{z}\right)^{\frac{1}{2} - i\beta} \left(1 - \frac{l}{z}\right)^{\frac{1}{2} + i\beta} =$$

$$= z\left\{1 + (\tfrac{1}{2} - i\beta)\frac{l}{z} + \ldots\right\}\left\{1 - (\tfrac{1}{2} + i\beta)\frac{l}{z} + \ldots\right\} =$$

$$= z - 2i\beta l + O\left(\frac{1}{z}\right),$$

and (110.40) gives

$$\int\limits_{-l}^{l} \frac{dt}{X(t)\,(t - z)} = \frac{2\pi i}{\varkappa + 1}\left\{\frac{1}{X(z)} - z + 2i\beta l\right\}.$$

Hence (114.12a) becomes

$$\Phi(z) = \frac{2\mu\varepsilon i}{\varkappa + 1}\{1 - (z - 2i\beta l)X(z)\}, \tag{114.13a}$$

and the problem is solved.

The value of $P(t) + iT(t)$ will now be calculated for a point t underneath the stamp. One has

$$P(t) + iT(t) = \Phi^+(t) - \Phi^-(t) = -\frac{2\mu\varepsilon i}{\varkappa + 1}(t - 2i\beta l)\{X^+(t) - X^-(t)\} =$$

$$\frac{2\mu\varepsilon i}{\varkappa + 1}(t - 2i\beta l)\frac{\varkappa + 1}{\varkappa} X^+(t),$$

i.e., in the present notation,

$$P(t) + iT(t) = -\frac{2\mu\varepsilon i}{\varkappa}(t - 2i\beta l)X(t). \tag{114.14a}$$

Substituting for $X(t)$ from (114.6a) and separating real and imaginary parts, one may obtain closed expressions for $P(t)$ and $T(t)$.

Hitherto it has been assumed that the angle ε is known and, consequently, that the stamp sustains in the given position some couple of the external forces which is not known beforehand. However, the problem may be stated differently; in fact, it may be assumed that the moment M of the couple of the external forces, acting on the stamp, is given and that it is required to find the corresponding angle of tilt ε.

For this purpose the moment M, corresponding to a known angle ε, will be calculated; this moment is given by

$$M = -\int_{-l}^{l} tP(t)dt \tag{114.15a}$$

(the positive direction of rotation being assumed to be counter-clockwise), where $P(t)$ has the value, given by (114.14).

The integral (114.15a) is the real part of the integral

$$I = -\int_{-l}^{l} t[P(t) + iT(t)]dt, \tag{114.16a}$$

which is readily evaluated in closed form. In fact, substituting from (114.14a), one finds

$$I = \frac{2\mu\varepsilon i}{\varkappa}\int_{-l}^{l} t(t - 2i\beta l)X(t)dt. \tag{114.17a}$$

This integral may be calculated by the same method as the integral in (110.42). Consider the integral

$$I_0 = \int_{\Lambda} \zeta(\zeta - 2i\beta l)X(\zeta)d\zeta, \tag{114.18a}$$

taken in counter-clockwise direction over a contour Λ surrounding the segment $L'(-l < t < l)$. Shrinking the contour Λ into the segment L',

one obtains

$$I_0 = \int_{-l}^{+l} t(t - 2i\beta l)X^+(t)dt + \int_{+l}^{-l} t(t - 2i\beta l)X^-(t)dt$$

or, remembering that $X^-(t) = -(1/\varkappa)\, X^+(t) = -(1/\varkappa)\, X(t)$,

$$I_0 = \frac{1+\varkappa}{\varkappa} t \int_{-l}^{+l} t(t - 2i\beta l)X(t)dt,$$

and hence

$$I = \frac{2\mu\varepsilon i}{\varkappa + 1} I_0.$$

On the other hand, one has for large $|\zeta|$

$$X(\zeta) = \frac{1}{\zeta}\left(1 + \frac{l}{\zeta}\right)^{-\frac{1}{2}+i\beta}\left(1 - \frac{l}{\zeta}\right)^{-\frac{1}{2}-i\beta} =$$

$$= \frac{1}{\zeta}\left\{1 - (\tfrac{1}{2} - i\beta)\frac{l}{\zeta} + \frac{1}{2}(\tfrac{1}{2} - i\beta)\left(\frac{3}{2} - i\beta\right)\frac{l^2}{\zeta^2} + \ldots\right\} \times$$

$$\times \left\{1 + (\tfrac{1}{2} + i\beta)\frac{l}{\zeta} + \frac{1}{2}(\tfrac{1}{2} + i\beta)\left(\frac{3}{2} + i\beta\right)\frac{l^2}{\zeta^2} + \ldots\right\} =$$

$$= \frac{1}{\zeta} + \frac{2i\beta l}{\zeta^2} + \frac{(1 - 4\beta^2)l^2}{2\zeta^3} + \ldots,$$

whence it follows that the coefficient of ζ^{-1} in the expansion for $\zeta(\zeta - 2i\beta l)X(\zeta)$ is equal to

$$\frac{l^2(1 + 4\beta^2)}{2}.$$

Therefore, applying the residue theorem to the integral in (114.18*a*), one finds

$$I_0 = -\pi i(1 + 4\beta^2)l^2,$$

and hence

$$I = \frac{2\pi\mu(1 + 4\beta^2)l^2}{\varkappa + 1}\,\varepsilon.$$

It is seen that I is real and thus $M = \Re I = I$, so that

$$M = \frac{2\pi\mu(1 + 4\beta^2)l^2}{\varkappa + 1}\,\varepsilon. \tag{114.19a}$$

For a given moment M, the angle of tilt ε of the stamp will be determined by

$$\varepsilon = \frac{\varkappa + 1}{2\pi\mu(1 + 4\beta^2)l^2}\,M. \tag{114.20a}$$

Substituting this value of ε in (114.13a), one obtains the solution of the problem of the equilibrium of a stamp, subject to a given couple.

3°. Effect of asymmetrically distributed forces

Let asymetrically distributed forces act vertically downward on a stamp with a straight base which is not restrained to move vertically. The effect of these forces is equivalent to that of the same forces, applied symmetrically, and of a certain couple. Hence the solution of this problem will be obtained by adding the solutions of the problems 1° and 2°, treated above.

§ 115. The problem of pressure of rigid stamps in the absence of friction.

This problem was first solved by M. A. Sadovski [1] for one particular case. In the earlier editions of this book, the general solution for the case $n = 1$ was given, while A. I. Begiashvili [1] gave a simple solution for arbitrary n, using the methods of this book and a formula, due to M. V. Keldysh and L. I. Sedov. The much simpler solution, reproduced here, was given by the Author in his paper [22]; simultaneously (and independently) A. V. Bitzadze found a solution which is essentially closely related to the Author's solution.

Consider now the problem of pressure of one or several stamps on the boundary of an elastic half-plane under the supposition that *there is no friction*. For greater clarity, it will first be assumed that there is only one stamp with a given profile. Let $y = f(x)$ be the equation of the profile before it is pressed into the elastic half-plane. If it is assumed that the stamp may only move vertically, i.e., in the direction normal to the boundary, then its profile, after pressure has been applied, will have the equation $y = f(x) + c$, where c is a real constant. It will be assumed that the segment ab of the boundary of the body comes into contact with the stamp. Since a point of the elastic body, occupying before deformation the position $(t, 0)$ and after deformation the position $(t + u, v)$, where

u, v are the components of its displacement, must lie on the line $y = f(x) + c$, one must have $v = f(t + u) + c$. Assuming, as always, that u, v are small quantities and that $f(x)$, $f'(x)$ are likewise small (this supposition being a consequence of the requirement of small deformations), one has, omitting small quantities of higher order, $v = f(t) + c \quad (a \leqslant t \leqslant b)$, where $v = v^-$ is the normal displacement of points of the boundary of the elastic half-plane. The reasoning for the case of several stamps is quite analogous.

Correspondingly, the boundary conditions for the problem of pressure due to a system of stamps which may only move vertically and are completely frictionless may be formulated similarly to the conditions for the problems of § 114, the only difference being (using the notation of § 114) that now

$$X_y^- = 0 \text{ everywhere on } L, \quad Y_y^- = 0 \text{ on } L'', \tag{115.1}$$

while on L' only *the normal component of displacement*

$$v^- = f(t) + c(t) \text{ on } L' \tag{115.2}$$

is given; as before, L denotes here the entire real axis, L' the union of segments $L_k = a_k b_k$ $(k = 1, \ldots, m)$ and L'' the remaining (unloaded) part of L. The first of the conditions (115.1) applies to the whole boundary L, since in the absence of friction the tangential stress at the boundary is also zero underneath the stamps.

In (115.2), the function $f(t)$, given on L', characterizes the profiles of the stamps, and, in fact, $y = f(x)$, where x belongs to L', represents the equation of the union of the profiles of the stamps before their displacement; $c(t)$ is determined as follows: either $c(t) = c$ on L' (rigidly interconnected stamps) or $c(t) = c_k$ on $L_k = a_k b_k$ (free stamps), where c and c_k are now *real* constants. Without affecting generality, one may assume in the first case that $c = 0$, and in the second case, for example, $c_1 = 0$; the remaining constants will not be known beforehand.

In the first case, the resultant vector $(0, Y)$ of the external forces, pressing the system of stamps into the elastic body, will be given, and in the second the resultant vectors $(0, Y_k)$ will be known separately for each stamp.

It had been assumed that the stamps may only move vertically; the case, where they may also rotate, may be reduced to the preceding one in quite the same manner as in §§ 114, 114*a*. In that case one must, in addition, be given either the angles of tilt of the stamps or the resultant moments of the external forces, acting on them.

It is easily shown that the problems, as formulated above, can only have one solution, neglecting vertical rigid body displacements of the entire system; the proof is analogous to that for the case of § 114.

It should still be noted that a translation of the stamps, parallel to the boundary L, has no influence on the elastic equilibrium (within the accuracy to which one is always restricted).

It will be assumed that $f'(t)$ satisfies the H condition on each of the segments $L_k = a_k b_k$.

As in § 114, the boundary conditions (115.1) show that $\Phi(z)$ is holomorphic in the plane, cut along L'. In addition, it is easily deduced that the first of the conditions (115.1) gives by (112.14)

$$\Phi^+(t) + \bar{\Phi}^+(t) = \Phi^-(t) + \bar{\Phi}^-(t) \text{ everywhere on } L,$$

whence it follows that the function $\Phi(z) + \bar{\Phi}(z)$ is holomorphic in the entire plane; further, since it vanishes at infinity, it must vanish everywhere. Consequently

$$\bar{\Phi}(z) = -\Phi(z). \tag{115.3}$$

By (112.14)

$$Y_y^- - iX_y^- = \Phi^-(t) - \Phi^+(t),$$

whence, going to the conjugate complex expression and remembering (§ 112) that $\overline{\Phi^-(t)} = \bar{\Phi}^+(t)$, $\overline{\Phi^+(t)} = \bar{\Phi}^-(t)$,

$$Y_y^- + iX_y^- = \bar{\Phi}^+(t) - \bar{\Phi}^-(t);$$

subtracting these equations, one obtains

$$-2iX_y^- = \Phi^-(t) + \bar{\Phi}^-(t) - \Phi^+(t) - \bar{\Phi}^+(t),$$

and the above statement follows.

On the basis of this result, the boundary condition (115.2), which will now be written [cf. remarks following (113.5)]

$$v^{-\prime} = f'(t) \text{ on } L', \tag{115.4}$$

gives by (112.15)

$$\Phi^+(t) + \Phi^-(t) = \frac{4\mu i f'(t)}{\varkappa + 1} \text{ on } L'. \tag{115.5}$$

Thus one has arrived at the boundary problem of § 110, and, in fact, at the particular case, where $g = -1$. Applying (110.31) and (110.33),

one finds

$$\Phi(z) = \frac{2\mu}{\pi(\varkappa + 1)X(z)} \int_{L'} \frac{X(t)f'(t)dt}{t - z} + \frac{iP_{n-1}(z)}{X(z)}, \qquad (115.6)$$

where $iP_{n-1}(z)$ denotes an arbitrary polynomial of degree not higher than $n - 1$ and

$$X(z) = \sqrt{(z - a_1)(z - b_1)\ldots(z - a_n)(z - b_n)}; \qquad (115.7)$$

here $X(z)$ is the branch, single-valued in the plane cut along L', for which $z^{-n}X(z) \to 1$ as $z \to \infty$. In future, $X^+(t)$ will be simply denoted by $X(t)$ so that, by definition,

$$X(t) = \sqrt{(t - a_1)(t - b_1)\ldots(t - a_n)(t - b_n)} = X^+(t); \quad (115.8)$$

note also that

$$X^-(t) = -X^+(t) = -X(t). \qquad (115.9)$$

The condition (115.3), i.e., $\overline{\Phi}(z) = -\Phi(z)$, has still to be satisfied. It is readily verified that the first term on the right-hand side of (115.6) satisfies this condition; the second term will satisfy it if, and only if, all the coefficients of the polynomial $P_{n-1}(z)$ are *real*.

The first of the preceding statements may be proved as follows. Denote, for the time being, the first term of (115.6) by $\Phi_0(z)$, i.e.,

$$\Phi_0(z) = \frac{2\mu}{\pi(\varkappa + 1)X(z)} \int_{L'} \frac{X(t)f'(t)dt}{t - z};$$

remembering that, by definition, $\overline{\Phi}_0(z) = \overline{\Phi_0(\bar{z})}$, one obtains

$$\overline{\Phi}_0(z) = \frac{2\mu}{\pi(\varkappa + 1)\overline{X}(z)} \int_{L'} \frac{\overline{X(t)}f'(t)dt}{t - z},$$

because $f'(t)$ is a real function. It is easily seen that $\overline{X}(z) = X(z)$, because by (115.7) $\overline{X}(z)$ represents the same root as $X(z)$ and doubt may only arise with regard to its sign: $\overline{X}(z) = \pm X(z)$; it is seen from the behaviour of $X(z)$ and $\overline{X}(z)$ at infinity (both functions behaving for large $|z|$ like z^n) that the upper sign must be chosen. Finally, on the basis of these results and of (112.10″),

$$\overline{X(t)} = \overline{X^+(t)} = \overline{X}^-(t) = X^-(t) = -X(t),$$

and hence

$$\overline{\Phi}_0(z) = -\Phi_0(z).$$

The second statement follows because, as has just been shown,

$$\overline{X}(z) = X(z).$$

Thus the general solution of the original problem is given by (115.6), where

$$P_{n-1}(z) = D_0 z^{n-1} + D_1 z^{n-2} + \ldots + D_{n-1} \qquad (115.9)$$

must be *a polynomial with real coefficients.*

The pressure $P(t)$, exerted by the stamps on the boundary of the half-plane, will now be determined. By (112.14),

$$P(t) = -Y_y^- = \Phi^+(t) - \Phi^-(t), \qquad (115.10)$$

whence follows, applying the Plemelj formula and remembering that $X^-(t_0) = -X^+(t_0) = -X(t_0)$,

$$P(t_0) = \frac{4\mu}{\pi(\varkappa + 1)X(t_0)} \int_{L'} \frac{X(t)f'(t)dt}{t - t_0} + \frac{2iP_{n-1}(t_0)}{X(t_0)} \qquad (115.11)$$

The coefficients D_j of the polynomial $P_{n-1}(z)$ will be determined from the additional conditions, stated above when formulating the problems in the same manner as in § 114.

For example, consider the case where the resultant vectors $(0, Y_k)$ of the forces applied to the stamps separately are given. Then

$$Y_k = \int P(t_0)dt_0, \quad k = 1, 2, \ldots, n, \qquad (115.12)$$

where $P(t_0)$ is given by (115.11). One thus obtains a system of n linear equations in the unknowns $D_0, D_1, \ldots, D_{n-1}$. It is easily shown, based on the uniqueness of the solution as a whole, that this system of linear equations also has always a unique solution.

A method, completely analogous to that studied here, leads very simply to the solution of a problem, connected with the investigation of the stresses in a stratum above layers of coal; this problem was stated and solved (using more complicated means) by S. G. Mikhlin [12].

Hitherto, it was assumed that the stamps may only move vertically. As stated earlier, the case, where the stamps can rotate, is easily reduced to the preceding one.

§ 116. Application. The solution, obtained in the preceding section, will now be studied in somewhat greater detail. To simplify the in-

vestigation, the case of one stamp will be considered which comes into contact with the axis Ox along one continuous segment ab; the general case may be considered in an analogous manner.

1°. In this case one will have, instead of (115.6) and (115.11),

$$\Phi(z) = \frac{2\mu}{\pi(\varkappa + 1)\sqrt{(z-a)(b-z)}} \int_a^b \frac{\sqrt{(t-a)(b-t)}f'(t)dt}{t-z} + \frac{D}{\sqrt{(z-a)(b-z)}} \qquad (116.1)$$

and

$$P(t_0) = \frac{4\mu}{\pi(\varkappa + 1)\sqrt{(t_0-a)(b-t_0)}} \int \frac{\sqrt{(t-a)(b-t)}f'(t)dt}{t-t_0} + \frac{2D}{\sqrt{(t_0-a)(b-t_0)}}, \qquad (116.2)$$

where D is a real constant. In these formulae, the function $X(z) = \sqrt{(z-a)(z-b)}$ has been replaced by the function $\sqrt{(z-a)(b-z)}$ [cf. Note 2 at the end of § 110) and, as a consequence, the expression for $P(t)$ becomes real. For $a < t < b$, the root $\sqrt{(t-a)(b-t)}$ must be taken as a positive quantity, while $\sqrt{(z-a)(b-z)}$ must be taken as the branch, holomorphic in the plane cut along ab and taking positive values on the upper side of ab. This branch is easily seen (cf. Note 2, § 110) to be characterized by the condition that for large $|z|$

$$\sqrt{(z-a)(b-z)} = -iz + O(1). \qquad (116.3)$$

The constant D is determined by the condition

$$P(t)dt = P_0, \qquad (116.4)$$

where P_0 is the given magnitude of the forces, applied to the stamp, and $P(t)$ is given by (116.2). The constant D may be determined in a simpler manner by noting that it follows from (116.1) and (116.3) that for large $|z|$

$$\Phi(z) = \frac{Di}{z} + O\left(\frac{1}{z^2}\right),$$

whence, comparing with (112.4),

$$D = \frac{P_0}{2\pi} \tag{116.5}$$

In order that a solution may be physically possible, one must obviously have $P(t) \geqslant 0$ (for $a \leqslant t \leqslant b$). Thus, after the solution has been obtained, it must be verified whether this condition is satisfied.

It will be assumed for the solution of the problem that the segment ab of contact between the stamp and the elastic half-plane is given beforehand. This corresponds, for example, to the case where the stamp has the form shown in Fig. 53 and where the force, applied to the stamp, is sufficiently large to ensure that the corners A and B of the stamp come into contact with the elastic body. The presence of the corners A, B also explains the occurrence of infinitely large stresses at the points a, b of the elastic body which coincide with the corners A, B of the stamp.

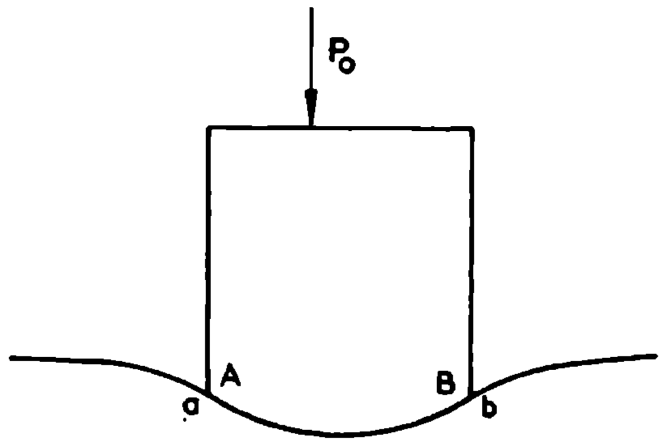


Fig. 53

2°. Considerable interest attaches to the case, where the rigid profile, pressed into the elastic half-plane, has no corners (e.g. circular disc) or where the force is not sufficiently large for the corners A and B to come into contact with the elastic body, as in Fig. 54. In that case the ends a and b of the region of contact are unknown beforehand. However, the formulae obtained above also permit solution of this problem. In fact, the general formula (116.2) for the presure $P(t_0)$ underneath the stamp now contains the two constants a, b which will not be known

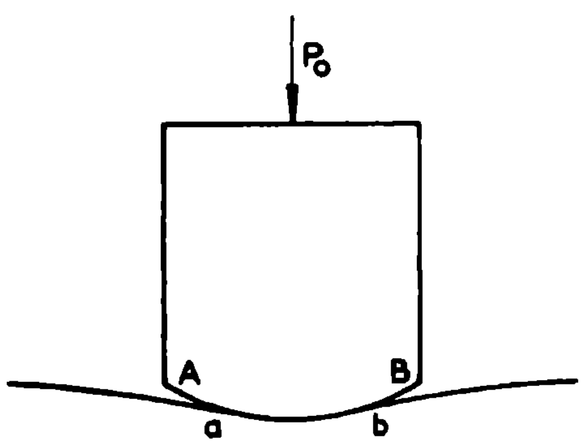


Fig. 54.

beforehand. [The constant D is given by (116.5).] For the determination of these constants one has the two relations

$$P(a) = 0, \quad P(b) = 0, \tag{116.6}$$

which express the condition that $P(t_0)$ tends to zero continuously as t_0 leaves the area of contact. This condition may be replaced by the more general one (which, in addition, is physically more obvious) that $P(t_0)$ remains bounded near the ends a, b, provided these points are not corners of the profile of the stamp. In fact, it is seen that the condition of boundedness of $P(t_0)$ near a, b entails also the relations (116.6).

In order to express the condition of boundedness of the pressure $P(t_0)$ in the neighbourhoods of the points a, b, introduce temporarily the notation

$$Q(t) = (t - a)(b - t)$$

and write (116.2) in the form

$$P(t_0) = \frac{4\mu}{\pi(\varkappa + 1)\sqrt{Q(t_0)}} \int_a^b \frac{Q(t) f'(t)\,dt}{\sqrt{Q(t)}\,(t - t_0)} + \frac{2D}{\sqrt{Q(t_0)}} =$$

$$= \frac{4\mu\sqrt{Q(t_0)}}{\pi(\varkappa + 1)} \int_a^b \frac{f'(t)\,dt}{\sqrt{Q(t)}\,(t - t_0)} +$$

$$+ \frac{4\mu}{\pi(\varkappa + 1)\sqrt{Q(t_0)}} \int_a^b \frac{Q(t) - Q(t_0)}{t - t_0} \frac{f'(t)\,dt}{\sqrt{Q(t)}} + \frac{2D}{\sqrt{Q(t_0)}},$$

or, noting that

$$\frac{Q(t) - Q(t_0)}{t - t_0} = -t - t_0 + a + b,$$

in the form

$$P(t_0) = \frac{4\mu\sqrt{Q(t_0)}}{\pi(\varkappa + 1)} \int_a^b \frac{f'(t)\,dt}{\sqrt{Q(t)}\,(t - t_0)} + \frac{At_0 + B + 2D}{\sqrt{Q(t_0)}}, \tag{a}$$

where

$$A = -\frac{4\mu}{\pi(\varkappa+1)} \int_a^b \frac{f'(t)dt}{\sqrt{Q(t)}},$$

$$B = -\frac{4\mu}{\pi(\varkappa+1)} \int_a^b \frac{tf'(t)dt}{\sqrt{Q(t)}} + \frac{4\mu(a+b)}{\pi(\varkappa+1)} \int_a^b \frac{f'(t)dt}{\sqrt{Q(t)}}.$$

The first term on the right-hand side of (a) is not only bounded near the points a, b, but it also vanishes there, as may easily be shown using the bounds for the value of a Cauchy integral near ends which are given in the Author's book [25]. Hence it is necessary and sufficient for the boundedness of $P(t_0)$ near a, b that $A = 0$, $B + 2D = 0$, or, by (116.5), that

$$\int_a^b \frac{f'(t)dt}{\sqrt{(t-a)(b-t)}} = 0, \quad \int_a^b \frac{tf'(t)dt}{\sqrt{(t-a)(b-t)}} = \frac{\varkappa+1}{4\mu} P_0. \qquad (116.7)$$

If the conditions (116.7) are satisfied, the formula (a) gives

$$P(t_0) = \frac{4\mu\sqrt{(t_0-a)(b-t_0)}}{\pi(\varkappa+1)} \int_a^b \frac{f'(t)dt}{\sqrt{(t-a)(b-t)}\,(t-t_0)} \qquad (116.8)$$

as has been stated earlier, this expression vanishes for $t_0 = a$, $t_0 = b$.

It should still be noted that it is readily verified by transforming (116.1) in the same manner as (116.2) that, under the conditions (116.7),

$$\Phi(z) = \frac{2\mu\sqrt{(z-a)(b-z)}}{\pi(\varkappa+1)} \int_a^b \frac{f'(t)dt}{\sqrt{(t-a)(b-t)}\,(t-z)} \qquad (116.9)$$

It may be added that (116.9) as well as (116.7) could have been obtained by seeking a solution (for the particular case $n = 1$) of the boundary problem (115.5), which remains bounded near the ends, and by applying the relevant formulae of § 110.

In that approach the first condition of (116.7) coincides with the condition for the existence of such a solution, while the second condition of (116.7) expresses that the coefficient of z^{-1} in the expansion of $\Phi(z)$ for large $|z|$ must be equal to a given quantity, determined by the magnitude P_0 of the force applied to the stamp.

Thus one has for the determination of a and b the two conditions (116.7) which, in general, determine a, b uniquely, provided the condition $P(t) \geqslant 0$ underneath the stamp has been observed (cf. § 116a).

Hitherto the stamp was restrained to move vertically. The case, where it may tilt, can be studied in an anologous manner (cf. § 116a).

§ 116a. Examples.

1°. Stamp with straight horizontal base

In this case $f'(t) = 0$ and (116.1) gives, using (116.5) and writing $a = -l$, $b = l$, where $2l$ is the width of the base,

$$\Phi(z) = \frac{P_0}{2\pi\sqrt{l^2 - z^2}}. \tag{116.1a}$$

One finds for the pressure $P(t)$ underneath the stamp, using the formula $P(t) = \Phi^+(t) - \Phi^-(t)$,

$$P(t) = \frac{P_0}{\pi\sqrt{l^2 - t^2}}. \tag{116.2a}$$

This solution was obtained (by other means) by M. A. Sadovski [1].

2°. Stamp with straight inclined base

Let ε be the angle of tilt (cf. § 114a, 2° and Fig. 52). In that case $f'(t) = i\varepsilon$ and, by (116.1) and (116.5) (assuming $a = -l$, $b = l$),

$$\Phi(z) = \frac{2\mu\varepsilon}{\pi(\varkappa + 1)\sqrt{l^2 - z^2}} \int_{-l}^{+l} \frac{\sqrt{l^2 - t^2}dt}{t - z} + \frac{P_0}{2\pi\sqrt{l^2 - z^2}}.$$

The integral on the right-hand side may be calculated in closed form by (110.40). In the present case, $g = -1$, $1/X_p(z) = \sqrt{l^2 - z^2}$; since, by (116.3),

$$iz\left(1 - \frac{l^2}{z^2}\right)^{\frac{1}{2}} = -iz\left(1 - \frac{l^2}{2z^2} + \dots\right) =$$
$$= -iz + \frac{il^2}{2z} + \dots,$$

formula (110.40) gives

$$\int_{-l}^{+l} \frac{\sqrt{l^2 - t^2}}{t - z} dt = \pi i(\sqrt{l^2 - z^2} + iz).$$

Hence

$$\Phi(z) = \frac{2\mu\varepsilon z}{(\varkappa + 1)\sqrt{l^2 - z^2}} + \frac{2\mu\varepsilon i}{\varkappa + 1} + \frac{}{2\pi\sqrt{l^2 - z^2}} \tag{116.3a}$$

Using the relation $P(t) = \Phi^+(t) - \Phi^-(t)$, one obtains for the pressure $P(t)$ underneath the stamp

$$P(t) = \frac{P_0}{\pi\sqrt{l^2 - t^2}} \qquad \frac{4\mu\varepsilon t}{(\varkappa + 1)\sqrt{l^2 - t^2}} \tag{116.4a}$$

The solution will be physically possible, if $P(t) > 0$ for $-l < t < l$, i.e., if

$$P_0 \geqslant \frac{4\pi\mu l}{\varkappa + 1}\varepsilon. \tag{116.5a}$$

The resultant moment of the external forces restraining the stamp in the given position is likewise easily calculated by the formula

$$M = -\int_{-l}^{+l} tP(t)dt.$$

In fact, applying the same method as in § 114*a*, 2° or evaluating the integral by ordinary means, it is found that

$$M = \frac{2\pi\mu l^2}{\varkappa + 1}\varepsilon. \tag{116.6a}$$

3°. Stamp with curved base

Let the stamp be represented by a strip, bounded by the vertical straight lines $x = -l$, $x = +l$ and by an arc AB of a circle with radius R and convex downwards (cf. Figs. 53 and 54). It will be assumed that the radius R is very large. This assumption is necessary, because small deformations are being considered. With the usual degree of approximation, one may write

$$f(t) = \frac{t^2}{2R}.$$

This implies replacement of the arc of the circle by that of a parabola, having the same curvature at the vertex.

Then (116.1) and (116.5) give, under the supposition that the entire

arc AB is in contact with the elastic body,

$$\Phi(z) = \frac{2\mu}{R\pi(\varkappa + 1)\sqrt{l^2 - z^2}} \int_{-l}^{+l} \frac{t\sqrt{l^2 - t^2}}{t - z}\, dt + \frac{P_0}{2\pi\sqrt{l^2 - z^2}}$$

The integral on the right-hand side may again be evaluated in closed form; in fact, since for large $|z|$

$$z\sqrt{l^2 - z^2} = -iz^2 + \frac{il^2}{2} + O\left(\frac{1}{z}\right),$$

one obtains, by (110.40),

$$\int_{-l}^{+l} \frac{t\sqrt{l^2 - t^2}}{t - z}\, dt = \pi i \left\{z\sqrt{l^2 - z^2} + iz^2 - \frac{il^2}{2}\right\},$$

and hence

$$\Phi(z) = \frac{\mu(l^2 - 2z^2)}{R(\varkappa + 1)\sqrt{l^2 - z^2}} + \frac{2\mu iz}{R(\varkappa + 1)} + \frac{P_0}{2\pi\sqrt{l^2 - z^2}}. \qquad (116.7a)$$

By the formula $P(t) = \Phi^+(t) - \Phi^-(t)$, the pressure underneath the stamp becomes

$$P(t) = \frac{2\mu(l^2 - 2t^2)}{R(\varkappa + 1)\sqrt{l^2 - t^2}} + \frac{P_0}{\pi\sqrt{l^2 - t^2}}. \qquad (116.8a)$$

The solution will be physically possible, if $P(t) > 0$ for $-l < t < l$, i.e., if

$$P_0 > \frac{2\pi\mu}{R(\varkappa + 1)}\, l^2. \qquad (116.9a)$$

If P_0 does not satisfy the preceding condition, this means that the force of magnitude P_0 is not sufficient to bring the arc AB of the stamp into complete contact with the elastic body. The arc $A'B'$, which actually engages the elastic body for some given $P_0 < 2\pi\mu l^2/R(\varkappa + 1)$, will now be found.

From symmetry, it is obvious that the segment $a'b'$ of the boundary of the elastic half-plane which enters into contact has its centre at the origin, so that one may write $a' = -l'$, $b' = l'$, where $2l'$ is the length of the segment $a'b'$. The function $\Phi(z)$ and the pressure $P(t)$, correspond-

ing to a given l', will be obtained by replacing in (116.7*a*) and (116.8*a*) l by l'. Expressing that $P(t) = 0$ for $t = \pm l'$, one obtains

$$l' = \frac{\sqrt{P_0 R(\varkappa + 1)}}{\sqrt{2\pi\mu}}. \qquad (116.10a)$$

It is sufficient to express that $P(t)$ remains bounded for $t = \pm l'$; the result would have been the same, as was to be expected on the basis of the statements in § 116.

Alternatively, one may assume that l' is known and calculate the magnitude P_0 of the force necessary to make the length of the line of contact equal to $2l'$. Corresponding to a given l', the functions $\Phi(z)$ and $P(t)$ are determined by the formulae

$$\Phi(z) = \frac{2\mu\sqrt{l'^2 - z^2}}{R(\varkappa + 1)} + \frac{2\mu i z}{R(\varkappa + 1)}, \qquad (116.11a)$$

$$P(t) = \frac{4\mu\sqrt{l'^2 - t^2}}{R(\varkappa + 1)}. \qquad (116.12a)$$

§ 117. Equilibrium of a rigid stamp on the boundary of an elastic half-plane in the presence of friction.

This section reproduces, with minor modifications, the Author's paper [24]. At about the same time, N. I. Glagolev [1] published a paper in which he gave the solution of the problem under consideration for the particular case of a stamp with a straight base. Somewhat later, N. I. Glagolev [2] gave the solution for the case where the profile of the stamp is of arbitrary shape and where the friction may depend on the area of contact. In a recently published paper, L. A. Galin [1] gave a somewhat different method of solution (also applicable to an anisotropic body).

The problem of the equilibrium of a rigid stamp on the boundary of an elastic half-plane has been solved in the preceding sections for the two extreme cases, where the coefficient of friction is zero (§§ 115—116) or infinite (§ 114); in the latter case, a further condition had to be imposed, namely, that the elastic material could not leave the stamp and that thus the presence of negative pressures, even arbitrarily large ones, is admissible.

Using the method of the preceding sections, one may also solve the problem for a finite coefficient of friction such as will occur in reality. In this context, consideration will be limited to the case where the stamp is on the verge of equilibrium; obviously, the solution for this case will be

an approximation to the case, where the stamp slides slowly along the boundary of the half-plane. More exactly, it will be assumed that $T = kP$ underneath the stamp on the boundary of the half-plane, where P and T are respectively the pressure and the tangential stress, applied to points of the boundary of the half-plane, and k is the coefficient of friction which will be assumed constant.

Recently L. A. Galin [2] gave a clever solution of the problem of the impression made by a rigid stamp with a plane base under the supposition that the segment of contact consists of three parts: a centre section with cohesion and two outer sections on which slip occurs. In a simultaneously published paper, S. V. Falkovicz [1] gave the solution of the same problem under the supposition that on the parts, where slips occurs, friction is absent.

As before, let the Ox axis be the boundary of the elastic half-plane and the Oy axis perpendicular to it, so that the elastic body occupies the lower half-plane $y < 0$. For this choice of axes, $P = -Y_x^-$, $T = X_y^-$.

Further, assume that the stamp engages the elastic half-plane along one continuous segment $L' = ab$. The result below is easily generalized to the case, where the region of contact consists of a finite number of individual segments (cf. preceding sections).

In addition, it will be assumed that the stamp may only move vertically. As in the preceding sections, the case where the stamp may tilt is readily reduced to this case; cf. § 116*a*, 2°.

The boundary conditions of the present problem have the form

$$T(t) = kP(t), \tag{117.1}$$

$$v^- = f(t) + \text{const.} \tag{117.2}$$

on L', $T(t) = P(t) = 0$ outside L' on Ox. As before, t denotes here the abscissa of a point on the Ox axis, v is the projection of the displacement on the Oy axis, $f(t)$ is a given function for the profile of the stamp, i.e., $y = f(x)$ is the equation of this profile. It will be assumed that $f(t)$ has a derivative $f'(t)$, satisfying the H condition.

In addition, it will be assumed that the quantity

$$P_0 = \int_{L'} P(t)dt, \tag{117.3}$$

i.e., the total pressure exerted by the stamp on the half-plane, is known. The total tangential stress will then obviously be $T_0 = kP_0$. Thus the resultant vector $(X, Y) = (T_0, -P_0)$ of the external forces, acting

on the stamp and balanced by the reactions of the elastic half-plane, will be given.

In the notation and under the general suppositions of § 112, the boundary conditions (117.1) and (117.2) of the present problem may be written, by (112.14) and (112.15) and using, as before, instead of (117.2) the condition $v^{-\prime} = f'(t)$,

$$(1 - ik)\Phi^+(t) + (1 + ik)\overline{\Phi}^+(t) = (1 - ik)\Phi^-(t) + (1 + ik)\overline{\Phi}^-(t), \quad (117.4)$$

$$\varkappa\Phi^-(t) + \Phi^+(t) - \varkappa\overline{\Phi}^+(t) - \overline{\Phi}^-(t) = 4i\mu f'(t) \quad (117.5)$$

on L', while the condition $P(t) = T(t) = 0$ on Ox outside L' is equivalent to the condition that the function $\Phi(z)$ must be holomorphic outside the segment $L' = ab$.

The formula (117.4) shows that the function $(1 - ik)\Phi(z) + (1 + ik)\overline{\Phi}(z)$ is holomorphic in the entire plane; since it must vanish at infinity, one has

$$(1 - ik)\Phi(z) + (1 + ik)\overline{\Phi}(z) = 0 \quad (117.6)$$

in the whole plane. Expressing by the help of (117.6) the function $\overline{\Phi}(z)$ in terms of $\Phi(z)$ and substituting this value in (117.5), one obtains the boundary condition for $\Phi(z)$

$$\Phi^+(t) = g\Phi^-(t) + f_0(t) \text{ on } L', \quad (117.7)$$

where

$$g = -\frac{\varkappa + 1 + ik(\varkappa - 1)}{\varkappa + 1 - ik(\varkappa - 1)}, \quad f_0(t) = \frac{4i\mu(1 + ik)f'(t)}{\varkappa + 1 - ik(\varkappa - 1)}.$$

These last formulae may be simplified by the introduction of the constant α, determined by the conditions (remembering that $\varkappa > 1$, $k > 0$)

$$\tan \pi\alpha = k\frac{\varkappa - 1}{\varkappa + 1}, \quad 0 \leqslant \alpha < \tfrac{1}{2}. \quad (117.8)$$

Then

$$\varkappa + 1 \pm ik(\varkappa - 1) = \sqrt{(\varkappa + 1)^2 + k^2(\varkappa - 1)^2}\, e^{\pm\pi i\alpha} = \frac{(\varkappa + 1)e^{\pm\pi i\alpha}}{\cos \pi\alpha}$$

and hence

$$g = -e^{2\pi i\alpha}, \quad f_0(t) = \frac{4i\mu(1 + ik)e^{\pi i\alpha}\cos \pi\alpha}{\varkappa + 1} f'(t). \quad (117.9)$$

Applying now the method of § 110 to solve the problem (117.7), noting that in the present case

$$\gamma = \frac{\log g}{2\pi i} = \tfrac{1}{2} + \alpha$$

and that one may take for the function $X_0(z)$ in (110.18) the expression $(z-a)^{-\frac{1}{2}-\alpha}\,(b-z)^{-\frac{1}{2}+\alpha}$ (cf. § 110, Note 2), one obtains

$$\Phi(z) = \frac{2\mu(1+ik)e^{\pi i\alpha}\cos\pi\alpha}{\pi(\varkappa+1)\,(z-a)^{\frac{1}{2}+\alpha}\,(b-z)^{\frac{1}{2}-\alpha}}\int_a^b \frac{(t-a)^{\frac{1}{2}+\alpha}\,(b-t)^{\frac{1}{2}-\alpha}f'(t)dt}{t-z} +$$

$$+ \frac{C_0}{(z-a)^{\frac{1}{2}+\alpha}\,(b-z)^{\frac{1}{2}-\alpha}}, \qquad (117.10)$$

where C_0 is a constant and where $(z-a)^{\frac{1}{2}+\alpha}\,(b-z)^{\frac{1}{2}-\alpha}$ must be understood to be that branch which is holomorphic outside the segment ab and which takes on the upper side of this segment the real, positive value $(t-a)^{\frac{1}{2}+\alpha}\,(b-t)^{\frac{1}{2}-\alpha}$; as is easily seen, this branch is characterized by the fact that

$$\lim \frac{(z-a)^{\frac{1}{2}+\alpha}\,(b-z)^{\frac{1}{2}-\alpha}}{z} = -ie^{\pi i\alpha}. \qquad (117.11)$$

For the determination of γ, it must be remembered that by the condition, introduced in § 110, the value of the logarithm must be taken for which

$$0 < \Re\frac{\log g}{2\pi i} < 1.$$

The quantity, denoted in § 110 by α, is now denoted by $\frac{1}{2}+\alpha$, and the function $f(t)$ by $f_0(t)$.

The constant C_0 is directly determined by (112.4) which gives

$$\lim_{z\to\infty} z\Phi(z) = \frac{-T_0 + iP_0}{2\pi} = \frac{iP_0(1+ik)}{2\pi},$$

whence, by (117.11),

$$C_0 = \frac{P_0(1+ik)e^{\pi i\alpha}}{2\pi}$$

and (117.10) becomes

$$\Phi(z) = \frac{2\mu(1 + ik)e^{\pi i\alpha}\cos\pi\alpha}{\pi(\varkappa + 1)\,(z - a)^{\frac{1}{2}+\alpha}\,(b - z)^{\frac{1}{2}-\alpha}} \int^{b} \frac{(t - a)^{\frac{1}{2}+\alpha}\,(b - t)^{\frac{1}{2}-\alpha} f'(t)dt}{t - z} +$$

$$+ \frac{P_0(1 + ik)e^{\pi i\alpha}}{2\pi(z - a)^{\frac{1}{2}+\alpha}(b - z)^{\frac{1}{2}-\alpha}}. \qquad (117.12)$$

It is readily verified that all the conditions of the problem will be satisfied, provided, as has been assumed, $f'(t)$ satisfies the H condition on L'. Thus the problem is solved, because $\Phi(z)$ completely characterizes the state of stress.

Naturally, the solution will be physically possible only in the case where the pressure $P(t)$ at the points t underneath the stamp satisfies the condition $P(t) \geqslant 0$. The pressure is easily calculated on the basis of (117.12). In fact, by (112.14),

$$P(t_0) + iT(t_0) = P(t_0)(1 + ik) = \Phi^+(t_0) - \Phi^-(t_0).$$

This last difference, using the Plemelj formula, gives

$$P(t_0) \qquad \frac{4\mu\sin\pi\alpha\cos\pi\alpha}{\varkappa + 1} f'(t_0) +$$

$$+ \frac{4\mu\cos^2\pi\alpha}{\pi(\varkappa + 1)\,(t_0 - a)^{\frac{1}{2}+\alpha}(b - t_0)^{\frac{1}{2}-\alpha}} \int^{b} \frac{(t - a)^{\frac{1}{2}+\alpha}(b - t)^{\frac{1}{2}-\alpha} f'(t)dt}{t - t_0} +$$

$$+ \frac{P_0\cos\pi\alpha}{\pi(t_0 - a)^{\frac{1}{2}+\alpha}(b - t_0)^{\frac{1}{2}-\alpha}}. \qquad (117.13)$$

For $k = 0$ (when also $\alpha = 0$), one obtains again the solution for the idealized case without friction.

§ 117a Examples.

1°. Stamp with straight, horizontal base.
In this case $f'(t) = 0$ and (117.12), (117.13) give

$$\Phi(z) = \frac{P_0(1 + k)e^{\pi}}{2\pi(z - a)^{\frac{1}{2}+\alpha}(b - z)^{\frac{1}{2}-\alpha}}, \qquad (117.1a)$$

$$P(t) = \frac{P_0\cos\pi\alpha}{\pi(t - a)^{\frac{1}{2}+\alpha}(b - t)^{\frac{1}{2}-\alpha}}. \qquad (117.2a)$$

2°. Stamp with straight, inclined base.

Let the base of the stamp form a (small) angle ε with the axis Ox. Then $f(t) = \varepsilon t + \text{const.}$, $f'(t) = \varepsilon$. Substituting in (117.2) and putting, for simplicity, $a = -l$, $b = +l$, one obtains by (110.40)

$$\Phi(z) = \frac{(1+ik)e^{\pi i\alpha}}{2\pi(\varkappa+1)} \frac{P_0(\varkappa+1) - 8\pi\mu\varepsilon\alpha l - 4\pi\mu\varepsilon z}{(l+z)^{\frac{1}{2}+\alpha}(l-z)^{\frac{1}{2}-\alpha}} + \frac{2\mu\varepsilon i(1+ik)}{\varkappa+1}. \quad (117.3a)$$

In the present case, one has for large $|z|$

$$\frac{1}{X_p(z)} = (l+z)^{\frac{1}{2}+\alpha}(l-z)^{\frac{1}{2}-\alpha} = -ie^{\pi i\alpha}(z+2l\alpha) + O(1/z).$$

The pressure $P(t)$ is given by

$$P(t) = \frac{\Phi^+(t) - \Phi^-(t)}{1+ik} = \frac{\cos\pi\alpha}{\pi(\varkappa+1)} \frac{P_0(\varkappa+1) - 8\pi\mu\varepsilon\alpha l - 4\pi\mu\varepsilon t}{(l+t)^{\frac{1}{2}+\alpha}(l-t)^{\frac{1}{2}-\alpha}}. \quad (117.4a)$$

The solution is physically possible, i.e., $P(t) \geqslant 0$ for $-l \leqslant t \leqslant l$, when

$$-\frac{P_0(\varkappa+1)}{4\pi\mu l(1-2\alpha)} \leqslant \varepsilon \leqslant \frac{P_0(\varkappa+1)}{4\pi\mu l(1+2\alpha)}. \quad (117.5a)$$

The moment

$$M = -\int_{-l}^{+l} tP(t)dt$$

is likewise easily determined; it is equal to the resultant moment of the external forces, acting on the stamp; in fact, proceeding as in § 114*a*, 2°, one obtains

$$M = 2\alpha l P_0 + \frac{2\pi\mu(1-4\alpha^2)l^2}{\varkappa+1}\varepsilon. \quad (117.6a)$$

This last formula determines ε for given M and P_0. In particular, if $M = 0$, i.e., if the external forces, acting on the stamp, are equivalent to a force applied to the centre of the base, then

$$\varepsilon = -\frac{\alpha(\varkappa+1)P_0}{\pi\mu l(1-4\alpha^2)}; \quad (117.7a)$$

since $0 < \alpha < \frac{1}{2}$, this value of ε leads, by (117.5*a*), to a possible solution.

§ 118. An alternative method for the solution of the boundary problems for the half-plane. In the preceding sections the solution of the boundary problems for the lower half-plane has been reduced to the search for a function $\Phi(z)$, suitably extended into the upper half-plane.

However, it is obvious (cf. the methods of solution, studied in the preceding Part) that these problems could be reduced directly to the search for a function

$$\varphi(z) \quad \int \Phi(z)dz,$$

likewise extended into the upper half-plane [cf. (112.16)]. Some inconvenience would result from the fact that, in general, $\varphi(z)$ is multi-valued. However, this inconvenience may be removed by separating from $\varphi(z)$ the multi-valued part, which is very simply done. On the other hand, introduction of the function $\varphi(z)$ instead of $\Phi(z)$ has the advantage that in constructing the boundary conditions, involving the boundary values of the displacements, one is not obliged beforehand to differentiate these values.

§ 119. Problem of contact of two elastic bodies (generalized plane problem of Hertz).

Consider two elastic bodies S_1, S_2 the shapes of which approximate to half-planes and which are in contact along a segment ab of their boundaries (Fig. 55). The segment of contact ab will not be given beforehand,

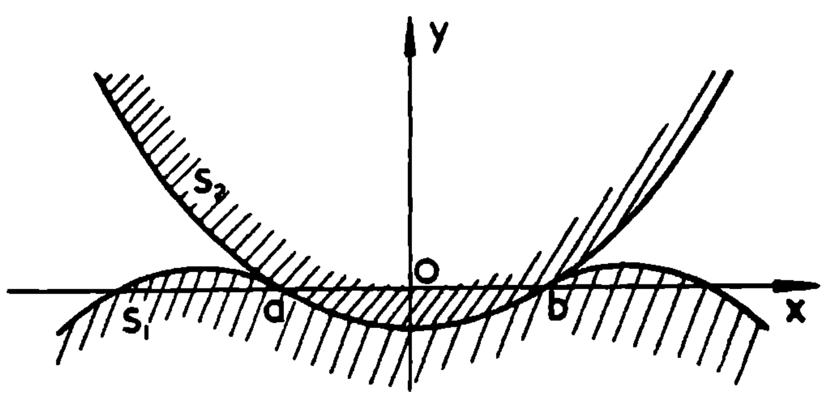


Fig. 55.

but will be subject to determination. The shapes of the boundaries (approximately straight lines) before deformation and the resultant vector of the external forces, applied, for example, by the body S_1 to the body S_2, will be initially known. It will be assumed that there is no

friction and, in addition, that the stresses and rotations in S_1 and S_2 vanish at infinity.

This problem, which is of great independent interest, is even more important, because it constitutes the problem of contact of two bodies of arbitrary shape (i.e., having the two-dimensional case in mind), provided the area of contact is very small in comparison with the dimensions of the bodies; in that case, if one is interested in the stresses and displacements near the region of contact, one can assume without appreciable error that the bodies under consideration are in shape close to half-planes.

In three dimensions, the problem of contact of two elastic bodies was first formulated and solved by Hertz under several restrictive assumptions; in particular, he assumed that the area of contact is very small and that the equations of the undeformed surfaces near the region of contact could be approximated sufficiently accurately by functions of the form $z = Ax^2 + 2Bxy + Cy^2$ with a suitable choice of coordinate axes. I. Ya. Shtaerman [2] reduced the three-dimensional Hertz problem, under more general suppositions, to an integral equation.

Thus the problem of contact of two bodies, approximating to half-planes, which was stated above is the two-dimensional analogue of Hertz' problem, however in a somewhat generalized form, since it has not been assumed that the region of contact is small and, correspondingly, no assumption regarding the shapes of the boundaries has been made, except the condition that they should approximate to straight lines (and be sufficiently smooth).

In recent times, several authors have considered this problem. I. Ya. Shtaerman [1, 3] (cf. also his recent book [4]) reduced it to a Fredholm Equation of the first kind which in the present notation may be written

$$\int_a^b P(t) \log |t - t_0| \, dt = f(t_0) + \text{const.}, \tag{a}$$

where $P(t)$ is the unknown pressure, exerted by one body on the other at a point t of the region of contact of the bodies, and $f(t)$ is a given function. The problem of pressure of a rigid stamp on an elastic half-plane, as treated in the earlier editions of this book (§ 87), led to just that equation; this equation is easily solved by quadrature for given a and b (cf. the earlier editions of this book, § 88).

A. V. Bitzadze [1] reduced this problem to a singular integral equation

which is immediately solved in closed form. His equation may be obtained by differentiating equation (a).

The following solution of the problem is obtained by a method, completely analogous to that used earlier in the case where one of the contacting bodies was absolutely rigid (§ 115).

It will be assumed that the body S_1 occupies the lower half-plane S^- and the body S_2 the upper half-plane S^+ and the corresponding stresses and displacements, and likewise the constants λ, μ, $\varkappa$, will be provided with the subscripts 1 and 2.

Let $\Phi_1(z)$ be a sectionally holomorphic function which corresponds to the body S_1 and is defined as in § 112; let $\Phi_2(z)$ be the analogous function for the body S_2. These functions are holomorphic throughout the plane, except at the segment ab of the Ox axis, because outside ab the boundaries of the bodies are free from external stresses. As, by supposition, there is no friction, one will have $[X_y^-]_1 = 0$ on Ox; hence, as in § 115, one may conclude that $\overline{\Phi}_1(z) = -\Phi_1(z)$; similarly, one finds that $\overline{\Phi}_2(z) = -\Phi_2(z)$. Further, if $P(t)$ is the pressure exerted by one body on the other at the point t, then, as in § 115,

$$P(t) = \Phi_1^+(t) - \Phi_1^-(t); \tag{119.1}$$

similarly, one has

$$P(t) = \Phi_2^-(t) - \Phi_2^+(t). \tag{119.2}$$

It is seen from these conditions that $[\Phi_1 + \Phi_2]^+ = [\Phi_1 + \Phi_2]^-$, i.e., that the sum $\Phi_1(z) + \Phi_2(z)$ is holomorphic in the entire plane; further, since it vanishes at infinity, one must have $\Phi_1(z) + \Phi_2(z) = 0$. Thus

$$\overline{\Phi}_1(z) = -\Phi_1(z), \quad \overline{\Phi}_2(z) = -\Phi_2(z), \quad \Phi_2(z) = -\Phi_1(z). \tag{119.3}$$

If now

$$y = f_1(t), \quad y = f_2(t)$$

are the equations of the boundaries of the bodies S_1 and S_2 before deformation, where $f_1(t)$, $f_2(t)$ as well as their derivatives $f_1'(t)$, $f_2'(t)$ must be small, one will have in the region of contact, after deformation,

$$f_1(t) + v_1^-(t) = f_2(t) + v_2^+(t),$$

whence

$$v_1^- - v_2^+ = f(t) \text{ on } ab \tag{119.4}$$

or

$$v_1^{-\prime} - v_2^{+\prime} = f'(t), \tag{119.5}$$

where

$$f(t) = f_2(t) - f_1(t). \tag{119.6}$$

It will be assumed that $f'(t)$ satisfies the H condition.

With regard to the deduction of the condition (119.4), reference should be made to § 115. Strictly speaking, one should have written $f_1(t) + v_1^- = F(t + u_1^-)$, $f_2(t) + v_2^+ = F(t + u_2^+)$, where $y = F(t)$ is the equation of the line of contact after deformation; however, within the accuracy considered here, it may be assumed that $F(t + u_1^-) = F(t + u_2^+)$, in which case one obtains the stated relation.

Expressing now the boundary condition (119.4) by means of (112.7), applied to S_1 and S_2 respectively, one finds, in view of (119.3),

$$\Phi_1^+(t) + \Phi_1^-(t) = \frac{if'(t)}{K}, \tag{119.7}$$

where

$$K = \frac{\varkappa_1 + 1}{4\mu_1} + \frac{\varkappa_2 + 1}{4\mu_2}. \tag{119.8}$$

One has thus arrived at the same mathematical boundary problem as in the case of the problem of pressure of an absolutely rigid stamp on a half-plane, i.e., at the problem, corresponding to the boundary condition (115.5); the only difference is that in this formula $\Phi_1(z)$ and $1/K$ take the place of $\Phi(z)$ and $4\mu/\varkappa + 1$ respectively. In addition, in the present case, the segment of contact is not known beforehand and, as in § 116, 2°, it is required to find the solution $\Phi_1(z)$, vanishing at infinity and bounded near the ends a, b.

Using the formulae of § 116 or directly those of § 110, one arrives at the following conclusions.

The function $\Phi_1(z)$ is given by [cf. (116.9)]

$$\Phi_1(z) = \frac{\sqrt{(z-a)(b-z)}}{2\pi K} \int_a^b \frac{f'(t)dt}{\sqrt{(t-a)(b-t)}\,(t-z)}$$

For the determination of a and b one has the two relations [cf. (116.7)]

$$\int_a^b \frac{f'(t)dt}{\sqrt{(t-a)(b-t)}} = 0 \tag{119.10}$$

and

$$\int_a^b \frac{tf'(t)dt}{\sqrt{(t-a)(b-t)}} = KP_0, \tag{119.11}$$

where P_0 is the magnitude of the resultant vector of the external forces applied by the body S_2 to S_1 (or S_1 to S_2) which will be assumed known.

As in § 116, $\sqrt{(z-a)(b-z)}$ must be interpreted as a branch such that for large $|z|$

$$\sqrt{(z-a)(b-z)} = -iz + O(1), \tag{119.12}$$

and $\sqrt{(t-a)(b-t)}$ for $a < t < b$ refers to the positive value of the root.

The pressure $P(t_0) = \Phi_1^+(t_0) - \Phi_1^-(t_0)$ is given by

$$P(t_0) = \frac{\sqrt{(t_0-a)(b-t_0)}}{\pi K} \int_a^b \frac{f'(t)dt}{\sqrt{(t-a)(b-t)}} \tag{119.13}$$

If the function $f(t)$ is even, i.e., if

$$f(-t) = f(t), \tag{119.14}$$

one may, from considerations of symmetry, write from the beginning $a = -l$, $b = +l$, where l is subject to determination. In this case the condition (119.10) is automatically satisfied and l may be determined from the equation

$$\int^l \frac{tf'(t)dt}{\sqrt{l^2-t^2}} = \tfrac{1}{2} KP_0. \tag{119.11'}$$

The final formulae, obtained for (119.14) and $a = -b$, agree with those deduced by A. V. Bitzadze [1].

As shown in § 110, Note 2, the integrals in the preceding formulae may be evaluated by elementary means, provided $f'(t)$ is a rational function or, in particular, a polynomial. For example, let

$$f(t) = At^{2n},$$

where A is a constant and n a positive integer; one then obtains immediately the solution, found by I. Ya. Shtaerman [1]. Putting

$$f(t) = \frac{t^2}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

which corresponds to the case, where S_1 and S_2 are bounded by circles with radii R_1 and R_2 (which are large as compared with the region of contact), one obtains the solution, found by L. Föppl [1] by other means. (As regards the calculations, this case is the same as in § 116a, 3°.)

Several other examples may be found in the paper [3] and the book [4] by I. Ya. Shtaerman.

In the presence of friction between the bodies in contact, the problem is considerably more complicated. Solution of several problems of contact in the presence of friction which are of special practical interest has been given by N. I. Glagolev; some of these results are published in his paper [3].

§ 120. Boundary problems for the plane with straight cuts *).

The fundamental boundary problems, and likewise some other problems, for the case where the region, occupied by the body, is the entire plane with straight cuts, distributed along one and the same straight line, are easily solved by methods, analogous to those used in the preceding sections. Let the Ox axis be the locus of the cuts. A beginning will be made with the deduction of several formulae, analogous to those of § 112.

1°. General formulae

Let the region S', occupied by the elastic body, be the entire plane, cut along n segments $L_k = a_k b_k$ $(k = 1, \ldots, n)$ of the Ox axis; the union of these segments will now be denoted by L.

It will not be assumed in this section that the stresses vanish, but only that they are *bounded at infinity*.

Then $\Phi(z)$ and $\Psi(z)$ are holomorphic in S', including the point at infinity, and for large $|z|$, by (36.4) and (36.5),

$$\begin{aligned} \Phi(z) &= \Gamma - \frac{X + iY}{2\pi(1 + \varkappa)} \frac{1}{z} + O\left(\frac{1}{z^2}\right), \\ \Psi(z) &= \Gamma' + \frac{\varkappa(X - iY)}{2\pi(1 + \varkappa)} \frac{1}{z} + O\left(\frac{1}{z^2}\right), \end{aligned} \qquad (102.1)$$

where (X, Y) is the resultant vector of the external forces, applied to the edges of L,

$$\Gamma = B + iC, \quad \Gamma' = B' + iC' \qquad (120.2)$$

*) This section reproduces, almost without modifications, the contents of the Author's paper [23].

are constants, determined by (§ 36)

$$B = \tfrac{1}{4}(N_1 + N_2), \quad C = \frac{2\mu\varepsilon_\infty}{1 + \varkappa}, \quad \Gamma' = -\tfrac{1}{2}(N_1 - N_2)e^{-2i\alpha} \tag{120.3}$$

N_1, N_2 being the values of the principal stresses at infinity, α the angle between N_1 and the Ox axis and ε_∞ the magnitude of the rotation at infinity.

In the usual notation, introduce the function

$$\Omega(z) = \overline{\Phi}(z) + z\overline{\Phi}'(z) + \overline{\Psi}(z), \tag{120.4}$$

which is also holomorphic in S' and has, by (120.1), for large $|z|$ the form

$$\Omega(z) = \overline{\Gamma} + \overline{\Gamma}' + \frac{\varkappa(X + iY)}{2\pi(1 + \varkappa)} \frac{1}{z} + O\left(\frac{1}{z^2}\right). \tag{120.5}$$

Substituting in (120.4) $\bar{z}$ for z and taking the conjugate complex value, one obtains

$$\Psi(z) = \overline{\Omega}(z) - \Phi(z) - z\Phi'(z). \tag{120.6}$$

Since the stress components are expressed in terms of the functions $\Phi(z)$, $\Psi(z)$, one may also express them in terms of $\Phi(z)$ and $\Omega(z)$.

In particular, one has by (32.8)

$$Y_y - iX_y = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}. \tag{120.7}$$

Similarly, one may express the components of displacement by introducing instead of $\psi(z)$ the function

$$\omega(z) = \int \Omega(z)dz = z\overline{\Phi}(z) + \overline{\psi}(z) + \text{const.} \tag{120.8}$$

which, like the functions $\varphi(z)$, $\psi(z)$, is determined by $\Phi(z)$, $\Psi(z)$, apart from an additive constant. Thus (32.1) takes the form

$$2\mu(u + iv) = \varkappa\varphi(z) - \omega(\bar{z}) - (z - \bar{z})\overline{\Phi(z)} + \text{const.} \tag{120.9}$$

It will be assumed in the sequel that $\Phi(z)$, $\Omega(z)$ are sectionally holomorphic in the sense of the definition in § 106, so that, in particular, near the ends a_k, b_k

$$\Phi(z) < \frac{A}{|z - c|^\alpha}, \quad |\Omega(z)| < \frac{A}{|z - c|^\alpha}. \tag{120.10}$$

where A, α are positive constants, $0 \leqslant \alpha < 1$, and c denotes the cor-

responding end. Further, it will be assumed that for all t on L which do not coincide with ends

$$\lim_{y\to 0} y\Phi'(t+iy) = 0. \tag{120.11}$$

2°. First fundamental problem

Consider now the solution of the first fundamental problem, i.e., assume the values of Y_y^+, X_y^+, Y_y^-, X_y^- on L to be given, where the $(+)$ and $(-)$ signs, as always, refer to the boundary values on the upper and lower edges of the cuts. A (less simple) solution of this problem was given by D. I. Sherman [12].

In addition, it will be assumed that the constants $\Re\Gamma = B$ and $\Gamma' = B' + iC'$, i.e., the values of the stresses at infinity, are known. Since one is concerned with the stress distribution, one may, without affecting generality, assume that $C = 0$, i.e., that

$$\Gamma = \bar{\Gamma} = B.$$

By (120.7) and (120.11), the boundary conditions take the form

$$\Phi^+(t) + \Omega^-(t) = Y_y^+ - iX_y^+, \quad \Phi^-(t) + \Omega^+(t) = Y_y^- - iX_y^- \tag{120.12}$$

on L. Adding and subtracting, one obtains

$$[\Phi(t) + \Omega(t)]^+ + [\Phi(t) + \Omega(t)]^- = 2p(t), \tag{120.13}$$

$$[\Phi(t) - \Omega(t)]^+ + [\Phi(t) - \Omega(t)]^- = 2q(t) \tag{120.14}$$

on L, where $p(t)$, $q(t)$ are the following functions, given on L:

$$\begin{aligned} p(t) &= \frac{1}{2}[Y_y^+ + Y_y^-] - \frac{i}{2}[X_y^+ + X_y^-], \\ q(t) &= \frac{1}{2}[Y_y^+ - Y_y^-] - \frac{i}{2}[X_y^+ - X_y^-]. \end{aligned} \tag{120.15}$$

It will be assumed that $p(t)$ and $q(t)$ satisfy the H condition on L.

Since $\Phi(\infty) - \Omega(\infty) = -\bar{\Gamma}'$, the general solution of the boundary problem (120.14) is given by (§ 108)

$$\Phi(z) - \Omega(z) = \frac{1}{\pi i}\int_L \frac{q(t)dt}{t-z} - \bar{\Gamma}'. \tag{120.16}$$

Further, writing

$$X(z) = \prod_{k=1}^{n} (z-a_k)^{\frac{1}{2}} (z-b_k)^{\frac{1}{2}} \tag{120.17}$$

and applying (110.33), one obtains the general solution of the boundary problem (120.13), bounded at infinity [as follows from (120.1) and (120.5)],

$$\Phi(z) + \Omega(z) = \frac{1}{\pi i X(z)} \int \frac{X(t)p(t)dt}{t - z} + \frac{2P_n(z)}{X(z)} \tag{120.18}$$

where $P_n(z)$ is the polynomial of degree not greater than n

$$P_n(z) = C_0 z^n + C_1 z^{n-1} + \ldots + C_n; \tag{120.19}$$

$X(t)$ must be interpreted as the value, taken by $X(z)$ on the upper (left) side of L.

The formulae (120.16) and (120.18) give

$$\Phi(z) = \Phi_0(z) + \frac{P_n(z)}{X(z)} - \tfrac{1}{2}\bar{\Gamma}', \quad \Omega(z) = \Omega_0(z) + \frac{P_n(z)}{X(z)} + \tfrac{1}{2}\bar{\Gamma}', \tag{120.20}$$

where

$$\Phi_0(z) = \frac{1}{2\pi i X(z)} \int \frac{X(t)p(t)dt}{t - z} + \frac{1}{2\pi i} \int \frac{q(t)dt}{t - z} \tag{120.21}$$

$$\Omega_0(z) = \frac{1}{2\pi i X(z)} \int \frac{X(t)p(t)dt}{t - z} - \frac{1}{2\pi i} \int_L \frac{q(t)dt}{t} \tag{120.22}$$

It is easily seen that under the present conditions with regard to $p(t)$ and $q(t)$ the condition (120.11) is fulfilled. (Cf. Author's book [25]).

The polynomial $P_n(z)$ has still to be determined. It will be assumed that $X(z)$ is a branch which for large $|z|$ has the form

$$X(z) = + z^n + \alpha_{n-1} z^{n-1} + \ldots. \tag{120.17'}$$

The coefficient C_0 follows immediately from the first of the formulae (120.20) and from the condition $\Phi(\infty) = \Gamma$ which give

$$C_0 = \Gamma + \tfrac{1}{2}\bar{\Gamma}'. \tag{120.23}$$

The remaining coefficients must be determined from the condition of single-valuedness of the displacements. By (120.9), this condition implies that the expression $\varkappa\varphi(z) - \omega(\bar{z})$ must revert to its original value as the point z describes a contour Λ_k, surrounding the segment $a_k b_k = L_k$. By contracting the contour Λ_k into the segment L_k, it is readily verified that the following relations express the condition of single-valuedness,

where the differences $\Phi_0^+ - \Phi_0^-$ and $\Omega_0^+ - \Omega_0^-$ are easily determined by the Plemelj formula,

$$2(\varkappa + 1) \int_{L_k} \frac{P_n(t)dt}{X(t)} + \varkappa \int_{L_k} [\Phi_0^+(t) - \Phi_0^-(t)]dt + \\ + \int_{L_k} [\Omega_0^+(t) - \Omega_0^-(t)]dt = 0, \quad (120.24)$$

$$k = 1, \ldots, n,$$

which give a system of n linear equations for the constants $C_1, C_2, \ldots, C_n$.

This system has always a solution. In fact, the homogeneous system, obtained in the case $\Gamma = \Gamma' = 0$, $Y_y^+ = X_y^+ = Y_y^- = X_y^- = 0$, can have no other solution except $C_1 = C_2 = \ldots = C_n = 0$, because the original problem, as is easily established by ordinary means, has in this case only the trivial solution $\Phi(z) = \Omega(z) = 0$. Therefore the non-homogeneous system (120.24) always has a unique solution and the problem is solved.

In the particular case, where the edges of the cuts are free from stresses (*problem of extension of plates weakened by cracks*), $\Phi_0(z) = \Omega_0(z) = 0$ and the solution takes the extraordinarily simple form

$$\Phi(z) = \frac{P_n(z)}{X(z)} - \tfrac{1}{2}\bar{\Gamma}', \quad \Omega(z) = \frac{P_n(z)}{X(z)} + \tfrac{1}{2}\bar{\Gamma}', \quad (120.25)$$

and the coefficients of $P_n(z)$ are determined by the conditions

$$C_0 = \Gamma + \tfrac{1}{2}\bar{\Gamma}', \qquad \int_{L_k} \frac{P_n(t)dt}{X(t)} = 0, \quad k = 1, 2, \ldots, n. \quad (120.26)$$

For $n = 1$ (single crack), letting $a_1 = -a$, $b_1 = a$, one obtains the formulae

$$\Phi(z) = \frac{(2\Gamma + \bar{\Gamma}')z}{2\sqrt{z^2 - a^2}} - \tfrac{1}{2}\bar{\Gamma}', \quad \Omega(z) = \frac{(2\Gamma + \bar{\Gamma}')z}{2\sqrt{z^2 - a^2}} + \tfrac{1}{2}\bar{\Gamma}'. \quad (120.27)$$

A solution of the (less simple) problem for the particular case $n = 1$ is effectively contained in § 82*a*, as a particular case of the problem of the equilibrium of a plate with an elliptic hole under the influence of external forces, applied to its boundary.

3°. Second fundamental problem.

Consider now the second fundamental problem, i.e., assume that the values of the displacements $u^+(t)$, $v^+(t)$ on the upper edges and $u^-(t)$, $v^-(t)$ on the lower edges of L are given; also, if $u(a_k)$, $v(a_k)$ and $u(b_k)$, $v(b_k)$ denote the (given) displacements of the points a_k, b_k, assume that

$$\begin{aligned} u^+(a_k) &= u^-(a_k) = u(a_k), \quad v^+(a_k) = v^-(a_k) = v(a_k), \\ u^+(b_k) &= u^-(b_k) = u(b_k), \quad v^+(b_k) = v^-(b_k) = v(b_k). \end{aligned} \tag{120.28}$$

In addition, let also the constants Γ and Γ' (without assuming this time $C = 0$) and the resultant vector (X, Y) of the external forces, applied to L, be known.

In order to avoid having to consider directly the functions $\varphi(z)$, $\omega(z)$ which may be multi-valued, the boundary conditions will not be constructed beginning from (120.9), but from a formula, obtained from (120.9) by differentiation with respect to x, i.e., from

$$2\mu(u' + iv') = \varkappa\Phi(z) - \Omega(\bar{z}) - (z - \bar{z})\overline{\Phi'(z)}, \tag{120.29}$$

where u', v' are the partial derivatives $\partial u/\partial x$, $\partial v/\partial x$. Accordingly the boundary conditions may be written

$$\begin{aligned} \varkappa\Phi^+(t) - \Omega^-(t) &= 2\mu(u'^+ + iv'^+), \\ \varkappa\Phi^-(t) - \Omega^+(t) &= 2\mu(u'^- + iv'^-). \end{aligned} \tag{120.30}$$

Adding and subtracting, one finds

$$[\varkappa\Phi(t) - \Omega(t)]^+ + [\varkappa\Phi(t) - \Omega(t)]^- = 2f(t), \tag{120.31}$$

$$[\varkappa\Phi(t) + \Omega(t)]^+ - [\varkappa\Phi(t) + \Omega(t)]^- = 2g(t) \tag{120.32}$$

on L, where $f(t)$, $g(t)$ are the following functions, given on L (cf. § 113):

$$\begin{aligned} f(t) &= \mu[(u^{+\prime} + u^{-\prime}) + i(v^{+\prime} + v^{-\prime})], \\ g(t) &= \mu[(u^{+\prime} - u^{-\prime}) + i(v^{+\prime} - v^{-\prime})]. \end{aligned} \tag{120.33}$$

It will be assumed that these functions satisfy the H condition on L.

In the same way as in the preceding problem, the general solutions of the boundary problems (120.32) and (120.31) are given by

$$\varkappa\Phi(z) + \Omega(z) = \frac{1}{\pi i}\int \frac{g(t)dt}{t - z} + \Gamma' + \varkappa\Gamma + \Gamma, \tag{120.34}$$

$$\varkappa\Phi(z) - \Omega(z) = \frac{1}{\pi i X(z)}\int \frac{X(t)f(t)dt}{t - z} + \frac{2P_n(z)}{X(z)}, \tag{120.35}$$

where $X(z)$ is determined by (120.17) and $X(t)$ is its value on the left side of L.

The preceding formulae determine the unknown functions $\Phi(z)$, $\Psi(z)$ apart from an additive term, containing the polynomial

$$P_n(z) = C_0 z^n + C_1 z^{n-1} + \ldots + C_n.$$

The first two coefficients C_0 and C_1 of this polynomial are immediately determined by (120.35), if one takes into consideration that for large $|z|$, by (120.1) and (120.5),

$$\varkappa\Phi(z) - \Omega(z) = \varkappa\Gamma - \bar{\Gamma} - \bar{\Gamma}' - \frac{\varkappa(X + iY)}{\pi(\varkappa + 1)} \frac{1}{z} + O\left(\frac{1}{z^2}\right). \quad (120.36)$$

It is readily verified, on the basis of (120.28) and (120.30), that the displacements u, v, calculated from (120.9) using the functions $\Phi(z)$, $\Omega(z)$ just found, will be single-valued. However, these displacements will assume on the cuts L_k the given values, apart from some constant terms which may be different on different cuts. Let the constants, by which the expression $2\mu(u + iv)$, calculated from $\Phi(z)$, $\Omega(z)$, differs on the cuts $L_1, L_2, \ldots, L_n$ from the given values, be denoted by c_1, $c_2, \ldots, c_n$. The functions $\Phi(z)$ and $\Omega(z)$ will satisfy the conditions of the problem only when $c_1 = c_2 = \ldots = c_n$.

These constants will be equal on the upper and lower edges of each cut, because, as is easily seen from the conditions introduced earlier, the expression $2\mu(u + iv)$ tends to a definite limit as z approaches one of the ends a_k, b_k. If $c_1 = c_2 = \ldots = c_n$, the condition $c_k = 0$ may be attained at the expense of an arbitrary constant entering into the right-hand side of (120.9).

It is easily seen, by (120.29), that the conditions for the c_k may be expressed in the form

$$\int_{b_k}^{a_{k+1}} [\varkappa\Phi(t) - \Omega(t)]dt = 2\mu\{[u(a_{k+1}) - u(b_k)] + i[v(a_{k+1}) - v(b_k)]\}, \quad (120.37)$$

$$k = 1, 2, \ldots, n - 1,$$

where the quantities on the right-hand side are given [cf. (120.28)].

Substituting for $\varkappa\Phi(t) - \Omega(t)$ from (120.35), one obtains a system of $n - 1$ linear equations for the determination of the $n - 1$ coefficients $C_2, \ldots, C_n$ which were so far undefined; similarly as before, it is easily seen that this system always has a unique solution. Thus the problem

is solved. The solution for the particular case $n = 1$ was obtained in § 83 by different means.

The problem, where the displacements are only given apart from constant terms which are different on different cuts, may be solved in an analogous manner; however, in that case, the resultant vectors of the external forces, acting on the individual cuts separately, must also be given.

4°. A certain mixed problem

In conclusion, a certain problem will be solved which was considered by D. I. Sherman [13]. In this problem the external stresses, applied, say, to the upper edges of the cuts, and the displacements on the lower edges are given. D. I. Sherman solved this problem by rather complicated means, reducing it to a system of singular integral equations (which admittedly is simple), and there is one omission in his solution about which more will be said later.

By (120.7) and (120.29), the boundary conditions may be written

$$\Phi^+(t) + \Omega^-(t) = Y_y^+ - iX_y^+, \quad \varkappa\Phi^-(t) - \Omega^+(t) = 2\mu(u'^- + iv'^-) \quad (120.38)$$

on L. Multiplying the second of these equations first by $-i/\sqrt{\varkappa}$, then by $+i/\sqrt{\varkappa}$, and adding to the first (cf. D. I. Sherman [13], p. 333), one obtains

$$\left[\Phi(t) + \frac{i}{\sqrt{\varkappa}}\,\Omega(t)\right]^+ - i\sqrt{\varkappa}\left[\Phi(t) + \frac{i}{\sqrt{\varkappa}}\,\Omega(t)\right]^- = 2f_1(t), \quad (120.39)$$

$$\left[\Phi(t) - \frac{i}{\sqrt{\varkappa}}\,\Omega(t)\right]^+ + i\sqrt{\varkappa}\left[\Phi(t) - \frac{i}{\sqrt{\varkappa}}\,\Omega(t)\right]^- = 2f_2(t) \quad (120.40)$$

on L, where $f_1(t)$, $f_2(t)$ are functions, given on L; it will be assumed that these functions satisfy the H condition on L.

Thus the functions

$$\Phi(z) + \frac{i}{\sqrt{\varkappa}}\,\Omega(z), \quad \Phi(z) - \frac{i}{\sqrt{\varkappa}}\,\Omega(z)$$

determine the solutions of the boundary problems (120.39) and (120.40) which are particular cases of the problem, solved in § 110. In the notation of that section, one has for the problem (120.39): $g = i\sqrt{\varkappa}$, while for the problem (120.40): $g = -i\sqrt{\varkappa}$.

Solving these problems by the method of § 110 and taking into con-

sideration the behaviour of $\Phi(z)$, $\Omega(z)$ at infinity, one finds

$$\Phi(z) + \frac{i}{\sqrt{\varkappa}}\,\Omega(z) = \frac{X_1(z)}{2\pi i}\int_L \frac{f_1(t)dt}{X_1^+(t)(t-z)} + X_1(z)P_n^{(1)}(z), \qquad (120.41)$$

$$\Phi(z) - \frac{i}{\sqrt{\varkappa}}\,\Omega(z) = \frac{X_2(z)}{2\pi i}\int_L \frac{f_2(t)dt}{X_2^+(t)(t-z)} + X_2(z)P_n^{(2)}(z), \qquad (120.42)$$

where

$$X_1(z) = \prod_1^n (z-a_k)^{-\gamma_1}(z-b_k)^{\gamma_1-1}, \quad X_2(z) = \prod_1^n (z-a_k)^{-\gamma_2}(z-b_k)^{\gamma_2-1}, \quad (120.43)$$

$$\gamma_1 = \frac{\log (i\sqrt{\varkappa})}{2\pi i} = \tfrac{1}{4} + \frac{\log \varkappa}{4\pi i}, \quad \gamma_2 = \frac{\log (-i\sqrt{\varkappa})}{2\pi i} = \tfrac{3}{4} + \frac{\log \varkappa}{4\pi i} \quad (120.44)$$

For $X_1(z)$, $X_2(z)$ one must select branches, holomorphic in the plane cut along L.

By adding and subtracting (120.41) and (120.42), one may obtain closed expressions for $\Phi(z)$ and $\Omega(z)$, but this will not be done here.

For the determination of the $2n+2$ coefficients of the polynomials $P_n^{(1)}$ and $P_n^{(2)}$ one has the following conditions. Firstly, that the functions $\Phi(z)$ and $\Omega(z)$ must behave at infinity in accordance with (120.1) and (120.5); in this connection it will be assumed that the constants Γ, Γ', X, Y, entering into these formulae, are given.

The resultant vector of the forces, applied to the upper edges of the cuts, are determined by the values Y_y^+, X_y^+ on L; in addition, it is assumed that the resultant vector of the forces, applied to the lower edges, is known. The sum of these vectors is the vector (X, Y).

Secondly, that the displacements must be single-valued, as in the case of the first fundamental problem. Finally, that on the lower edges of the cuts the expression $2\mu(u+iv)$ assumes given values, and not only apart from certain constants; as in 3°, it is sufficient for this purpose to express that $2\mu(u+iv)$ assumes on the lower edges of the cuts the given values apart from a constant which is the same for all cuts. In this way one obtains a system of $2n+2$ linear equations, since the first group renders four, the second n equations one of which is a consequence of all the others by the strength of the equations of the first group, while the last group contains $n-1$ equations. These equations determine the $2n+2$ unknown coefficients and it is easily verified, on the basis of the uniqueness theorem (which obviously holds under the given conditions), that this system has always a unique solution.

D. I. Sherman [13] did not subject the unknown solution to the conditions, securing uniqueness, in contrast to what he did in his paper [12] in which he solved the first fundamental problem. Therefore his solution of the problem, considered just now, contains constants which cannot be determined without additional conditions. Sherman gave no study of his solution and assumed these constants to be arbitrary.

For example, consider the case where there is only one cut $L = ab$ the lower edge of which does not move ($u^- = v^- = 0$ on L), while the upper edge is free from stresses ($Y_y^+ = X_y^+ = 0$ on L), and where the stresses and rotation vanish at infinity ($\Gamma = \Gamma' = 0$). Further, let it be assumed that the vector of the external forces, applied to the lower edge, is equal to $(0, -P_0)$.

This problem may be interpreted as follows: a rigid straight strip has been welded to the lower edge on which acts a symmetrically distributed force of magnitude P_0 in the negative y direction.

In the present case, $n = 1$, $f_1(t) = f_2(t) = 0$,

$$X_1(z) = (z-a)^{-\gamma_1}(z-b)^{\gamma_1-1}, \quad X_2(z) = (z-a)^{-\gamma_2}(z-b)^{\gamma_2-1}$$

and, since $\Phi(z)$ and $\Omega(z)$ must vanish at infinity (because $\Gamma = \Gamma' = 0$), one has, by (120.41) and (120.42),

$$\Phi(z) = C_1X_1(z) + C_2X_2(z), \quad \Omega(z) = -i\sqrt{\varkappa}\,C_1X_1(z) + i\sqrt{\varkappa}\,C_2X_2(z),$$

where C_1 and C_2 are constants. These constants are determined on the basis of the conditions, following from (120.1) and (120.5), i.e., for large $|z|$

$$\Phi(z) = \frac{iP_0}{2\pi(1+\varkappa)}\,\frac{1}{z} + O\left(\frac{1}{z^2}\right), \quad \Omega(z) = \frac{-i\varkappa P_0}{2\pi(1+\varkappa)}\,\frac{1}{z} + O\left(\frac{1}{z^2}\right),$$

whence, assuming that $X_1(z)$ and $X_2(z)$ refer to branches for which $\lim_{z\to\infty} zX_1(z) = \lim_{z\to\infty} zX_2(z) = 1$,

$$C_1 + C_2 = \frac{iP_0}{2\pi(1+\varkappa)}, \quad -i\sqrt{\varkappa}\,C_1 + i\sqrt{\varkappa}\,C_2 = -\frac{i\varkappa P_0}{2\pi(1+\varkappa)},$$

i.e.,

$$C_1 = \frac{iP_0(1-i\sqrt{\varkappa})}{4\pi(1+\varkappa)}, \quad C_2 = \frac{iP_0(1+i\sqrt{\varkappa})}{4\pi(1+\varkappa)}.$$

SOLUTION OF BOUNDARY PROBLEMS FOR REGIONS, BOUNDED BY CIRCLES, AND FOR THE INFINITE PLANE, CUT ALONG CIRCULAR ARCS

Important boundary problems for the circle and for the infinite plane with circular holes may be easily solved in a manner analogous to that used in the preceding chapter. The solutions of the first, second and mixed problems for these cases, and likewise for a more general case to be considered in Chap. 21, were given by I. N. Kartzivadze in his dissertation parts of which have been published in his papers [1, 2]; only finite regions are considered, since the case of infinite regions may be solved by analogous means. Those results of Kartzivadze which refer to regions bounded by circles will be studied in §§ 121—123.

B. L. Mintzberg [1] published recently a solution of the mixed problem for an infinite region with circular holes; he was apparently only acquainted with the first of the above papers by Kartzivadze.

In § 124, the solution of the fundamental problems for an infinite region cut along circular arcs will be given.

§ 121. Transformation of the general formulae for regions, bounded by a circle. Let L be the unit circle with centre at the origin and let S^+ be the inside of this circle and S^- the remaining part of the plane (excluding L).

Let the elastic body occupy one of the regions S^-, S^+. Introduce polar coordinates r, ϑ by the relation

$$z = x + iy = re^{i\vartheta}$$

and, as in § 39, denote by $\widehat{rr}$, $\widehat{\vartheta\vartheta}$, $\widehat{r\vartheta}$ the stress components in polar coordinates. The formulae, expressing these components in terms of the functions $\Phi(z)$ and $\Psi(z)$ (§ 39), will now be written as follows [(39.4), (39.5), where in the latter $e^{2i\vartheta}$ has been replaced by $(z/\bar{z})$]:

$$\widehat{rr} + \widehat{\vartheta\vartheta} = 2[\Phi(z) + \overline{\Phi(z)}], \tag{121.1}$$

$$rr + ir\vartheta = \Phi(z) + \overline{\Phi(z)} - \bar{z}\overline{\Phi'(z)} - \frac{\bar{z}}{z}\,\overline{\Psi(z)}. \qquad (121.2)$$

Further, the formula

$$2\mu(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} + \text{const.} \qquad (121.3)$$

will be recalled which expresses the displacement components u, v (in rectangular coordinates) in terms of the functions $\varphi(z)$, $\psi(z)$, related to $\Phi(z)$ and $\Psi(z)$ by the formulae $\varphi'(z) = \Phi(z)$, $\psi'(z) = \Psi(z)$; differentiating (121.3) with respect to ϑ, one finds the formula

$$2\mu(u' + iv') = iz\left[\varkappa\Phi(z) - \overline{\Phi(z)} + \bar{z}\overline{\Phi'(z)} + \frac{\bar{z}}{z}\,\overline{\Psi(z)}\right], \qquad (121.4)$$

where now

$$u' = \frac{\partial u}{\partial \vartheta}, \quad v' = \frac{\partial v}{\partial \vartheta}.$$

The functions $\Phi(z)$ and $\Psi(z)$ are holomorphic in the region under consideration (S^+ or S^-). When this region is S^-, these functions have for large $|z|$ the form

$$\Phi(z) = \Gamma - \frac{X + iY}{2\pi(1 + \varkappa)}\,\frac{1}{z} + O\left(\frac{1}{z^2}\right), \qquad (121.5)$$

$$\Psi(z) = \Gamma' + \frac{\varkappa(X - iY)}{2\pi(1 + \varkappa)}\,\frac{1}{z} + O\left(\frac{1}{z^2}\right), \qquad (121.6)$$

where in the former notation

$$\Gamma = B + iC, \quad \Gamma' = B' + iC', \qquad (121.7)$$

$$B = \tfrac{1}{4}(N_1 + N_2), \quad C = \frac{2\mu\varepsilon_\infty}{1 + \varkappa}, \quad \Gamma' = -\tfrac{1}{2}(N_1 - N_2)e^{-2i\alpha}. \qquad (121.8)$$

Using a previous notation (cf. § 76), the definition of $\Phi(z)$, originally defined in S^+ [or in S^-], will now be extended to the region S^-[or S^+] by writing

$$\Phi(z) = -\overline{\Phi}\left(\frac{1}{z}\right) + \frac{1}{z}\,\overline{\Phi}'\left(\frac{1}{z}\right) + \frac{1}{z^2}\,\overline{\Psi}\left(\frac{1}{z}\right) \qquad (121.9)$$

in S^-[or S^+], i.e., for $|z| > 1$ [or $|z| < 1$].

This extension has been selected in such a way that the values of $\Phi(z)$ from the right and left of L continue each other analytically through the unloaded parts

of the boundary (cf. § 112); with this aim in mind, one readily arrives at (121.9), remembering (121.2) and noting that on L

$$\bar{z} = \frac{1}{z}.$$

Replacing in (121.9) z by $1/\bar{z}$, as one is, by supposition, justified to do for $|z| > 1$ [or for $|z| < 1$], and assuming now $|\bar{z}| = |z| < 1$ [or $|\bar{z}| = |z| > 1$], one obtains

$$\Phi\left(\frac{1}{\bar{z}}\right) = -\Phi(\bar{z}) + \bar{z}\Phi'(z) + \bar{z}^2\Psi(\bar{z}),$$

whence, taking conjugate values,

$$\Psi(z) = \frac{1}{z^2}\,\Phi(z) + \frac{1}{z^2}\,\overline{\Phi}\left(\frac{1}{z}\right) - \frac{1}{z}\,\Phi'(z). \tag{121.10}$$

Since the components of stress and displacement may be expressed in terms of $\Phi(z)$ and $\Psi(z)$, they may also, by use of (121.10), be expressed in terms of $\Phi(z)$ only, which is now defined throughout the plane (excluding L).

When S^+ is the region occupied by the body, the function $\Phi(z)$ is holomorphic in S^+ as well as in S^-, including the point at infinity; this follows from (121.6) and (121.9). However, the behaviour of $\Phi(z)$ at infinity must be subject to several conditions, in order that the corresponding function $\Psi(z)$ will be holomorphic in S^+. In fact, let

$$\begin{aligned} \Phi(z) &= A_0 + A_1 z + A_2 z^2 + \ldots \qquad (\text{for } |z| < 1),\\ \Phi(z) &= B_0 + \frac{B_1}{z} + \frac{B_2}{z^2} + \ldots \qquad (\text{for } |z| > 1). \end{aligned} \tag{121.11}$$

In order that $\Psi(z)$, determined by (121.10), will also be holomorphic at $z = 0$, one easily deduces the condition

$$A_0 + \bar{B}_0 = 0, \quad B_1 = 0. \tag{121.12}$$

In the sequel it will be assumed that these conditions are satisfied.

When S^- is the region occupied by the body, the function $\Phi(z)$ is holomorphic in S^- (including the point at infinity) as well as in S^+, except at the point $z = 0$ where it may have a pole. In fact, (121.9) together with (121.5), (121.6) shows that near $z = 0$

$$\Phi(z) = \frac{\overline{\Gamma}'}{z^2} + \frac{\varkappa(X + iY)}{2\pi(1 + \varkappa)}\,\frac{1}{z} + \text{a holomorphic function.} \tag{121.13}$$

The stress components will be obtained in terms of the function $\Phi(z)$ from the formulae (121.1) and (121.2), if one understands by $\Psi(z)$ in these formulae the expression (121.10). In order to give (121.10) the form which is most convenient for the present purpose, it may be noted that by (121.10)

$$\Phi(z) - \bar{z}\overline{\Phi'(z)} = \bar{z}^2\overline{\Psi(z)} - \Phi\left(\frac{1}{\bar{z}}\right);$$

substituting this expression in (121.2), one finds

$$\widehat{rr} + i\,\widehat{r\vartheta} = \Phi(z) - \Phi\left(\frac{1}{\bar{z}}\right) + \bar{z}\left(\bar{z} - \frac{1}{z}\right)\overline{\Psi(z)}, \qquad (121.14)$$

where on the right-hand side one should understand by $\Psi(z)$ the expression (121.10).

Similarly, one obtains from (121.4)

$$2\mu(u' + iv') = iz\left[\varkappa\Phi(z) + \Phi\left(\frac{1}{\bar{z}}\right) - \bar{z}\left(\bar{z} - \frac{1}{z}\right)\overline{\Psi(z)}\right], \qquad (121.15)$$

where again $\Psi(z)$ is given by (121.10) and, as before,

$$u' = \frac{\partial u}{\partial \vartheta}, \quad v' = \frac{\partial v}{\partial \vartheta}. \qquad (121.16)$$

It will now be assumed that $\Phi(z)$ is continuous at L from S^+ and from S^-, except possibly at a finite number of points c_k of L near which

$$|\,\Phi(z)\,| < \frac{\text{const.}}{|\,z - c_k\,|^{\alpha}}, \qquad 0 \leqslant \alpha < 1; \qquad (121.17)$$

in addition, it will be assumed that

$$\lim_{r\to 1}\,(1 - r)\,\Phi'(z) = 0, \quad z = re^{i\vartheta} \qquad (121.18)$$

for all values of ϑ, except possibly for those which correspond to the points c_k. It is easily seen, on the basis of (121.10), that

$$\lim_{r\to 1}\left(\bar{z} - \frac{1}{z}\right)\Psi(z) = \lim_{r\to 1} e^{-i\vartheta}\left(r - \frac{1}{r}\right)\Psi(z) = 0. \qquad (121.19)$$

If L contains unloaded sections L', i.e., if on L': $\widehat{rr} = \widehat{r\vartheta} = 0$, then, as shown by (121.14), $\Phi^+(t) - \Phi^-(t) = 0$ on L'. Consequently, the values of $\Phi(z)$ inside and outside of L are analytic continuations through

the unloaded sections of the boundary, as in the case of the half-plane. In fact, the definition (121.9) of $\Phi(z)$ in S^-[or S^+] was chosen in order to ensure this property.

A number of fundamental boundary problems for the circle are easily solved by use of the preceding formulae in a manner, similar to that used in Chap. 19 in the case of the half-plane.

§ 122. Solution of the first and second fundamental probler for the region, bounded by a circle. These problems have previously solved by different methods. Their solution will be outlined here as an illustration of the new method.

1°. First fundamental problem for the circle

In this case S^+ is the region, occupied by the body, and the boundary condition has the form

$$\widehat{\rho\rho}^+ + i\,\widehat{\rho\vartheta}^+ = N(t) + iT(t), \tag{122.1}$$

where N and T are the normal and tangential stresses on L which will be assumed known. By (121.14), this boundary condition takes the form [cf. remarks following (113.1)]

$$\Phi^+(t) - \Phi^-(t) = N(t) + iT(t). \tag{122.2}$$

One has thus arrived at the problem, solved in § 108; in the present case it is required to find the solution, bounded at infinity. Applying (108.2), one finds

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{N(t) + iT(t)}{t - z}\,dt + B_0, \tag{122.3}$$

where $B_0 = \Phi(\infty)$ is a constant, at present unknown. In order to determine B_0 and also to ascertain whether the problem is possible, consider (121.12) which must, by supposition, be satisfied.

For this purpose the constants A_0 and B_1 of (121.11) will be calculated. One has

$$A_0 = \Phi(0) = \frac{1}{2\pi i}\int_L [N(t) + iT(t)]\frac{dt}{t} + B_0 = \frac{1}{2\pi}\int_0^{2\pi} (N + iT)d\vartheta + B_0,$$

$$B_1 = \lim_{z\to\infty} [z\Phi(z)] = -\frac{1}{2\pi i}\int [N(t) + iT(t)]dt = -\frac{1}{2\pi}\int^{2\pi} (N + iT)e^{i\vartheta}\,d\vartheta.$$

The conditions (121.12) give then

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi}(N - iT)d\vartheta + B_0 + \bar{B}_0 = 0, \\ &\int_0^{2\pi}(N + iT)e^{i\vartheta}\, d\vartheta = 0. \end{aligned} \tag{122.4}$$

The first of these formulae shows that one must have

$$\int_0^{2\pi} T\, d\vartheta = 0; \tag{122.5}$$

if this condition is satisfied, then

$$\Re B_0 = -\frac{1}{4\pi}\int_0^{2\pi} N d\vartheta. \tag{122.6}$$

The conditions (122.4) and (122.5) express that the resultant vector and moment of the external forces vanish, and so must clearly hold in this problem.

The formula (122.6) determines the real part of the constant B_0; the imaginary part of B_0 remains undetermined, as was to be expected, because it only influences the rigid body motion. Thus the problem is solved.

2°. First fundamental problem for the plane with a circular hole

This problem may be solved in the same way as the preceding one. In this case

$$\widehat{\rho\rho}^- + i\,\widehat{\rho\vartheta}^- = N(t) + iT(t), \tag{122.7}$$

where $N(t)$ and $T(t)$ are the given external normal and tangential stresses; as in § 87*a* and § 56, N is the projection on the normal, directed towards the centre, while T is the projection on the tangent which points to the left as one looks along the positive normal.

On the basis of (121.14), this condition takes the form

$$\Phi^+(t) - \Phi^-(t) = -[N(t) + iT(t)] \tag{122.8}$$

which is of a similar form as (122.2). However, in the present case, one has to find a solution which has at infinity the given value Γ and at $z = 0$ a pole with the principal part equal to

$$\frac{\bar{\Gamma}'}{z^2} + \frac{\varkappa(X + iY)}{2\pi(1 + \varkappa)} \frac{1}{z};$$

this follows from (121.5) and (121.13). Therefore, applying the results of § 108, one finds immediately

$$\Phi(z) = -\frac{1}{2\pi i} \int_L \frac{N(t) + iT(t)}{t - z} dt + \Gamma + \frac{\varkappa(X + iY)}{2\mu(1 + \varkappa)} \frac{1}{z} + \frac{\bar{\Gamma}'}{z^2}. \quad (122.9)$$

The quantities X, Y (i.e., the components of the resultant vector of the external forces) may be calculated directly from the given data; in fact,

$$X + iY = -\int_0^{2\pi} (N + iT)e^{i\vartheta} d\vartheta.$$

The constants Γ and Γ', however, which determine the stresses and the rotation at infinity must be assumed known. It is readily verified that the displacements will be single-valued.

The problem is thus solved. It is easily seen that for $\Gamma = \Gamma' = 0$ the present expression for $\Phi(z)$ in S^- agrees with that obtained in § 87*a* (where it had been assumed that the stresses and rotation vanish at infinity).

3°. The second fundamental problem for S^+ and S^- may be solved in an analogous manner, beginning with (121.15). This will be left to the reader.

§ 123. The mixed fundamental problem for a region, bounded by a circle. This problem has so far not been solved. Let $L_k = a_k b_k$ $(k = 1, 2, \ldots, n)$ be given arcs of the circle L, denoted in such a way that the ends are encountered in the order $a_1, b_1, \ldots, a_n, b_n, a_1$, when passing around the circle in the counter-clockwise direction. The union of these arcs will be denoted by L', so that

$$L' = L_1 + L_2 + \ldots + L_n,$$

and the remaining part of L by L''.

Let the components of displacement and of stress be given on L' and L'' respectively.

Since the problem of the case, where the external stresses are given along the entire boundary, has already been solved, the mixed problem under consideration may be reduced to the case where the displacements are given on the segments $L_k = a_k b_k$, while the remaining part of the boundary is free from external stresses. (The solution of the general problem may also be obtained directly; cf. the Note at the end of this section.)

1°. Solution of the mixed problem for the circle

Consider first the case where S'^{+}, i.e., the inside of the unit circle, is the region occupied by the body. The boundary condition then takes the form

$$u^{+} + iv^{+} \quad = g(t) \quad \text{on } L', \tag{123.1}$$

$$\widehat{rr}^{+} + i\,\widehat{r\vartheta}^{+} = 0 \quad \text{on } L'', \tag{123.2}$$

where $g(t)$ is a function, given on L'. It will be assumed that the derivative $g'(t)$ satisfies the H condition.

On the basis of (121.15), one obtains from (123.1)

$$\varkappa\Phi^{+}(t) + \Phi^{-}(t) = 2\mu g'(t) \quad \text{on } L', \tag{123.3}$$

where

$$g'(t) = \frac{dg}{dt} = -\, ie^{-i\vartheta}\frac{dg}{d\vartheta}. \tag{123.4}$$

On the other hand, (123.2) leads, as noted earlier, to the condition $\Phi^{+}(t) - \Phi^{-}(t) = 0$ on L'' which expresses that $\Phi(z)$ is holomorphic in the entire plane, cut along L'.

Thus the problem of finding $\Phi(z)$ is reduced to the determination of a solution of the problem, considered in § 110, which must be bounded at infinity. In the present case the constant g of § 110 is equal to $-\,1/\varkappa$ and

$$f(t) = \frac{2\mu}{\varkappa}\, g'(t).$$

By (110.5),

$$\gamma = \frac{1}{2\pi i}\log\left(-\frac{1}{\varkappa}\right) = -\frac{\log(-\varkappa)}{2\pi i} = -\frac{\log\varkappa}{2\pi i} + \tfrac{1}{2},$$

i.e.,

$$\gamma = \tfrac{1}{2} + i\beta,$$

where

$$\beta = \frac{\log \varkappa}{2\pi}. \tag{123.5}$$

Therefore, by (110.2),

$$X_0(z) = \prod_{k=1}^{n} (z - a_k)^{-\frac{1}{2}-i\beta} (z - b_k)^{-\frac{1}{2}+i\beta}, \tag{123.6}$$

where one has to understand by $X_0(z)$ that branch which for large $|z|$ has the form

$$X_0(z) = \frac{1}{z^n} + \frac{\alpha_{-n+1}}{z^{n-1}} + \dots. \tag{123.7}$$

Applying (110.18) and remembering that $\Phi(z)$ is to be bounded at infinity, one finds

$$\Phi(z) = \frac{\mu X_0(z)}{\pi i \varkappa} \int_{L'} \frac{g'(t)dt}{X_0^+(t)\,(t-z)} + X_0(z) P_n(z), \tag{123.8}$$

where $P_n(z)$ is a polynomial of degree not higher than n:

$$P_n(z) = C_0 z^n + C_1 z^{n-1} + \dots + C_n. \tag{123.9}$$

The constants $C_0, C_1, \dots, C_n$ have still to be determined so that they satisfy all the requirements of the original problem, i.e., the conditions (121.12) and the boundary condition (123.1); it will not be sufficient to satisfy only (123.3) which was obtained from (123.1) by differentiation with respect to ϑ. It should be noted that (123.1) has to be fulfilled exactly, apart from a constant which must be the same for all L_k, because then it may be satisfied exactly by a suitable choice of the arbitrary constant on the right-hand side of (121.3). It is easily seen that this last condition may be expressed by the relations

$$\begin{gathered}\int_{b_k a_{k+1}} [\varkappa\Phi^+(t_0) + \Phi^-(t_0)]\, dt_0 = 2\mu[g(a_{k+1}) - g(b_k)], \\ k = 1, 2, \dots, n, \quad (a_{k+1} = a_1),\end{gathered} \tag{123.10}$$

where $\Phi^+(t_0)$ and $\Phi^-(t_0)$ must be obtained from (123.8). Since $\Phi^+(t_0) = \Phi^-(t_0)$ on the arcs $b_k a_{k+1}$, the conditions (123.10) give

$$(\varkappa + 1) \int_{b_k a_{k+1}} \Phi_0(t_0) dt_0 + \sum_{j=0}^{n} A_{kj} C_j = 2\mu[g(a_{k+1}) - g(b_k)], \tag{123.11}$$

where

$$\Phi_0(t_0) = \frac{\mu X_0^+(t_0)}{\pi i \varkappa} \int\limits_{L'} \frac{g'(t)dt}{X_0^+(t)\,(t-t_0)}, \tag{123.12}$$

$$A_{kj} = (\varkappa + 1) \int\limits_{b_k a_{k+1}} \frac{t^{n-j}dt}{X_0^+(t)}. \tag{123.13}$$

One has thus obtained n linear equations in $C_0, C_1, \ldots, C_n$. The conditions (121.12) have still to be satisfied. It is readily verified that the second of these is a consequence of the conditions (123.11), obtained above. In fact, it follows from (123.11), equivalent to (123.10), that

$$\int\limits_{L} [\varkappa\Phi^+(t_0) + \Phi^-(t_0)]dt_0 = 0,$$

since, by (123.3),

$$\int\limits_{a_k b_k} [\varkappa\Phi^+(t_0) + \Phi^-(t_0)]dt_0 = 2\mu[g(b_k) - g(a_k)].$$

However, since $\Phi(z)$ is holomorphic in S^+, the integral over the first term vanishes; hence

$$\int\limits_{L} \Phi^-(t_0)dt_0 = 0,$$

and this means that the coefficient B_1 in the expansion for $\Phi(z)$ in decreasing powers of z near the point at infinity is equal to zero.

Thus there remains only the first condition of (121.12) which may be written

$$\Phi(0) + \overline{\Phi(\infty)} = 0,$$

so that, by (123.8),

$$\bar{C}_0 + X_0(0)C_n + \frac{\mu}{\pi i \varkappa} \int\limits_{L'} \frac{g'(t)}{X_0^+(t)} \frac{dt}{t} = 0. \tag{123.14}$$

Consequently, one has finally the $n + 1$ linear equations (123.11) and (123.14) for the determination of the constants $C_0, C_1, \ldots, C_n$, or, more correctly, a system of $2n + 2$ linear equations for the determination of the real and imaginary parts of these constants.

It has still to be shown that this system has always a unique solution. But for this purpose it is sufficient to verify that the homogeneous system, obtained for $g(t) = \text{const.}$, has no other solution except $C_0 = C_1 = \ldots = C_n = 0$. However, this is a direct consequence of the uniqueness theorem for the mixed problem.

2°. Solution of the mixed problem for the plane with a circular hole

This may be treated in quite an analogous manner. As mentioned in the introduction to this chapter, the solution of this problem was recently published by B. L. Mintzberg [1]; his solution (for the particular case $n = 1$) is somewhat more complicated than that given here.

In the present case, the boundary condition has the form

$$u^- + iv^- = g(t) \quad \text{on } L', \tag{123.15}$$

$$\widehat{rr}^- + i\,\widehat{r\vartheta}^- = 0 \quad \text{on } L''; \tag{123.16}$$

it follows from (123.15), using (121.15), that

$$\Phi^+(t) + \varkappa\Phi^-(t) = 2\mu g'(t), \tag{123.17}$$

while (123.16) gives, as before, $\Phi^+ - \Phi^- = 0$ on L''.

It is now required to find a solution $\Phi(z)$ which (as in the preceding case) is bounded at infinity and has at the point $z = 0$ a pole of not higher than second order [cf. (121.13)].

It will be assumed that the stress components at infinity, i.e., the constants Γ, Γ' in (121.5), (121.6), as well as the resultant vector (X, Y) of the external forces, applied to L', are given.

As before, the results of § 110 will be applied for the solution of the problem (123.17). This time

$$f(t) = 2\mu g'(t),$$

$$\gamma = \frac{\log(-\varkappa)}{2\pi i} = \frac{\log \varkappa}{2\pi i} + \tfrac{1}{2},$$

i.e.,

$$\gamma = \tfrac{1}{2} - i\beta,$$

where β is the same as previously, viz.

$$\beta = \frac{\log \varkappa}{2\pi}.$$

In the present case

$$X_0(z) = \prod_{k=1}^{n} (z - a_k)^{-\frac{1}{2}+i\beta} (z - b_k)^{-\frac{1}{2}-i\beta}, \tag{123.18}$$

where one has again to select a branch, satisfying (123.7); the general solution of the problem (123.17), satisfying the conditions stated above, is given by [cf. (110.26)]

$$\Phi(z) = \frac{\mu X_0(z)}{\pi i} \int_{L'} \frac{g'(t)dt}{X_0^+(t)\,(t - z)} + \left\{ \frac{D_2}{z^2} + \frac{D_1}{z} + P_n(z) \right\} X_0(z), \tag{123.19}$$

where $P_n(z)$ is a polynomial of degree not higher than n and D_1, D_2 are constants. These constants are immediately determined from the condition [cf. (121.13)] that near $z = 0$

$$X_0(z) \left\{ \frac{D_2}{z^2} + \frac{D_1}{z} \right\} = \frac{\overline{\Gamma}'}{z^2} + \frac{\varkappa(X + iY)}{2\pi(1 + \varkappa)} \frac{1}{z} + O(1). \tag{123.20}$$

Similarly, the coefficients C_0 and C_1 of z^n and z^{n-1} in the polynomial $P_n(z)$ are determined by the conditions [cf. (121.5)] that for large $|z|$

$$\Phi(z) = \Gamma - \frac{X + iY}{2\pi(\varkappa + 1)} \frac{1}{z} + O\left(\frac{1}{z^2}\right); \tag{123.21}$$

in particular, $C_0 = \Gamma$. The values of the remaining coefficients $C_2, \ldots, C_n$ are found from conditions, completely analogous to the conditions (123.10). It is easily seen that the required single-valuedness of the displacements will then be ensured.

NOTE. It has been assumed that the part L'' of the boundary is free from external stresses. However, the solution of the case, where L'' is subjected to arbitrarily given loads, is easily written down; for this purpose it is sufficient to refer to the statements in § 111 (cf. also the Note at the end of § 114).

§ 123a Example.

This example was presented in the paper [1] by B. L. Mintzberg who started from his own general formulae which are more complicated than those deduced here; he was therefore obliged to evaluate several integrals, in contrast to the present method where the solution is obtained almost without any calculations. Other problems, treated in Mintzberg's paper, may likewise be solved in this way.

Let a rigid stamp be applied to the arc $L' = ab$ of the circular hole of radius 1; let the profile of this stamp be an arc of a circle of the same radius and let it be rigidly attached to the elastic body. A normal force of magnitude P_0 which is distributed symmetrically is applied through the stamp. It will be assumed that the stresses vanish at infinity.

Further, let the centre of the arc ab lie on the positive part of the Oy axis, so that $X = 0$, $Y = P_0$. In the present case, $n = 1$, $g(t) = \text{const.}$, $g'(t) = 0$, $\Gamma = \Gamma' = 0$. Therefore, by (123.19),

$$\Phi(z) = X_0(z)\left\{C_0 z + C_1 + \frac{D_1}{z} + \frac{D_2}{z^2}\right\},$$

where

$$X_0(z) = (z - a)^{-\frac{1}{2}+i\beta}\ (z - b)^{-\frac{1}{2}-i\beta}\ ,$$

with the supplementary condition $\lim_{z\to\infty} zX_0(z) = 1$.

It follows from (123.20), since $\Gamma' = 0$, $X + iY = iP_0$, that

$$D_2 = 0, \quad D_1X_0(0) = \frac{i\varkappa P_0}{2\pi(\varkappa + 1)}\ ,$$

and from (123.21), since $\Gamma = 0$ and, for large $|z|$, $X_0(z) = z^{-1} + O(1)$, that

$$C_0 = 0, \quad C_1 = -\frac{iP_0}{2\pi(\varkappa + 1)}\ .$$

It is also easily established by investigating the changes in the arguments of $(z - a)$ and $(z - b)$ as z moves along the Ox axis from infinity to the point $z = 0$ that

$$X_0(0) = ie^{\omega\beta},$$

where ω is the central angle subtended by the arc ab, and consequently

$$D_1 = \frac{\varkappa P_0 e^{-\omega\beta}}{2\pi(\varkappa + 1)}\ .$$

With those values for the constants one finally obtains

$$\Phi(z) = \frac{P_0}{2\pi(\varkappa + 1)}\ (z - a)^{-\frac{1}{2}+i\beta}\ (z - b)^{-\frac{1}{2}-i\beta}\left\{\frac{\varkappa e^{-\omega\beta}}{z} - i\right\},$$

and the problem is solved.

§ 124. Boundary problems for the plane, cut along circular arcs *). Let an elastic body occupy the plane, cut along the arcs $L_1 = a_1b_1, \ldots, L_k = a_kb_k$ of one and the same circle. As previously, it will be assumed that the ends of the arcs are encountered in the order $a_1, b_1, \ldots, a_n, b_n, a_1$ as one moves around the circle in an anti-clockwise direction. The union of these arcs will now be denoted by L, so that

$$L = L_1 + L_2 + \ldots + L_n.$$

Let the radius of the circle be of unit length and its centre coincide with the origin.

The solution of the boundary problems for such a body may be obtained in a similar manner as in the case of the plane with straight cuts (§ 120).

A beginning will be made with (121.1)—(121.3) in which now $\Phi(z)$ and $\Psi(z)$ are defined throughout the plane, cut along L, and instead of $\Psi(z)$ a function $\Omega(z)$ will be introduced which will be defined in the following manner:

$$\Omega(z) = \bar{\Phi}\left(\frac{1}{z}\right) - \frac{1}{z}\bar{\Phi}'\left(\frac{1}{z}\right) - \frac{1}{z^2}\bar{\Psi}\left(\frac{1}{z}\right), \tag{124.1}$$

whence

$$\Psi(z) = \frac{1}{z^2}\Phi(z) - \frac{1}{z^2}\bar{\Omega}\left(\frac{1}{z}\right) - \frac{1}{z}\Phi'(z). \tag{124.2}$$

It follows from (124.1) that $\Omega(z)$ is holomorphic everywhere in the plane cut along L (including the point $z = \infty$), except at $z = 0$ where it has a pole of not higher than second order. In fact, it is easily seen from (121.5) and (121.6) that near $z = 0$

$$\Omega(z) = -\frac{\bar{\Gamma}'}{z^2} - \frac{\varkappa(X + iY)}{2\pi(\varkappa + 1)}\frac{1}{z} + \text{a holomorphic function}; \tag{124.3}$$

if this condition is satisfied, the function $\Psi(z)$, defined by (124.2), will satisfy the condition (121.6).

Further, in order that $\Psi(z)$, as given by (124.2), will be holomorphic near the point $z = 0$, the function $\Omega(z)$ must satisfy certain conditions; in fact, if

$$\begin{aligned} \Phi(z) &= A_0 + A_1z + \ldots \quad (\text{for } |z| < 1), \\ \Omega(z) &= B_0 + \frac{B_1}{z} + \ldots \quad (\text{for } |z| > 1), \end{aligned} \tag{124.4}$$

*) To the Author's knowledge, no solution of these problems has been published previously.

then, for $\Psi(z)$ to be holomorphic near $z = 0$, it is necessary and sufficient that

$$A_0 = \bar{B}_0, \quad B_1 = 0. \tag{124.5}$$

It will also be recalled that for large $|z|$

$$\Phi(z) = \Gamma - \frac{X + iY}{2\pi(\varkappa + 1)} \frac{1}{z} + O\left(\frac{1}{z^2}\right). \tag{124.6}$$

Since the components of stress and displacement may be expressed in terms of $\Phi(z)$, $\Psi(z)$, they may likewise be expressed in terms of $\Phi(z)$ and $\Omega(z)$. In fact,

$$\widehat{rr} + i\,\widehat{r\vartheta} = \Phi(z) + \Omega\left(\frac{1}{\bar{z}}\right) + \bar{z}\left(\bar{z} - \frac{1}{z}\right)\overline{\Psi(z)}, \tag{124.7}$$

$$2\mu(u' + iv') = iz\left[\varkappa\Phi(z) - \Omega\left(\frac{1}{\bar{z}}\right) - \bar{z}\left(\bar{z} - \frac{1}{z}\right)\Psi(z)\right], \tag{124.8}$$

where $\Psi(z)$ is now given by (124.2) and

$$u' = \frac{\partial u}{\partial \vartheta}, \quad v' = \frac{\partial v}{\partial \vartheta}.$$

It will now be assumed that for all points $t = e^{i\vartheta}$ on L, except the ends a_k, b_k, the functions $\Phi(z)$ and $\Omega(z)$ are continuous at L from the left and from the right and that

$$\lim_{r \to 1} (1 - r)\Psi(z) = 0. \tag{124.9}$$

In addition, it will be assumed that near any end c

$$|\Phi(z)| < \frac{\text{const.}}{|z - c|^\alpha}, \quad |\Omega(z)| < \frac{\text{const.}}{|z - c|^\alpha}, \quad 0 \leqslant \alpha < 1. \tag{124.10}$$

By means of the above formulae, all the boundary problems, solved in § 120 for the case of straight cuts, may now be solved for the present type of cuts by methods, quite analogous to those used in § 120. In view of this analogy, consideration will here be restricted to the solution of *the first fundamental* problem, when the stresses on both sides of L are given, i.e., when the values of $\widehat{rr}^+ + i\widehat{r\vartheta}^+$ and of $\widehat{rr}^- + i\widehat{r\vartheta}^-$ on L are known.

In addition, it will be assumed that the values of the stresses at infinity, i.e., the values of the constants B and Γ', as defined by (121.8), are known.

Let the rotation vanish at infinity, i.e., assume that $C = 0$ and therefore $\Gamma = \overline{\Gamma} = B$.

By (124.7) and (124.9),

$$\Phi^+(t) + \Omega^-(t) = \widehat{rr}^+ + i\widehat{r\vartheta}^+, \ \Phi^-(t) + \Omega^+(t) = \widehat{rr}^- + i\widehat{r\vartheta}^- \text{ on } L, \tag{124.11}$$

whence, adding and subtracting,

$$[\Phi(t) + \Omega(t)]^+ + [\Phi(t) + \Omega(t)]^- = 2p(t) \quad \text{on } L, \tag{124.12}$$

$$[\Phi(t) - \Omega(t)]^+ - [\Phi(t) - \Omega(t)]^- = 2q(t) \quad \text{on } L, \tag{124.13}$$

where

$$\begin{aligned} p(t) &= \tfrac{1}{2}[\widehat{rr}^+ + \widehat{rr}^-] + \frac{i}{2}[\widehat{r\vartheta}^+ + \widehat{r\vartheta}^-], \\ q(t) &= \tfrac{1}{2}[\widehat{rr}^+ - \widehat{rr}^-] + \frac{i}{2}[\widehat{r\vartheta}^+ - \widehat{r\vartheta}^-]. \end{aligned} \tag{124.14}$$

Taking into consideration that the function $\Phi(z) - \Omega(z)$ is bounded at infinity and has, in accordance with (124.3), at $z = 0$ a pole with the principal part

$$\frac{\overline{\Gamma}'}{z^2} + \frac{\varkappa(X + iY)}{2\pi(\varkappa + 1)} \cdot \frac{1}{z},$$

one obtains from (124.13), using (108.5),

$$\Phi(z) - \Omega(z) = \frac{1}{\pi i}\int_L \frac{q(t)dt}{t - z} + D_0 + \frac{\varkappa(X + iY)}{2\pi(\varkappa + 1)} \cdot \frac{1}{z} + \frac{\overline{\Gamma}'}{z^2},$$

where D_0 is a constant.

Similarly, one obtains from (124.12), using (110.26),

$$\Phi(z) + \Omega(z) = \frac{1}{\pi i X(z)}\int_L \frac{X(t)p(t)dt}{t - z} + \frac{1}{X(z)}\left\{P_n(z) + \frac{D_1}{z} + \frac{D_2}{z^2}\right\},$$

where $X(z)$ denotes one of the branches of

$$X(z) = \prod_{k=1}^{n}(z - a_k)^{\frac{1}{2}} (z - b_k)^{\frac{1}{2}} \tag{124.15}$$

which is single-valued in the plane, cut along L, and $X(t)$ has been written for $X^+(t)$, i.e., the value of $X(z)$ on the left side of L; further, D_1 and D_2 are constants and

$$P_n(z) = C_0 z^n + C_1 z^{n-1} + \ldots + C_n$$

is a polynomial of degree not higher than n. [Note that, in applying (110.26), use has been made of the fact that $X_0(z) = 1/X(z)$].

Thus

$$\Phi(z) = \frac{1}{2\pi i X(z)} \int\limits_L \frac{X(t)p(t)dt}{t-z} + \frac{1}{2\pi i} \int\limits_L \frac{q(t)dt}{t-z} +$$

$$+ \frac{1}{2X(z)} \left\{ P_n(z) + \frac{D_1}{z} + \frac{D_2}{z^2} \right\} + \frac{D_0}{2} + \frac{\varkappa(X+iY)}{4\pi(\varkappa+1)} \frac{1}{z} + \frac{\overline{\Gamma}'}{2z^2}, \quad (124.17)$$

$$\Omega(z) = \frac{1}{2\pi i X(z)} \int\limits_L \frac{X(t)p(t)dt}{t-z} - \frac{1}{2\pi i} \int\limits_L \frac{q(t)dt}{t-z} +$$

$$+ \frac{1}{2X(z)} \left\{ P_n(z) + \frac{D_1}{z} + \frac{D_2}{z^2} \right\} - \frac{D_0}{2} - \frac{\varkappa(X+iY)}{4\pi(\varkappa+1)} \frac{1}{z} - \frac{\overline{\Gamma}'}{2z^2}. \quad (124.18)$$

The constants D_1 and D_2 may be determined immediately from (124.3) which, by (124.18), takes the form

$$\frac{1}{2X(z)} \left\{ \frac{D_1}{z} + \frac{D_2}{z^2} \right\} = - \frac{\varkappa(X+iY)}{4\pi(\varkappa+1)} \frac{1}{z} - \frac{\overline{\Gamma}'}{2z^2} + O(1) \quad (124.19)$$

near the point $z = 0$. If D_1 and D_2 satisfy this condition (by which they are uniquely determined), the right-hand side of (124.17) will be holomorphic near $z = 0$.

The remaining constants in the above formulae, i.e.,

$$D_0, C_0, C_1, \ldots, C_n, \quad (124.20)$$

of which there are $n+2$, are determined by the following conditions: $\Phi(\infty) = \Gamma$, (124.5) and the single-valuedness of the displacements; this last condition (giving n equations) may be expressed in an analogous manner as in the case of straight cuts (§ 120, 2°). It is easily shown on the basis of the uniqueness theorem that these conditions determine the constants (124.20) uniquely.

There are actually $n + 3$ conditions for the determination of the $n + 2$ constants, i.e., there is one more condition than there are unknown constants. This is due to the fact that the quantity $X + iY$ in (124.3) had been assumed known, i.e., it had been calculated beforehand from the stresses, given on the boundary. However, one may assume that it is initially unknown and find its value together with those of the constants (124.20) from the above conditions. Similarly, one could have left the coefficient of z^{-2} in (124.3) indeterminate in which case one would have to retain the condition $\Psi(\infty) = \Gamma'$.

It will be left to the reader to construct the solutions of the second and mixed fundamental problems, where in the latter case the external stresses and the displacements are given on the left and right edges respectively.

§ 124a Example

Extension of the plane, cut along a circular arc.

Let the elastic plane be cut along the circular arc ab, let the edges of the cut be free from external stresses and let the stresses at infinity, i.e., the constants Γ and Γ', be given, while $\Gamma = \overline{\Gamma}$ (implying that the rotation vanishes at infinity). The radius of the arc will be taken as unity and its centre at the origin; the axis Ox will be assumed to pass through the midpoint of the arc ab which subtends at the origin an angle 2θ so that

$$a = e^{-i\theta}, \quad b = e^{i\theta}. \tag{124.1a}$$

In the present case, $n = 1$, $p(t) = q(t) = 0$, $X = Y = 0$. Hence one finds from (124.17) and (124.18)

$$\Phi(z) = \frac{1}{2X(z)}\left\{C_0 + C_1 + \frac{D_1}{z} + \frac{D_2}{z^2}\right\} + \frac{D_0}{2} + \frac{\overline{\Gamma}'}{2z^2}, \tag{124.2a}$$

$$\Omega(z) = \frac{1}{2X(z)}\left\{C_0 z + C_1 + \frac{D_1}{z} + \frac{D_2}{z^2}\right\} - \frac{D_0}{2} - \frac{\overline{\Gamma}'}{2z^2}, \tag{124.3a}$$

where now

$$X(z) = \sqrt{(z-a)(z-b)} = \sqrt{z^2 - 2z\cos\theta + 1}. \tag{124.4a}$$

It will be assumed that $z^{-1}X(z) \to 1$ for $z \to \infty$. It is easily seen that under this condition $X(0) = -1$ and hence, near $z = 0$,

$$\frac{1}{X(z)} = -\left(1 - \frac{z}{a}\right)^{-\frac{1}{2}}\left(1 - \frac{z}{b}\right)^{-\frac{1}{2}} =$$

$$= -\left(1 + \frac{z}{2a} + \frac{3}{8}\frac{z^2}{a^2} + \dots\right)\left(1 + \frac{z}{2b} + \frac{3}{8}\frac{z^2}{b^2} + \dots\right) =$$

$$-1 - z\cos\theta - \frac{1 + 3\cos 2\theta}{4} z^2 + \dots . \tag{124.5a}$$

Hence for small $|z|$

$$\frac{1}{2X(z)}\left(\frac{D_1}{z}+\frac{D_2}{z^2}\right)=-\frac{D_2}{2z^2}-\frac{D_1+D_2\cos\theta}{2z}-\frac{D_1\cos\theta}{2}-\frac{1+3\cos 2\theta}{8}D_2+\ldots, \tag{124.6a}$$

and (124.19) gives

$$D_2=\bar{\Gamma}',\quad D_1=-\bar{\Gamma}'\cos\theta. \tag{124.7a}$$

For the determination of D_0, C_0, C_1, one may use the formula (124.6), which now has the form

$$\Phi(z)=\Gamma+O\left(\frac{1}{z^2}\right)\ \text{for large}\ |z| \tag{124.8a}$$

and the conditions (124.5). Since for large $|z|$

$$\frac{1}{X(z)}=\frac{1}{z}+\frac{\cos\theta}{z^2} \tag{124.9a}$$

one obtains, by (124.2*a*) and (124.8*a*),

$$C_0+D_0=2\Gamma,\quad C_1+C_0\cos\theta=0. \tag{a}$$

The second condition of (124.5) does not give any new information (as it coincides with the second of the preceding conditions). In order to formulate the first condition of (124.5), it will be noted that, by (124.2*a*), (124.6*a*) and (124.7*a*),

$$A_0=\Phi(0)=\frac{C_1}{2X(0)}-\frac{D_1\cos\theta}{2}-\frac{1+3\cos 2\theta}{8}D_2+\frac{D_0}{2}=\frac{C_1}{2}+\frac{D_0}{2}+\frac{\bar{\Gamma}'\sin^2\theta}{4}$$

and that, by (124.3*a*),

$$B_0=\Omega(\infty)=\frac{C_0-D_0}{2}$$

therefore the first condition of (124.5) gives

$$C_0-D_0=-\bar{C}_1+\bar{D}_0+\tfrac{1}{2}\bar{\Gamma}'\sin^2\theta. \tag{b}$$

The relations (*a*) and (*b*) determine all the unknown constants; in

fact, one finds

$$C_0 = \tfrac{1}{2}(\Gamma' - \bar{\Gamma}') \sin^2\frac{\theta}{2} + \frac{4\Gamma + (\Gamma' + \bar{\Gamma}')\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{2\left(1 + \sin^2\frac{\theta}{2}\right)}. \tag{124.10a}$$

$$C_1 = -C_0 \cos\theta, \quad D_0 = 2\Gamma - C_0. \tag{124.11a}$$

It is easily verified that the displacements are single-valued; this condition has not been used above, since another condition, ensuring their single-valuedness, had been introduced into the preliminary analysis. Thus the problem is solved.

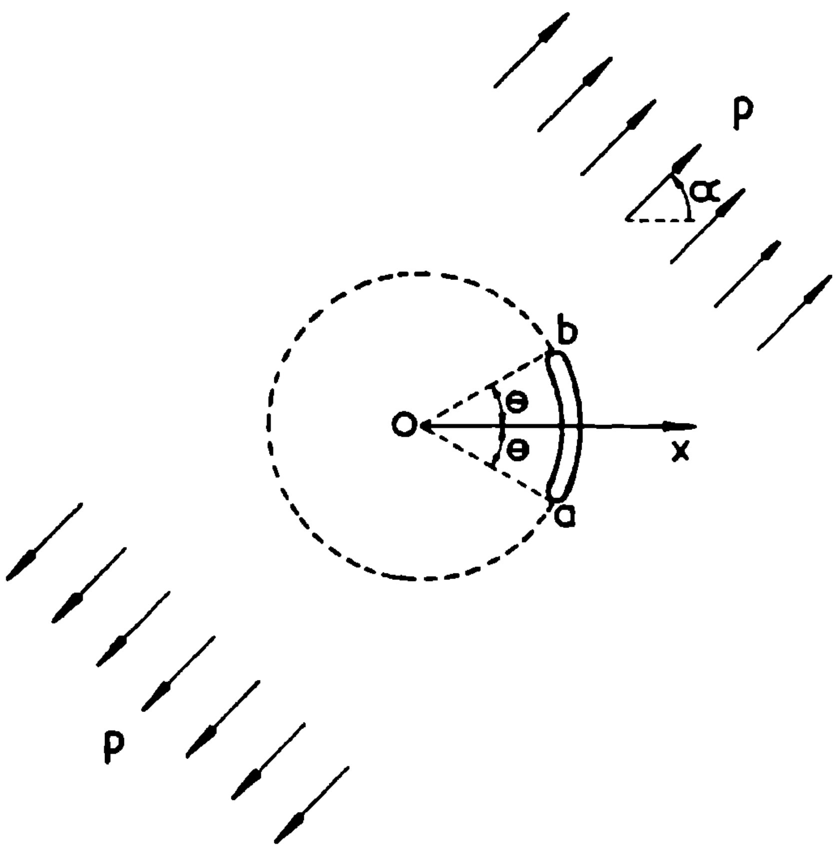


Fig. 56.

In particular, if the stresses at infinity reduce to a tension p in a direction, forming an angle α with the Ox axis (Fig. 56), then

$$\Gamma = \frac{t}{\cdot} \qquad \Gamma' \qquad -\frac{p}{2} e^{-2i\alpha}. \tag{124.12a}$$

If one has uniform tension p in all directions, then

$$\Gamma = \frac{p}{2}, \quad \Gamma' = 0. \tag{124.13a}$$

In this case

$$D_1 = D_2 = 0, \quad C_0 = \frac{p}{1 + \sin^2\frac{\theta}{2}},$$
$$C_1 = -\frac{p\cos\theta}{1 + \sin^2\frac{\theta}{2}}, \quad D_0 = \frac{p\sin^2\frac{\theta}{2}}{1 + \sin^2\frac{\theta}{2}}, \tag{124.14}$$

and hence

$$\Phi(z) = \frac{p}{2\left(1 + \sin^2\frac{\theta}{2}\right)}\left\{\frac{z - \cos\theta}{\sqrt{1 - 2z\cos\theta + z^2}} + \sin^2\frac{\theta}{2}\right\},$$
$$\Omega(z) = \frac{p}{2\left(1 + \sin^2\frac{\theta}{2}\right)}\left\{\frac{z - \cos\theta}{\sqrt{1 - 2z\cos\theta + z^2}} - \sin^2\frac{\theta}{2}\right\}. \tag{124.15a}$$

In particular, for a semi-circular cut ($\theta = \pi/2$), these formulae take the simple form

$$\Phi(z) = \frac{p}{3}\left\{\frac{z}{\sqrt{z^2 + 1}} + \frac{1}{2}\right\}, \quad \Omega(z) = \frac{p}{3}\left\{\frac{z}{\sqrt{z^2 + 1}} - \frac{1}{2}\right\}. \tag{124.16a}$$

CHAPTER 21

SOLUTION OF THE BOUNDARY PROBLEMS FOR REGIONS, MAPPED ON TO THE CIRCLE BY RATIONAL FUNCTIONS

The methods of solution of the preceding chapters are easily extended to the cases of regions, mapped on to the circle by rational functions. It has already been seen in the preceding Part that the first and second fundamental problems for such regions are easily solved in closed form.

The new method, studied below, leads to the same results, and about the same amount of calculation is involved in obtaining the final solution as would be required, when using the method of the preceding Part.

However, the present method offers the possibility of solving also *the fundamental mixed problem*, and likewise some other boundary problems.

§ 125. Transformation of the general formulae *). Let S be a finite or infinite region in the z plane, bounded by one simple smooth contour L, and let

$$z = \omega(\zeta) \tag{125.1}$$

be the function which maps S on to the circle $|\zeta| < 1$ of the ζ plane; the boundary of this circle will be denoted by γ and its positive direction will be assumed to be counter-clockwise.

If the region S is finite, the function $\omega(\zeta)$ is holomorphic inside γ; if S is infinite, $\omega(\zeta)$ is holomorphic everywhere inside γ, except at the point corresponding to $z = \infty$, where it has a simple pole. Without affecting generality, it may be assumed that this point is the centre of γ, i.e., the point $\zeta = 0$; under this supposition

$$z = \omega(\zeta) = \frac{c}{\zeta} + \omega_0(\zeta), \tag{125.1'}$$

where $\omega_0(\zeta)$ is holomorphic inside γ and c is a constant which is not zero.

* The results of §§ 125-127 are due to I.N. Kartzivadze [2]; some simplifications have been introduced by the Author.

The formulae will now be recalled which express the components of stress and displacement in the corresponding curvilinear coordinates in terms of the functions $\Phi(\zeta)$ and $\Psi(\zeta)$ of the complex variable $\zeta = \rho e^{i\vartheta}$ (§ 50):

$$\widehat{\rho\rho} + \widehat{\vartheta\vartheta} = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}], \tag{125.2}$$

$$\widehat{\rho\rho} + i\widehat{r\vartheta} = \Phi(\zeta) + \overline{\Phi(\zeta)} - \frac{\bar{\zeta}^2}{\rho^2\overline{\omega'(\zeta)}} \{\omega(\zeta)\overline{\Phi'(\zeta)} + \overline{\omega'(\zeta)}\overline{\Psi(\zeta)}\}, \tag{125.3}$$

$$2\mu \mid \omega'(\zeta) \mid (v_\rho + iv_\vartheta) = \frac{\zeta}{\rho} \overline{\omega'(\zeta)} \{\varkappa\varphi(\zeta) - \omega(\zeta)\overline{\Phi(\zeta)} - \overline{\psi(\zeta)}\}, \tag{125.4}$$

where $\varphi(\zeta)$, $\psi(\zeta)$ are related to $\Phi(\zeta)$, $\Psi(\zeta)$ by the formulae

$$\varphi'(\zeta) = \Phi(\zeta)\omega'(\zeta), \quad \psi'(\zeta) = \Psi(\zeta)\omega'(\zeta). \tag{125.5}$$

In the majority of cases it is more convenient to use instead of (125.4) the formula

$$2\mu(u + iv) = \varkappa\varphi(\zeta) - \omega(\zeta)\overline{\Phi(\zeta)} - \overline{\psi(\zeta)} \tag{125.6}$$

which gives the displacement components u, v in rectangular coordinates.

If $\varphi(\zeta)$, $\psi(\zeta)$ are given, the functions $\Phi(\zeta)$, $\Psi(z)$ are completely determined; however, if $\Phi(\zeta)$, $\Psi(\zeta)$ are given, the functions $\varphi(\zeta)$, $\psi(\zeta)$ are only determined apart from arbitrary constants. Hence one may in the last case rewrite (125.6)

$$2\mu(u + iv) = \varkappa\varphi(\zeta) - \omega(\zeta)\overline{\Phi(\zeta)} - \overline{\psi(\zeta)} + \text{const.}, \tag{125.6'}$$

stressing the presence of the arbitrary constant.

It will now be assumed that $\omega(\zeta)$ is a *rational function*; the definition of $\Phi(\zeta)$ will be extended into the region $|\zeta| > 1$ by writing

$$\omega'(\zeta)\Phi(\zeta) = -\omega'(\zeta)\bar{\Phi}\left(\frac{1}{\zeta}\right) + \frac{1}{\zeta^2}\omega(\zeta)\bar{\Phi}'\left(\frac{1}{\zeta}\right) +$$
$$+ \frac{1}{\zeta^2}\bar{\omega}'\left(\frac{1}{\zeta}\right)\bar{\Psi}\left(\frac{1}{\zeta}\right) \quad \text{for } |\zeta| > 1. \tag{125.7}$$

This extension has been chosen so that the values of $\Phi(\zeta)$ on the left and on the right of γ extend each other analytically through the unloaded parts of the boundary L of the region S, i.e., through the segments on which $\widehat{\rho\rho} = \widehat{\rho\vartheta} = 0$; in this choice one was guided by (125.3).

Replacing in (125.7) ζ by $\bar{\zeta}^{-1}$ ($|\bar{\zeta}|$ $|\zeta| < 1$) and going to the conjugate complex expression, one finds

$$\omega'(\zeta)\Psi(\zeta) = \frac{1}{\zeta^2}\,\bar{\omega}'\left(\frac{1}{\zeta}\right)\left\{\Phi(\zeta) + \bar{\Phi}\left(\frac{1}{\zeta}\right)\right\} - \bar{\omega}\left(\frac{1}{\zeta}\right)\Phi'(\zeta). \qquad (125.8)$$

This formula expresses $\Psi(\zeta)$ for $|\zeta| < 1$ (this function is not defined for other values) in terms of $\Phi(\zeta)$ for $|\zeta| < 1$ as well as for $|\zeta| > 1$.

The definition of $\varphi(\zeta)$ may likewise be extended to the region $|\zeta| > 1$ by imposing the condition that in this region

$$\varphi(\zeta) = \int \Phi(\zeta)\omega'(\zeta)d\zeta; \qquad (125.9)$$

integrating both sides of (125.7) with respect to ζ, one easily obtains, after omitting an arbitrary constant,

$$\varphi(\zeta) = -\,\omega(\zeta)\,\bar{\Phi}\left(\frac{1}{\zeta}\right) - \bar{\psi}\left(\frac{1}{\zeta}\right) \quad \text{for } |\zeta| > 1, \qquad (125.7')$$

whence, similarly as before,

$$\psi(\zeta) = -\,\bar{\varphi}\left(\frac{1}{\zeta}\right) - \bar{\omega}\left(\frac{1}{\zeta}\right)\Phi(\zeta) \quad \text{for } |\zeta| < 1. \qquad (125.8')$$

Thus the components of stress and displacement may be expressed in terms of the single function $\Phi(\zeta)$, defined for $|\zeta| < 1$ as well as for $|\zeta| > 1$.

The expression (125.2) for $\widehat{\rho\rho} + \widehat{\vartheta\vartheta}$ remains unchanged, while (125.3) is easily seen to take the form

$$\widehat{\rho\rho} + i\,\widehat{\rho\vartheta} = \Phi(\zeta) - \Phi\left(\frac{1}{\bar{\zeta}}\right) + \bar{\zeta}^2\left\{\frac{\omega(\bar{\zeta}^{-1})}{\omega'(\bar{\zeta}^{-1})} - \frac{\omega(\zeta)}{\rho^2\omega'(\zeta)}\right\}\overline{\Phi'(\zeta)} +$$
$$+ \zeta^2\,\overline{\omega'(\zeta)}\left\{\frac{1}{\omega'(\bar{\zeta}^{-1})} - \frac{1}{\rho^2\omega'(\zeta)}\right\}\overline{\Psi(\zeta)}, \qquad (125.10)$$

where now $\Psi(\zeta)$ is given by (125.8).

In order to deduce (125.10), one has to proceed as follows: add and subtract the function $\Phi(1/\bar{\zeta})$ on the right-hand side of (125.3) and replace the added term by the expression, obtained from (125.8) after going first to the conjugate complex value.

Replacing $\psi(\zeta)$ in (125.6') by the expression (125.8'), one obtains for

the displacement components u, v

$$2\mu(u + iv) = \varkappa\varphi(\zeta) + \varphi\left(\frac{1}{\bar{\zeta}}\right) - \left\{\omega(\zeta) - \omega\left(\frac{1}{\bar{\zeta}}\right)\right\}\overline{\Phi(\zeta)} + \text{const.} \quad (125.11)$$

From (50.4) follows the analogous formula

$$\frac{\partial U}{\partial x} + i\frac{\partial U}{\partial y} = \varphi(\zeta) - \varphi\left(\frac{1}{\bar{\zeta}}\right) + \left\{\omega(\zeta) - \omega\left(\frac{1}{\bar{\zeta}}\right)\right\}\overline{\Phi(\zeta)} + \text{const.} \quad (125.12)$$

In the sequel also the expression for $u' + iv'$ will be required, where

$$u' = \frac{\partial u}{\partial \vartheta}, \quad v' = \frac{\partial v}{\partial \vartheta}.$$

This expression will be deduced by differentiating both sides of (125.6) with respect to ϑ and by transforming the resulting expression in the same manner as (125.3) above. One thus obtains

$$2\mu(u' + iv') = i\zeta\omega'(\zeta)\left\{\varkappa\Phi(\zeta) + \Phi\left(\frac{1}{\bar{\zeta}}\right)\right\} -$$

$$- i\rho^2\omega'(\zeta)\left\{\frac{\bar{\zeta}\omega(\bar{\zeta}^{-1})}{\omega'(\bar{\zeta}^{-1})} - \frac{\omega(\zeta)}{\zeta\omega'(\zeta)}\right\}\overline{\Phi'(\zeta)} -$$

$$- i\rho^2\omega'(\zeta)\overline{\omega'(\zeta)}\left\{\frac{\bar{\zeta}}{\omega'(\bar{\zeta}^{-1})} - \frac{1}{\zeta\omega'(\zeta)}\right\}\overline{\Psi(\zeta)}. \quad (125.13)$$

Under the usual conditions the functions $\Phi(\zeta)$ and $\Psi(\zeta)$ are holomorphic inside γ. The behaviour of $\Phi(\zeta)$, extended by (125.7) to the region $|\zeta| > 1$, will now be studied outside γ; it is sufficient for this purpose to investigate the behaviour of $\Phi(\zeta)\omega'(\zeta) = \varphi'(\zeta)$.

Revert to the formula (125.7) which defines $\Phi(\zeta)$ for $|\zeta| > 1$. The rational function $\omega(\zeta)$ may have poles at a finite number of points; all these points lie outside γ, except in the case, where S is infinite and where, consequently, $\omega(\zeta)$ has a simple pole for $\zeta = 0$.

Denote by $\zeta_1, \zeta_2, \ldots, \zeta_r$ the poles of $\omega(\zeta)$ outside γ, not counting the point $\zeta = \infty$ which may likewise be a pole. If the orders of these poles are $m_1 - 1, m_2 - 1, \ldots, m_r - 1$, the function $\omega'(\zeta)$ will have at the same points poles of order $m_1, m_2, \ldots, m_r$; further, if $\omega(\zeta)$ has at infinity a pole of order $m + 1$, $\omega'(\zeta)$ will have there a pole of order m.

Thus $\Phi(\zeta)\omega'(\zeta)$ will have poles of order not greater than $m_1, m_2, \ldots, m_r$ at the points $\zeta_1, \zeta_2, \ldots, \zeta_r$; these poles originate from the first two terms on the right-hand side of (125.7), because, as is readily seen, the third

term represents a function, holomorphic outside γ, including the point at infinity. In addition, the point $\zeta = \infty$ may be a pole of order not greater than m.

It should also be noted that in the case of an infinite region $\Phi(\zeta)\omega'(\zeta)$ may have inside γ, and, in fact, at $\zeta = 0$, a pole of not greater than the second order.

Thus, *all possible poles of the function* $\Phi(\zeta)\omega'(\zeta)$ and *the maximum orders of these poles are known beforehand.*

Finally, it should be stressed that there will not always be a function $\Psi(\zeta)$, holomorphic inside γ as required by the present conditions, corresponding to a given $\Phi(\zeta)$, defined inside as well as outside γ and having poles of the stated type. In fact, formula (125.8) shows that $\Psi(\zeta)$, corresponding to a given $\Phi(\zeta)$, may have poles at the points

$$\zeta_1' = \frac{1}{\bar{\zeta}_1}, \ldots, \zeta_r' = \frac{1}{\bar{\zeta}_r},$$

and also at the point $\zeta = 0$, lying inside γ.

Expressing that $\Psi(\zeta)$ is also to remain holomorphic at the stated points, one obtains a known (finite) number of linear equations, relating a certain (finite) number of the first coefficients in the expansions of $\Phi(\zeta)$ near the points $\zeta_1', \ldots, \zeta_r'$

$$\Phi(\zeta) = A_{k0} + A_{k_1}(\zeta - \zeta_k') + A_{k2}(\zeta - \zeta_k')^2 + \ldots, k = 1, 2, \ldots, r$$

to the coefficients of the principal parts of the poles of the function $\Phi(\zeta)$ at the points $\zeta_1, \ldots, \zeta_r$; likewise, a known number of analogous linear relations corresponds to the point $\zeta = 0$.

In order to verify the above, one has to keep in mind that the principal part of the pole of

$$\bar{\omega}'\left(\frac{1}{\zeta}\right)\bar{\Phi}\left(\frac{1}{\zeta}\right)$$

at ζ_k' may be obtained directly from the principal part of the pole of the function

$$\omega'(\zeta)\Phi(\zeta)$$

at the point ζ_k. In fact, if near ζ_k

$$\omega'(\zeta)\Phi(\zeta) = \frac{B_l}{(\zeta - \zeta_k)^l} + \ldots + \frac{B_1}{\zeta - \zeta_k} + \text{a holomorphic function,}$$

then near $\zeta_k' = \bar{\zeta}_k^{-1}$

$$\bar{\omega}'\left(\frac{1}{\zeta}\right)\bar{\Phi}\left(\frac{1}{\zeta}\right) = \frac{(-1)^l \bar{B}_l \zeta_k'^l \zeta^l}{(\zeta - \zeta_k')^l} + \ldots - \frac{\bar{B}_1 \zeta_k' \zeta}{\zeta - \zeta_k'} + \text{a holomorphic function.}$$

Similarly for the pole at the point $\zeta = 0$.

These relations will not be written down here, but will simply be denoted by

$$F_1 = 0, \quad F_2 = 0, \quad \ldots, \quad F_N = 0. \tag{125.14}$$

They are always easily constructed for any given region, i.e., for any given function $\omega(\zeta)$.

The construction of these conditions is especially simple in the case, where $\omega(\zeta)$ is a polynomial, i.e.,

$$\omega(\zeta) = c_1\zeta + c_2\zeta^2 + \ldots + c_{m+1}\zeta^{m+1}, \tag{125.15}$$

when S is finite, or

$$\omega(\zeta) = \frac{c}{\zeta} + c_1\zeta + \ldots + c_{m+1}\zeta^{m+1}, \tag{125.16}$$

when S is infinite. The function $\Phi(\zeta)\omega'(\zeta)$ may then only have poles at the points $\zeta = \infty$ and $\zeta = 0$ (the last pole only being possible when S is infinite).

Note also that, if S is infinite, one has near $\zeta = 0$ [cf. (50.14), (50.15)]

$$\Phi(\zeta)\omega'(\zeta) = -\frac{\Gamma c}{\zeta^2} + \frac{X + iY}{2\pi(\varkappa + 1)}\frac{1}{\zeta} + O(1), \tag{125.17}$$

$$\Psi(\zeta)\omega'(\zeta) = -\frac{\Gamma' c}{\zeta^2} - \frac{\varkappa(X - iY)}{2\pi(\varkappa + 1)}\frac{1}{\zeta} + O(1), \tag{125.18}$$

where c is the same constant as in (125.1′) and, in the previous notation,

$$\begin{aligned}\Gamma &= B + iC = \tfrac{1}{4}(N_1 + N_2) + \frac{2\mu\varepsilon_\infty}{\varkappa + 1}i,\\ \Gamma' &= B' + iC' = -\tfrac{1}{2}(N_1 - N_2)e^{-2i\alpha},\end{aligned} \tag{125.19}$$

while X, Y are again the components of the resultant vector of the external forces, applied to the boundary of S.

In future, it will be assumed that $\Phi(\zeta)$, defined for $|\zeta| < 1$ and for $|\zeta| > 1$, is continuous at all points σ of the circle γ from the left as well as from the right, except possibly at a finite number of points $\gamma_k = e^{i\vartheta_k}$ near which

$$|\Phi(\zeta)| < \frac{\text{const.}}{|\zeta - \gamma_k|^\alpha}, \quad 0 \leqslant \alpha < 1; \tag{125.20}$$

in addition, it will be assumed that for all points $\sigma = e^{i\vartheta}$ of the circle γ,

except possibly for the same points $\gamma_k = e^{i\vartheta_k}$,

$$\lim_{\rho \to 1} (1 - \rho)\Phi'(\rho e^{i\vartheta}) = 0. \tag{125.21}$$

By (125.8), one will then also have

$$\lim_{\rho \to 1} (1 - \rho)\Psi'(\rho e^{i\vartheta}) = 0 \tag{125.22}$$

with the same reservations as before.

NOTE. 1. It is easily seen, on the basis of (125.21) and (125.22), that the last two terms on the right-hand sides of (125.10) and (125.13) tend to zero as $\rho \to 1$, except possibly for the values $\vartheta = \vartheta_k$.

NOTE. 2. In the case of infinite regions S, it is sometimes more convenient to make use of transformations on to the region $|\zeta| > 1$ rather than on to the circle $|\zeta| < 1$; however, this distinction is not of great importance. The reader will easily introduce the necessary modifications in some of the preceding formulae.

§ 126. Solution of the first and second fundamental problems. These problems have already been solved for regions of the type under consideration in Part V. The formulae of the preceding section offer the opportunity of solving these problems in a very simple manner. Consider, for example, the case of the first fundamental problem, where the boundary condition has the form

$$\widehat{\rho\rho}^{+} + i\,\widehat{\rho\vartheta}^{+} = N(\sigma) + iT(\sigma); \tag{126.1}$$

$N(\sigma)$ and $T(\sigma)$ are given functions of the point σ of the circle γ, since the normal and tangential stresses are given at the point t of the actual boundary L which correspond to the point σ. By (125.10), one then has

$$\Phi^{+}(\sigma) - \Phi^{-}(\sigma) = N(\sigma) + iT(\sigma). \tag{126.2}$$

Thus one has reached, for the determination of $\Phi(\zeta)$, the same boundary problem which was obtained in the case, when the region S is a circle (§ 122, 1°). The essential difference arises from the fact that the unknown function $\Phi(\zeta)$ may now have poles outside γ, and this circumstance must be taken into consideration when constructing the general solution of the boundary problem (126.2).

From the practical point of view it will be more convenient to somewhat modify the condition (126.2) by writing it in the form

$$[\Phi(\sigma)\omega'(\sigma)]^{+} - [\Phi(\sigma)\omega'(\sigma)]^{-} = [N(\sigma) + iT(\sigma)]\omega'(\sigma) \tag{126.2'}$$

and by choosing $\Phi(\zeta)\omega'(\zeta)$ as the unknown function. The distribution of poles of this function has been discussed in the preceding section; it will be recalled that in the case, where the region S is infinite, the function $\Phi(\zeta)\omega'(\zeta)$ may also have a pole (of not higher than second order) inside γ, and, in fact, at $\zeta = 0$.

The general solution of the problem (126.2′) has the form

$$\Phi(\zeta)\omega'(\zeta) = \frac{1}{2\pi i}\int_{\gamma} \frac{N(\sigma) + iT(\sigma)}{\sigma - \zeta}\, \omega'(\sigma)d\sigma + R(\zeta), \tag{126.3}$$

where $R(\zeta)$ is a rational function of ζ for which the general expression is easily written down, since all possible poles and their maximum orders of the function $\Phi(\zeta)\omega'(\zeta)$ are known.

The arbitrary constants of $R(\zeta)$ may be determined from the following supplementary conditions:

1. The function $\Psi(\zeta)$, determined by (125.8), must be holomorphic inside γ.

2. In the case, where the region S is infinite, the stresses must have given values at infinity and the displacements must be single-valued.

The first condition above is expressed by the relations (125.14) which give a set of linear algebraic equations involving the real and imaginary parts of the unknown coefficients; the second condition renders similar equations. These equations completely determine the unknown constants, except for one real constant, in agreement with the fact that $\Phi(\zeta)$ is only determined apart from an additive term iC, where C is an arbitrary real constant. In the case of finite regions, the above equations will only be compatible, provided the resultant vector and moment of the external forces vanishes.

The above statements are a direct consequence of the uniqueness and existence theorems.

By means of a more detailed analysis, it may be shown that these assertions are not based on the existence theorem (but only on the uniqueness theorem); this was done in the paper by I. N. Kartzivadze quoted above.

The second fundamental problem may be solved in quite an analogous manner; on the basis of (125.13), this problem leads to the determination of $\Phi(\zeta)$ from the boundary condition

$$[\varkappa\Phi(\sigma)\omega'(\sigma)]^{+} + [\Phi(\sigma)\omega'(\sigma)]^{-} = 2\mu g'(\sigma), \tag{126.4}$$

where

$$g'(\sigma) = \frac{dg}{d\sigma} = ie^{-i\vartheta}\frac{dg}{d\vartheta} \tag{126.5}$$

$g(\sigma) = g_1 + ig_2$, g_1 and g_2 being the boundary values of the displacement components u, v.

The amount of calculations, required for the solution of the first and second fundamental problems by the present method, is approximately equal to that required, when applying the method of Part V. Therefore no more will be said about it here, particularly, since the first and second fundamental problems are particular cases of the mixed fundamental problem which will be considered in more detail in the next section.

NOTE 1. In the case of the first fundamental problem for an infinite region S, the conditions 2 above are expressed by (125.17), (125.18), where the real part B of the constant Γ and the constant Γ', determined by the stress components at infinity, must be assumed known. The constants X and Y may remain undetermined, as their values will be found from the remaining conditions, referred to above. However, they may be calculated beforehand from the given boundary values of the stresses; then, when requiring $\Phi(\zeta)$ and $\Psi(\zeta)$ to satisfy the conditions (125.17), (125.18), one will obtain additional equations which may be used to replace some of the other, less simple relations between the unknown quantities.

In the case of the second fundamental problem for infinite regions, the constants X and Y as well as Γ, Γ' must be assumed known.

NOTE 2. In order to solve the first and second fundamental problems, one may, of course, begin from (125.12) and (125.11) respectively. This will be especially convenient in the case of a finite region, because the unknown function $\varphi(\zeta)$ will then be single-valued. However, in the case of an infinite region, the multi-valuedness of the unknown function is easily removed by separating the logarithmic term, just as it was done in Part V.

§ 127. Solution of the mixed fundamental problem.

Let $L_1 = a_1b_1$, $L_2 = a_2b_2$, $\ldots$, $L_n = a_nb_n$ be the arcs of the boundary L of the elastic body S, numbered in such a way that the ends are encountered in the order a_1, b_1, $\ldots$, a_n, b_n, when passing around L in the positive direction. Let $L' = L_1 + \ldots + L_n$ and L'' be the remaining part of the boundary.

Let the displacements be given on L' and the external stresses on L''. Without affecting generality, it may be assumed that L'' is free from external stresses. The general case is also easily solved directly (cf. Note at the end of this section).

Denote by α_k, β_k the points of the circle γ which correspond to the points a_k, b_k of L, by γ' the part of the circle, corresponding to L', and by γ'' the remaining part of γ. The points α_k, β_k will play the parts of the points γ_k, mentioned at the end of § 125.

On the basis of (125.10) and (125.13), the boundary conditions of the present problem may be written

$$\Phi^+(\sigma) - \Phi^-(\sigma) = 0 \text{ on } \gamma'', \tag{127.1}$$

$$[\Phi(\sigma)\omega'(\sigma)]^+ + \frac{1}{\varkappa}[\Phi(\sigma)\omega'(\sigma)]^- = f(\sigma) \text{ on } \gamma', \tag{127.2}$$

where, if g_1, g_2 are the given boundary values of the displacement components u, v on L',

$$f(\sigma) = -\frac{2\mu i}{\varkappa}\bar{\sigma}\left\{\frac{dg_1}{d\vartheta} + i\frac{dg_2}{d\vartheta}\right\} = \frac{2\mu}{\varkappa}\left\{\frac{dg_1}{d\sigma} + i\frac{dg_2}{d\sigma}\right\}. \tag{127.3}$$

The condition (127.1) shows that γ'' is not a line of discontinuity of the function $\Phi(\zeta)$, i.e., that $\Phi(\zeta)$ is holomorphic in the plane, cut along γ', except at a finite number of points, where it may have poles; the same is, of course, true with regard to the function $\Phi(\zeta)\omega'(\zeta)$.

For the determination of this last function one has the condition (127.2) which is exactly the same as the condition, obtained when dealing with the mixed fundamental problem for the case, where S is a circle (§ 123,1°); however, this time the unknown function may have poles at predetermined points $\zeta_1, \zeta_2, \ldots, \zeta_r, \infty$, the order of these poles not being higher than the given numbers $m_1, m_2, \ldots, m_r, m$ (§ 125). When the region S is infinite, there may also occur a pole of not higher than the second order at the point $\zeta = 0$.

As in § 123, let

$$\beta = \frac{\log \varkappa}{2\pi}, \tag{127.4}$$

$$X_0(\zeta) = \prod_{k=1}^{n}(\zeta - \alpha_k)^{-\frac{1}{2}-i\beta}(\zeta - \beta_k)^{-\frac{1}{2}+i\beta}, \tag{127.5}$$

where $X_0(\zeta)$ represents the branch for which

$$\lim_{\zeta\to\infty} \zeta^n X_0(\zeta) = +1; \tag{127.6}$$

one then obtains, by (110.26),

$$\omega'(\zeta)\Phi(\zeta) = \frac{X_0(\zeta)}{2\pi i}\int_{\gamma'} \frac{f(\sigma)d\sigma}{X_0^+(\sigma)\,(\sigma-\zeta)} + X_0(\zeta)R(\zeta), \tag{127.7}$$

where $R(\zeta)$ is a rational function of the form

$$R(\zeta) = \frac{D_2}{\zeta^2} + \frac{D_1}{\zeta} + C_0 + C_1\zeta + \ldots + C_{m+n}\zeta^{m+n} + \\ + \sum_{k=1}^{r}\sum_{k=1}^{m_r}\frac{B_{kl}}{(\zeta-\zeta_k)^l} \tag{127.8}$$

(in the case of a finite region S: $D_1 = D_2 = 0$).

The constants D_j, C_j, B_{kl} in (127.8) have still to be determined on the basis of the following conditions:

1. The function $\Psi(\zeta)$, corresponding to $\Phi(\zeta)$ and defined by (125.8), must be holomorphic inside γ. This condition is expressed by the relations (125.14).

2. In the case of an infinite region S, the components of stress and rotation must have given values at infinity, the components of the resultant vector of the external forces, applied to L', must likewise take given values and the displacement components should be single-valued. These last conditions are equivalent to (125.17), (125.18) for given values of Γ, Γ', X, Y.

3. Finally, the fact must be taken into consideration that, if all the above conditions are satisfied, the displacement components u, v will only take on the arcs $\alpha_k\beta_k$ the given values, apart from certain constants c_k $(k = 1, 2, \ldots, n)$, because, when solving the problem, it was only demanded that the derivatives of u and v with respect to ϑ have given values on these arcs. Thus, one has, in addition to the above conditions,

$$c_1 = c_2 = \ldots = c_n = 0 \tag{127.9}$$

which may be replaced by the weaker conditions

$$c_1 = c_2 = \ldots = c_n, \qquad (127.10$$

because, when (127.10) is fulfilled, the condition (127.9) may be satisfied by use of the arbitrary constant on the right-hand side of (125.6').

The condition (127.10) may be expressed in a manner, quite analogous to that followed in § 123, when S was a circle. Therefore it will be unnecessary to write out the corresponding formulae.

From the conditions 1—3 above one obtains a known number of linear algebraic equations for the determination of the unknown constants which will in this way be completely determined, as is easily seen on the basis of the theorems of uniqueness and existence of the solution.

The solution will be particularly simple, when the boundary L contains only two arcs $a_1b_1 = L'$ and $b_1a_1 = L''$, i.e., when $n = 1$; in that case the conditions (127.10) are superfluous.

NOTE 1. When the part L'' of the boundary L is not free from external stresses, but subject to external loads, the problem is likewise easily solved directly. In that case the boundary condition takes the form

$$\begin{aligned} \Big[\Phi(\sigma)\omega'(\sigma)\Big]^+ + \frac{1}{\varkappa}\Big[\Phi(\sigma)\omega'(\sigma)\Big]^- &= f(\sigma) \text{ on } \gamma', \\ \Big[\Phi(\sigma)\omega'(\sigma)\Big]^+ - \Big[\Phi(\sigma)\omega'(\sigma)\Big]^- &= f(\sigma) \text{ on } \gamma'', \end{aligned} \tag{127.11}$$

where

$$\begin{aligned} f(\sigma) &= \frac{2\mu}{\varkappa}\left\{\frac{dg_1}{d\sigma} + i\frac{dg_2}{d\sigma}\right\} \text{ on } \gamma', \\ f(\sigma) &= \omega'(\sigma)\Big\{N(\sigma) + iT(\sigma)\Big\} \text{ on } \gamma''; \end{aligned} \tag{127.12}$$

$N(\sigma)$ and $T(\sigma)$ denote here the same as in § 126. It will be assumed that $f(\sigma)$ satisfies the H condition on each of the parts γ' and γ'' (but that it may be discontinuous at the points α_k, β_k).

Applying the results of § 111, one obtains the formula, completely analogous to (127.7),

$$\Phi(\zeta) = \frac{X_0(\zeta)}{2\pi i}\int_\gamma \frac{f(\sigma)d\sigma}{X_0^+(\sigma)(\sigma-\zeta)} + X_0(\zeta)R(\zeta), \tag{127.13}$$

where $X_0(\zeta)$ and $R(\zeta)$ are the same as before, i.e., as determined by (127.5) and (127.8); however, the integral now extends over the entire

circle γ and $f(\sigma)$ is determined by (127.12). The remaining calculations are the same as before.

NOTE 2. One may (and sometimes this is more convenient) solve the present problem by beginning with the simpler formulae (125.11), (125.12) [cf. Note 2 at the end of § 126]. In that case it must not be overlooked that, when determining the function $\varphi(\zeta)$ from the corresponding boundary condition, this function has to remain bounded near the points α_k, β_k, as follows easily from the conditions, imposed earlier on the function $\Phi(\zeta)$.

§ 127a. Example. Solution of the mixed fundamental problem for the plane with an elliptic hole

In the notation of § 48,5°, one has in this case for the transformation on to the circle $|\zeta| < 1$

$$\omega(\zeta) = R\left(\frac{1}{\zeta} + m\zeta\right), \quad R > 0, \quad 0 \leqslant m < 1. \tag{127.1a}$$

One may also (and even somewhat more conveniently) use the transformation on to the region $|\zeta| > 1$; however, the transformation on to the circle will be used here, in order to be able to make direct use of the formulae of the preceding section.

The formulae (125.7) and (125.8) then take the form

$$\omega'(\zeta)\Phi(\zeta) = R\left(\frac{1}{\zeta^2} - m\right)\bar{\Phi}\left(\frac{1}{\zeta}\right) + \frac{R}{\zeta^2}\left(\frac{1}{\zeta} + m\zeta\right)\bar{\Phi}'\left(\frac{1}{\zeta}\right) - \\ - R\left(1 - \frac{1}{\zeta^2}\right)\bar{\Psi}\left(\frac{1}{\zeta}\right) \quad \text{for } |\zeta| > 1, \tag{127.2a}$$

$$\omega'(\zeta)\Psi(\zeta) = -R\left(1 - \frac{m}{\zeta^2}\right)\Phi(\zeta) - R\left(1 - \frac{m}{\zeta^2}\right)\bar{\Phi}\left(\frac{1}{\zeta}\right) - \\ - R\left(\zeta + \frac{m}{\zeta}\right)\Phi'(\zeta) \quad \text{for } |\zeta| < 1, \tag{127.3a}$$

while (125.17) and (125.18) become

$$\omega'(\zeta)\Phi(\zeta) = -\frac{R\Gamma}{\zeta^2} + \frac{X + iY}{2\pi(\varkappa + 1)}\frac{1}{\zeta} + O(1), \tag{127.4a}$$

$$\omega'(\zeta)\Psi(\zeta) = -\frac{R\Gamma'}{\zeta^2} - \frac{\varkappa(X - iY)}{2\pi(\varkappa + 1)}\frac{1}{\zeta} + O(1). \tag{127.5a}$$

For simplicity, assume that $n = 1$, i.e., that the contour L is divided into two parts a_1a_2, a_2a_1, the second of which is free from external stresses, while the displacements are given on the first.

Denoting by σ_1 and σ_2 the points of γ which correspond to the points a_1 and a_2 of the ellipse, one has

$$X_0(\zeta) = (\zeta - \sigma_1)^{-\frac{1}{2}-i\beta} (\zeta - \sigma_2)^{-\frac{1}{2}+i\beta}, \quad \beta = \frac{\log \varkappa}{2\pi}. \tag{127.6a}$$

Let

$$\sigma_1 = e^{i\vartheta_1}, \quad \sigma_2 = e^{i\vartheta_2}, \quad \vartheta_1 = \vartheta_0 - \frac{\omega}{2}, \quad \vartheta_2 = \vartheta_0 + \frac{\omega}{2}, \tag{127.7a}$$

where ϑ_0 is the argument of the midpoint of the arc $\sigma_1\sigma_2$ and ω is the angle subtended by this arc at the centre of the circle.

For large $|\zeta|$

$$X_0(\zeta) = \frac{1}{\zeta} + \frac{\alpha}{\zeta^2} + \dots, \tag{127.8a}$$

where

$$\alpha = \frac{\sigma_1 + \sigma_2}{2} + i\beta(\sigma_1 - \sigma_2) = e^{i\vartheta_0}\left(\cos\frac{\omega}{2} + 2\beta \sin\frac{\omega}{2}\right). \tag{127.9a}$$

Further, it is easily verified that

$$X_0(0) = -e^{-\beta\omega - i\vartheta_0}, \tag{127.10a}$$

so that for small $|\zeta|$

$$X_0(\zeta) = X_0(0)\left[1 - \frac{\zeta}{\sigma_1}\right]^{-\frac{1}{2}-i\beta}\left[1 - \frac{\zeta}{\sigma_2}\right]^{-\frac{1}{2}+i\beta} =$$

$$= X_0(0)[1 + \alpha_0\zeta + \dots], \tag{127.11a}$$

where

$$\alpha_0 = \frac{\bar\sigma_1 + \bar\sigma_2}{2} + i\beta(\bar\sigma_1 - \bar\sigma_2) = e^{-i\vartheta_0}\left(\cos\frac{\omega}{2} - 2\beta \sin\frac{\omega}{2}\right). \tag{127.12a}$$

The formula (127.2a) shows that $\omega'(\zeta)\Phi(\zeta)$ must be holomorphic outside γ, including the point at infinity; further, since $\Phi(\zeta)$ is holomorphic inside γ, the function $\omega'(\zeta)\Phi(\zeta)$ has at $\zeta = 0$ a pole of not higher than the second order. Therefore, in agreement with (127.7),

$$\omega'(\zeta)\Phi(\zeta) = \frac{X_0(\zeta)}{2\pi i}\int_{\sigma_1\sigma_2} \frac{f(\sigma)d\sigma}{X_0^+(\sigma)\,(\sigma - \zeta)} +$$

$$+ \left\{C_0 + C_1\zeta + \frac{D_1}{\zeta} + \frac{D_2}{\zeta^2}\right\} X_0(\zeta), \tag{127.13a}$$

where C_0, C_1, D_1, D_2 are constants, subject to determination.

The values of the constants D_1 and D_2 are determined directly from (127.4*a*); in fact, by (127.13*a*) and (127.11*a*), the principal part of the pole of the function $\Phi(\zeta)\omega'(\zeta)$ at $\zeta = 0$ is given by

$$X_0(0)\left\{\frac{D_2}{\zeta^2} + \frac{D_1 + \alpha_0 D_2}{\zeta}\right\},$$

whence, by comparison with (127.4*a*),

$$X_0(0)D_2 = -R\Gamma,\ X_0(0)(D_1 + \alpha_0 D_2) = \frac{X + iY}{2\pi(\varkappa + 1)}; \qquad (127.14a)$$

the values of D_1 and D_2 may be obtained from these formulae.

The coefficients C_0 and C_1 may be determined by the help of the conditions (127.5*a*). For this purpose the principal part of the pole at $\zeta = 0$ of the function $\Psi(\zeta)\omega'(\zeta)$, as determined by (127.3*a*), will now be calculated. It is given by

$$\frac{\bar{C}_1 - mD_2X_0(0)}{\zeta^2} + \frac{\bar{C}_0 + \bar{\alpha}\bar{C}_1}{\zeta}.$$

Comparison with (127.4*a*), taking into consideration (127.14*a*) and going to the conjugate complex value, yields

$$C_1 \qquad R(m\Gamma + \Gamma'), \qquad C_0 + \alpha C_1 = \frac{\varkappa(X + iY)}{2\pi(\varkappa + 1)} \qquad (127.15a)$$

Thus all the constants have been determined and the problem is solved. For $m = 0$, one obtains the solutions for the infinite plane with a circular hole. This case was considered independently in § 123, 2°.

§ 128. The problem of contact with a rigid stamp.

1°. Statement of the problem. Uniqueness of solution

In very many cases, occurring in practice, the boundary problems arise from the contact of the surface of the elastic body under consideration with the surfaces of other bodies. Several particular cases of problems of this type have been considered in § 58, §§ 115—119.

The case will be studied here where the given elastic body is in contact with an absolutely rigid body of given shape. It will be assumed that *contact occurs along the entire boundary of the elastic body* and that the surfaces of the bodies are perfectly smooth, so that there is no friction.

To the Author's knowledge this problem was first formulated and solved by J. Hadamard [2] for the case of an elastic sphere.

The solution of the problem for plane regions, mapped on to the circle by means of rational functions, was given in the Author's paper [19] and it was reproduced, with some additions, in the preceding editions of this book. In the second part of this section a solution will be deduced which is somewhat different in appearance, but essentially the same.

In the later work, consideration will be restricted to the plane case and it will be assumed that the boundary of the elastic body consists of one simple contour; however, the body may be finite or infinite (infinite plate with a hole). Hence one will have to deal with one of the following two cases:

A. Case of a finite region

An elastic disc is inserted into an opening of given shape in a rigid body (plate); the boundary and position of the disc, before deformation, differs slightly from the shape of the hole into which it is pressed (because, as always, displacements have to be small).

B. Case of an infinite region

Into a hole in an infinite elastic body (plate) a rigid disc is inserted whose boundary and known position, before deformation, differs somewhat from that of the hole. In this case it will be assumed that the values of the stresses and rotation at infinity are given (i.e., that the constants Γ, Γ' are known) as well as the resultant vector (X, Y) of the external forces, exerted by the disc on the surrounding material. This vector is obviously equal to the resultant vector of the forces, applied from outside the disc (the forces, exerted by the elastic body on the edges of the disc, are not included here).

The boundary conditions of these problems will now be constructed, although they could have been written down simply on the basis of the results of § 115 which hold for the particular case of a straight boundary. However, they will be approached here in a somewhat different, possibly slightly clearer, manner and certain additional observations will be made.

First of all, since there is no friction, one will have on the boundary of the elastic body

$$T = 0,$$

where T is the tangential stress, acting on the boundary.

Next, the condition of contact between the elastic and the rigid bodies will be stated. As indicated earlier, it will be assumed in the sequel that contact occurs along the entire boundary. For greater clarity, attention will be concentrated, for the time being, on the case A. Let the elastic disc originally lie on the hole in the rigid plate (like a lid), so that its edge somewhat overlaps the edge of the hole. Further, let the points of the boundary of the disc, as a result of suitable forces applied to this contour, execute normal displacements v_n of such magnitude that in the end the boundaries of the disc and of the hole will coincide. The disc will then be inserted into the hole. The disc will now be in some state of elastic equilibrium which is to be determined. Since the points of the edge of the disc can slip freely along the edge of the hole, the tangential displacements on the boundary will be initially unknown. However, the normal displacements v_n will be given, since they will be determined by the position of the boundary of the disc before deformation relative to the edge of the hole. Thus the boundary conditions of the present problem are

$$T = 0, \quad v_n = f \text{ on the boundary,} \tag{128.1}$$

where f is a given real function of the arc coordinate of the contour.

Consider now the following circumstance. The process of compressing the disc until it has the dimensions of the hole (by means of normal displacements v_n) may be performed, beginning from different positions of the disc before deformation; all these positions may be obtained from some fixed position by means of rigid body displacements of the disc (as always, one is here only concerned with small displacements). If one begins from some position of the disc (before deformation), different from that on which the second condition of (128.1) was based, the quantity f there will have a value f' which differs from f by the normal components of the rigid body displacement necessary to return the disc to its original position; the boundary conditions will now be

$$T = 0, \quad v_n = f' \text{ on the boundary.} \tag{128.1'}$$

However, it is obvious that the solution of the problem (128.1′) may be obtained from that of (128.1) by superimposing on the latter the above-mentioned rigid body displacement which is known not to affect the stress distributions.

Next consider Problem B (of an infinite region). Repeating the above reasoning almost word for word, one arrives again at (128.1) which has now to be supplemented by the conditions, stated earlier (i.e., that the

constants Γ, Γ', X, Y must be known). It should still be mentioned that the rigid body displacement of the elastic plate will be purely translatory, because the constant $C = \mathfrak{J}(\Gamma)$, characterizing the rotation at infinity, is, by supposition, given beforehand.

It is easily shown that the problem, corresponding to the boundary conditions (128.1), cannot have two different solutions. In fact, it will be remembered that in the proofs of the uniqueness of the solutions of the fundamental problems an important part was played by the fact that the expression

$$X_n u + Y_n v$$

for the "difference" of two solutions vanished on the boundary (§ 40). However, this expression is the scalar product of the vector (X_n, Y_n), representing the stresses applied to the boundary, and the vector (u, v), representing the displacements of the points of the boundary. Further, since for the "difference" of two solutions satisfying the boundary conditions (128.1) one has

$$T = 0, \quad v_n = 0,$$

the vectors (X_n, Y_n), (u, v) will be perpendicular to each other, and hence their scalar product will vanish.

Therefore, by the same reasoning as in § 40, it may be verified that the stress components in both solutions will be identical, and consequently the displacements may only differ by rigid body displacements.

Further, it is obvious that, if one excludes the case when the boundary is a circle, the displacements can likewise not be different. In the case of a circular disc, solutions may clearly differ from each other by rigid rotations about the centre of the circle; in the case of an infinite plate with a circular hole, one will again have complete definiteness, because it has been assumed that the rotation at infinity is given.

It has been shown that solutions of the stated problems, if they exist, are unique; the existence was recently proved by D. I. Sherman [22]. No space will be devoted to it here, but instead an effective method of solution will be stated for regions, mapped on to the circle by the help of rational functions.

2°. Solution for regions, mapped on to the circle by rational functions

The method below is completely analogous to that, studied in detail in § 126 for the cases of the first and second fundamental problems.

Therefore only general remarks will be made here and the application of the method will be demonstrated by means of examples.

In the Author's paper [19] (and likewise in the second edition of this book) the solution was obtained by the method, applied in Part V to the solution of the fundamental problems.

Let the region S be mapped on to the circle $|\zeta| < 1$ by the relation

$$z = \omega(\zeta), \tag{128.2}$$

where, by supposition, $\omega(\zeta)$ is a rational function (one may also use the transformation on to the region $|\zeta| > 1$); the circle $|\zeta| = 1$ will again be denoted by γ and the positive direction on it will be taken as counter-clockwise.

The boundary conditions, in the notation of § 50, take the form

$$\widehat{\rho\vartheta} = 0, \quad v_\rho = f \text{ on the boundary.} \tag{128.3}$$

The expressions for $\widehat{\rho\vartheta}$ and v_ρ in terms of complex functions may be obtained from (50.11) and (50.7) respectively. In order to deduce that for $\widehat{\rho\vartheta}$, it is sufficient to subtract (50.11) from its conjugate complex expression in which case one finds on the left-hand side $2i\widehat{\rho\vartheta}$. In a similar manner, v_ρ may be calculated from (50.7). With these expressions (128.3) leads to

$$\sigma^2\omega'(\sigma)\{\omega(\sigma)\Phi'(\sigma) + \omega'(\sigma)\Psi(\sigma)\} - \bar{\sigma}^2\overline{\omega'(\sigma)}\,\{\overline{\omega(\sigma)}\overline{\Phi'(\sigma)} + \overline{\omega'(\sigma)}\overline{\Psi(\sigma)}\} = 0,$$

$$\sigma\omega'(\sigma)\left\{\varkappa\overline{\varphi(\sigma)} - \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\varphi'(\sigma) - \psi(\sigma)\right\} +$$

$$+\ \bar{\sigma}\overline{\omega'(\sigma)}\left\{\varkappa\varphi(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\,\overline{\varphi'(\sigma)} - \overline{\psi(\sigma)}\right\} = 4\mu f(\sigma)\,|\,\omega'(\sigma)\,|,$$

where all terms are to be interpreted as the boundary values of the respective functions as $\zeta \to \sigma$ from inside γ; $f(\sigma)$ denotes a known real function of σ.

For the present, it will be assumed that in the case, where S is infinite, the resultant vector (X, Y) of the external forces, applied to the edge of the hole (i.e., to the boundary of S) is equal to zero. In addition, the stresses are to vanish at infinity.

Under these conditions, $\varphi(\zeta)$ and $\psi(\zeta)$ as well as $\Phi(\zeta)$ and $\Psi(\zeta)$ will be holomorphic inside γ.

Introduce now the following functions which are sectionally holomorphic, except for a finite number of poles:

$$\Omega_1(\zeta) = \begin{cases} \zeta^2\omega'(\zeta)\left[\bar{\omega}\left(\frac{1}{\zeta}\right)\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)\right] & \text{for } |\zeta|<1, \\ \frac{1}{\zeta}\,\bar{\omega}'\left(\frac{1}{\zeta}\right)\left[\omega(\zeta)\bar{\Phi}'\left(\frac{1}{\zeta}\right) + \bar{\omega}'\left(\frac{1}{\zeta}\right)\bar{\Psi}\left(\frac{1}{\zeta}\right)\right] & \text{for } |\zeta|>1, \end{cases} \tag{128.4}$$

$$\Omega_2(\zeta) = \begin{cases} \frac{\varkappa}{\zeta}\,\bar{\omega}'\left(\frac{1}{\zeta}\right)\varphi(\zeta) - \zeta\bar{\omega}\left(\frac{1}{\zeta}\right)\varphi'(\zeta) - \zeta\omega'(\zeta)\psi(\zeta) & \text{for } |\zeta| < 1, \\ -\varkappa\zeta\omega'(\zeta)\bar{\varphi}\left(\frac{1}{\zeta}\right) + \frac{1}{\zeta}\,\omega(\zeta)\bar{\varphi}'\left(\frac{1}{\zeta}\right) + & \\ \qquad + \frac{1}{\zeta}\,\bar{\omega}'\left(\frac{1}{\zeta}\right)\bar{\psi}\left(\frac{1}{\zeta}\right) & \text{for } |\zeta| > 1. \end{cases} \tag{128.5}$$

Obviously, the preceding boundary conditions may now be written

$$\Omega_1^+(\sigma) - \Omega_1^-(\sigma) = 0, \tag{128.6}$$

$$\Omega_2^+(\sigma) - \Omega_2^-(\sigma) = 4\mu\,|\,\omega'(\sigma)\,|\,f(\sigma). \tag{128.7}$$

As indicated above, the functions $\Omega_1(\zeta)$, $\Omega_2(\zeta)$ are sectionally holomorphic, except for a finite number of poles, i.e., they are holomorphic in each of the regions $|\zeta| < 1$, $|\zeta| > 1$, except for a finite number of points where they have poles. These poles and their maximum orders will be known beforehand, since they arise from the poles of the rational function $\omega(\zeta)$ and from the factors ζ^{-1}, ζ on the right-hand side of (128.5). It is readily seen that to each pole ζ_k inside (outside) γ there corresponds a pole $\zeta_k' = 1/\bar{\zeta}_k$ of the same order outside (inside) γ.

Applying now the results of § 108 to the solution of the boundary problems (128.6), (128.7), one finds

$$\Omega_1(\zeta) = R_1(\zeta), \tag{128.8}$$

$$\Omega_2(\zeta) = \frac{2\mu}{\pi i}\int_\gamma \frac{|\,\omega'(\sigma)\,|\,f(\sigma)d\sigma}{\sigma-\zeta} + R_2(\zeta), \tag{128.9}$$

where $R_1(\zeta)$ and $R_2(\zeta)$ are rational functions with undetermined coefficients which have at given points poles whose order is not greater than a known limit. The general expressions are easily written down, but this will not be done here and only the following observations will be

made. By the definitions (128.4) and (128.5) of the functions $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$, one must have

$$\bar{\Omega}_1\left(\frac{1}{\zeta}\right) = \Omega_1(\zeta), \quad \bar{\Omega}_2\left(\frac{1}{\zeta}\right) = -\Omega_2(\zeta),$$

whence, by (128.8) and (128.9), it is found that the rational functions $R_1(\zeta)$ and $R_2(\zeta)$ must satisfy the following identities:

$$\bar{R}_1\left(\frac{1}{\zeta}\right) = R_1(\zeta), \tag{128.10}$$

$$\bar{R}_2\left(\frac{1}{\zeta}\right) = -R_2(\zeta) - \frac{2\mu}{\pi i}\int_\gamma |\omega'(\sigma)|\, f(\sigma)\frac{d\sigma}{\sigma}; \tag{128.11}$$

in order to deduce the last condition, use has been made of the fact that, if $f(\sigma)$ is a real function and if

$$F(\zeta) = \frac{2\mu}{\pi i}\int_\gamma \frac{f\, d\sigma}{\sigma - \zeta},$$

then

$$\bar{F}\left(\frac{1}{\zeta}\right) = -\frac{2\mu}{\pi i}\int_\gamma \frac{f(\sigma)d\bar{\sigma}}{\bar{\sigma} - 1/\zeta}$$

or, noting that $\bar{\sigma} = 1/\sigma$,

$$\bar{F}\left(\frac{1}{\zeta}\right) = \frac{2\mu}{\pi i}\int_\gamma \frac{\zeta f(\sigma)d\sigma}{\sigma(\zeta - \sigma)} = -\frac{2\mu}{\pi i}\int_\gamma \frac{f(\sigma)d\sigma}{\sigma - \zeta} + \frac{2\mu}{\pi i}\int_\gamma f\frac{d\sigma}{\sigma}.$$

The relations (128.10), (128.11) impose definite conditions on the coefficients of $R_1(\zeta)$ and $R_2(\zeta)$; these conditions, together with others to be stated below, serve for the determination of the above coefficients.

Applying (128.8) and (128.9) to points inside γ, one deduces from (128.4) and (128.5)

$$\zeta^2\omega'(\zeta)\left\{\bar{\omega}\left(\frac{1}{\zeta}\right)\Phi'(\zeta) + \omega'(\zeta)\Psi(\zeta)\right\} = R_1(\zeta), \tag{128.12}$$

$$-\zeta\bar{\omega}\left(\frac{1}{\zeta}\right)\varphi'(\zeta) - \zeta\omega'(\zeta)\psi(\zeta) + \frac{\varkappa}{\zeta}\bar{\omega}'\left(\frac{1}{\zeta}\right)\varphi(\zeta) =$$

$$= \frac{2\mu}{\pi i}\int_\gamma \frac{f(\sigma)\,|\omega'(\sigma)|\,d\sigma}{\sigma - \zeta} + R_2(\zeta). \tag{128.13}$$

Application of (128.8) and (128.9) to points outside γ renders no new information and only leads to the conditions (128.10), (128.11) which will be assumed to be fulfilled. For this reason consideration may be restricted to the preceding equations.

Dividing these equations by $\zeta^2\omega'(\zeta)$ and $\zeta\omega'(\zeta)$ respectively and noting that

$$\omega'(\zeta)\Phi(\zeta) = \varphi'(\zeta), \quad \omega'(\zeta)\Psi(\zeta) = \psi'(\zeta),$$

these same equations may be rewritten

$$\psi'(\zeta) + \bar{\omega}\left(\frac{1}{\zeta}\right)\left[\frac{\varphi'(\zeta)}{\omega'(\zeta)}\right]' = G(\zeta), \tag{128.14}$$

$$-\psi(\zeta) - \frac{\bar{\omega}\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)}\varphi'(\zeta) + \frac{\varkappa}{\zeta^2}\frac{\bar{\omega}'\left(\frac{1}{\zeta}\right)}{\omega'(\zeta)}\varphi(\zeta) = H(\zeta), \tag{128.15}$$

where $G(\zeta)$, $H(\zeta)$ are known functions, containing linearly a certain number of constants, as yet undetermined.

The function $\psi(\zeta)$ is easily eliminated between the last two equations. In fact, differentiating the second equation and adding it to the first, one finds, after certain reductions,

$$(\varkappa + 1)\,\Omega(\zeta)\varphi'(\zeta) + \varkappa\Omega'(\zeta)\varphi(\zeta) = G(\zeta) + H'(\zeta), \tag{128.16}$$

where

$$\Omega(\zeta) = \frac{\bar{\omega}'\left(\frac{1}{\zeta}\right)}{\zeta^2\omega'(\zeta)}. \tag{128.17}$$

Thus the function $\varphi(\zeta)$ satisfies the linear, first order differential equation

$$\varphi'(\zeta) + \nu\frac{\Omega'(\zeta)}{\Omega(\zeta)}\varphi(\zeta) = F(\zeta), \tag{128.18}$$

where

$$F(\zeta) = \frac{G(\zeta) + H'(\zeta)}{(\varkappa + 1)\Omega(\zeta)} \tag{128.19}$$

is a known function, containing linearly a certain number of undetermined constants, and

$$\nu = \frac{\varkappa}{\varkappa + 1} \quad (\tfrac{1}{2} < \nu < 1). \tag{128.20}$$

Integrating (128.18), one obtains

$$\varphi(\zeta) = [\Omega(\zeta)]^{-\nu}[K + \int F(\zeta)[\Omega(\zeta)]^{\nu} d\zeta], \tag{128.21}$$

where K is a constant.

Having found $\varphi(\zeta)$, one may determine $\psi(\zeta)$ from (128.15). The unknown constants in the expressions for $\varphi(\zeta)$ and $\psi(\zeta)$ may be determined from the conditions (128.10), (128.11) and also from the requirement that these functions are to be holomorphic inside γ.

It has been assumed in the case, where the region S is infinite, that the stresses vanish at infinity. This condition is not essential. If it is assumed that the stresses have given finite values at infinity, the preceding reasoning will remain valid. It must only be noted that in the case under consideration the functions $\varphi(\zeta)$ and $\psi(\zeta)$ have first order poles at $\zeta = 0$ with known principal parts which can only affect the form of the rational functions $R_1(\zeta)$, $R_2(\zeta)$.

In addition, it has been assumed that in the case of infinite regions the resultant vector (X, Y) is equal to zero. If the vector (X, Y) is not zero, the corresponding problem is easily reduced to the preceding one (cf. the second example of the next section).

§ 128a. Examples.

1°. Circular disc. In this case

$$z = \omega(\zeta) = R\zeta, \tag{128.1a}$$

where R is the radius of the disc and the boundary conditions (128.6) (128.7), written explicitly, take the form (after dividing the first equation by R^2 and the second by R)

$$[\sigma\Phi'(\sigma) + \sigma^2\Psi(\sigma)]^+ - \left[\frac{1}{\sigma}\overline{\Phi}'\left(\frac{1}{\sigma}\right) + \frac{1}{\sigma^2}\overline{\Psi}\left(\frac{1}{\sigma}\right)\right]^- = 0, \tag{128.2a}$$

$$\left[\frac{\varkappa}{\sigma}\varphi(\sigma) - \varphi'(\sigma) - \sigma\psi(\sigma)\right]^+ -$$

$$- \left[-\varkappa\sigma\overline{\varphi}\left(\frac{1}{\sigma}\right) + \overline{\varphi}'\left(\frac{1}{\sigma}\right) + \frac{1}{\sigma}\overline{\psi}\left(\frac{1}{\sigma}\right)\right]^- = 4\mu f(\sigma). \tag{128.3a}$$

The subsequent calculations will be somewhat simplified, if it is assumed that

$$\varphi(0) = 0, \tag{128.4a}$$

and this may be done without affecting generality.

Solving the boundary problems (128.2*a*) and (128.3*a*) and taking into consideration that the functions

$$\frac{1}{\zeta}\bar{\Phi}'\left(\frac{1}{\zeta}\right)+\frac{1}{\zeta^2}\bar{\Psi}\left(\frac{1}{\zeta}\right),\quad -\varkappa\zeta\bar{\varphi}\left(\frac{1}{\zeta}\right)+\bar{\varphi}'\left(\frac{1}{\zeta}\right)+\frac{1}{\zeta}\bar{\Psi}\left(\frac{1}{\zeta}\right)$$

are holomorphic for $|\zeta|>1$ [where it follows from (128.4*a*) that the second of these functions is holomorphic for $\zeta=\infty$], that the first of these functions vanishes at infinity and that the functions

$$\zeta\Phi'(\zeta)+\zeta^2\Psi(\zeta),\quad \frac{\varkappa}{\zeta}\varphi(\zeta)-\varphi'(\zeta)-\zeta\psi(\zeta)$$

are holomorphic for $|\zeta|<1$, one finds that inside γ

$$\zeta\Phi'(\zeta)+\zeta^2\Psi(\zeta)=0, \tag{128.5a}$$

$$-\varphi'(\zeta)-\zeta\psi(\zeta)+\frac{\varkappa}{\zeta}\varphi(\zeta)=\frac{2\mu}{\pi i}\int_\gamma\frac{f(\sigma)d\sigma}{\sigma-\zeta}+a, \tag{128.6a}$$

where a is a constant. The condition (128.10) is automatically satisfied, while (128.11) gives

$$a+\bar{a}=-\frac{2\mu}{\pi i}\int_\gamma f(\sigma)\frac{d\sigma}{\sigma}=-\frac{2\mu}{\pi}\int^{2\pi} f(\sigma)d\vartheta. \tag{128.7a}$$

(The multiplier $|\omega'(\sigma)|=R$ does not appear on the right-hand side, since (128.3*a*) had been divided by R.)

Comparison of (128.5*a*), (128.6*a*) with (128.14), (128.15) shows that in the notation of § 128

$$G(\zeta)=0,\quad H(\zeta)=\frac{A(\zeta)+a}{\zeta}, \tag{128.8a}$$

where

$$A(\zeta)=\frac{2\mu}{\pi i}\int_\gamma\frac{f(\sigma)d\sigma}{\sigma-\zeta}. \tag{128.9a}$$

By (128.17) and (128.19), one has in the present case

$$\Omega(\zeta)=\frac{1}{\zeta^2},\quad F(\zeta)=\frac{\zeta A'(\zeta)-A(\zeta)}{\varkappa+1}-\frac{a}{\varkappa+1}$$

and, by (128.21),

$$\varphi(\zeta) = K\zeta^{2\nu} + \frac{\zeta^{2\nu}}{\varkappa + 1}\int [\zeta A'(\zeta) - A(\zeta)]\zeta^{-2\nu}\, d\zeta + \frac{a\zeta}{\varkappa -}$$

where K is a constant; this last formula may still be written

$$\varphi(\zeta) = K\zeta^{2\nu} + \frac{\zeta^{2\nu}}{\varkappa + 1}\int^{\zeta} [\zeta A'(\zeta) - A(\zeta) + A(0)]\zeta^{-2\nu} d\zeta +$$

$$+ \frac{A(0)}{\varkappa - 1}\,\zeta + \frac{a\zeta}{\varkappa - 1} \qquad (128.10a)$$

The lower integration limit in the last formula is justified, because, as it is easily seen, the expansion near the origin of the expression.

$$[A(\zeta) - A(0) - \zeta A'(\zeta)]\zeta^{-2\nu}$$

begins with a term multiplied by $\zeta^{-2\nu+2}$, and it is known that

$$1 < 2\nu = \frac{2\varkappa}{\varkappa + 1} < 2.$$

The constants K and a must be determined from the condition that $\varphi(\zeta)$ is holomorphic inside γ and from (128.7a), because (128.4a) is satisfied.

It is obvious that $\varphi(\zeta)$ will be holomorphic if, and only if, $K = 0$, because 2ν is not an integer.

The second term on the right-hand side of (128.10a) is easily seen to be a holomorphic function, because the multi-valued multiplier $\zeta^{-2\nu}$ under the integral sign compensates the multi-valued factor $\zeta^{2\nu}$ outside the integral. Finally, it should be realized that the branches of $\zeta^{2\nu}$ and $\zeta^{-2\nu}$ must be chosen such that

$$\zeta^{-2\nu} = \frac{1}{\zeta^{2\nu}}.$$

The condition (128.7a) determines the real part of a; its imaginary part remains arbitrary, as was to be expected (because it only affects rigid body motion). Assuming this imaginary part to be zero and noting that the right-hand side of (128.7a) is equal to $-A(0)$, one obtains

$$a = \bar{a} = -\tfrac{1}{2}A(0),$$

and (128.10a) finally gives

$$\varphi(\zeta) = \frac{\zeta^{2\nu}}{1 + \varkappa}\int^{\zeta} [\zeta A'(\zeta) - A(\zeta) + A(0)]\zeta^{-2\nu} d\zeta + \frac{A(0)}{2(\varkappa - 1)}\,\zeta; \quad (128.11a)$$

by (128.6*a*) one finds now

$$\psi(\zeta) = \frac{\varkappa}{\zeta^2}\varphi(\zeta) - \frac{1}{\zeta}\varphi'(\zeta) - \frac{1}{\zeta}A(\zeta) + \frac{A(0)}{2\zeta}, \tag{128.12a}$$

where it is readily verified that the right-hand side is holomorphic at $\zeta = 0$. Thus the problem is solved.

2°. Infinite plane with a circular hole

In this case the transformation on to the region $|\zeta| > 1$ will be applied, so that (128.1*a*) remains true.

If f denotes the normal displacement, assumed positive when it is directed inward, i.e., towards the centre, the boundary conditions take the form (after division by R^2 and R respectively)

$$[\sigma\Phi'(\sigma) + \sigma^2\Psi(\sigma)]^- - \left[\frac{1}{\sigma}\overline{\Phi}'\left(\frac{1}{\sigma}\right) + \frac{1}{\sigma^2}\overline{\Psi}\left(\frac{1}{\sigma}\right)\right]^+ = 0, \tag{128.13a}$$

$$\left[\frac{\varkappa}{\sigma}\varphi(\sigma) - \varphi'(\sigma) - \sigma\psi(\sigma)\right]^- -$$
$$- \left[-\varkappa\sigma\bar{\varphi}\left(\frac{1}{\sigma}\right) + \bar{\varphi}'\left(\frac{1}{\sigma}\right) + \frac{1}{\sigma}\bar{\psi}\left(\frac{1}{\sigma}\right)\right]^+ = 4\mu f(\sigma). \tag{128.14}$$

For the present, it will be assumed that the resultant vector (X, Y) is equal to zero and that the stresses and rotation vanish at infinity. Then $\varphi(\zeta)$, $\psi(\zeta)$ will be holomorphic for $|\zeta| > 1$, including the point at infinity, and for large $|\zeta|$

$$\Phi(\zeta) = O\left(\frac{1}{\zeta^2}\right), \quad \Psi(\zeta) = O\left(\frac{1}{\zeta^2}\right);$$

in addition, it may be assumed, without affecting generality, that $\psi(\infty) = 0$.

Normally it has been assumed in such cases that $\varphi(\infty) = 0$; however, one may put instead $\psi(\infty) = 0$. In the present case this last assumption is somewhat more convenient.

Taking into consideration the stated properties of the unknown functions and solving the boundary problems (128.13*a*) and (128.14*a*), one finds for points of the region $|\zeta| > 1$

$$\zeta\Phi'(\zeta) + \zeta^2\Psi(\zeta) = a, \tag{128.15a}$$

$$\varphi'(\zeta) + \zeta\psi(\zeta) - \frac{\varkappa}{\zeta}\varphi(\zeta) = \frac{2\mu}{\pi i}\int\frac{f(\sigma)d\sigma}{\sigma - \zeta} + b, \tag{128.16a}$$

where a, b are certain constants which may be determined in the following manner. By (128.10a)

$$a = a,$$

and by (128.11a) (remembering that R had been eliminated)

$$b + \bar{b} = \frac{2\mu}{\pi i}\int f(\sigma)\frac{d\sigma}{\sigma} = -\frac{2\mu}{\pi}\int^{2\pi} f(\sigma)d\vartheta.$$

In addition, letting $\zeta \rightarrow \infty$ in (128.15a), (128.16a) and noting that

$$[\zeta^2\Psi(\zeta)]_{\zeta=\infty} = -\frac{1}{R}[\zeta\psi(\zeta)]_{\zeta=\infty},$$

one finds

$$b = -Ra,$$

and hence

$$b = -Ra = -\frac{\mu}{\pi}\int_0^{2\pi} f(\sigma)d\vartheta. \qquad (128.17a)$$

The relation $[\zeta^2\Psi(\zeta)]_{\zeta=\infty} = -\frac{1}{R}[\zeta\psi(\zeta)]_{\zeta=\infty}$ above is obtained in the following manner. Remembering that

$$\Psi(\zeta) = \frac{\psi'(\zeta)}{\omega'(\zeta)} = \frac{\psi'(\zeta)}{R},$$

so that, if for large $|\zeta|$

$$\psi(\zeta) = \frac{A}{\zeta} + O\left(\frac{1}{\zeta^2}\right),$$

one has

$$\Psi(\zeta) = -\frac{A}{R\zeta^2} + O\left(\frac{1}{\zeta^3}\right).$$

Comparing (128.15a), (128.16a) with (128.14), (128.15), it is readily verified that in the present case

$$G(\zeta) = \frac{\bar{b}}{\zeta^2}, \quad H(\zeta) = \frac{A(\zeta)}{\zeta} - \frac{b}{\zeta},$$

where $A(\zeta)$ is given by (128.9); however, ζ lies now outside γ.

Thus one arrives again at (128.18), where now

$$F(\zeta) = \frac{G(\zeta) + H'(\zeta)}{(\varkappa + 1)\Omega(\zeta)} = \frac{A(\zeta) - \zeta A'(\zeta)}{\varkappa + 1}$$

By (128.21), one has

$$\varphi(\zeta) = K\zeta^{2\nu} + \frac{\zeta^{2\nu}}{\varkappa + 1}\int^{\zeta} [A(\zeta) - \zeta A'(\zeta)]\zeta^{-2\nu} d\zeta;$$

the choice of the lower integration limit is justified, because the integral is easily seen to converge; on the other hand, this limit may be chosen arbitrarily.

It is obvious that $\varphi(\zeta)$ will only be holomorphic, if $K = 0$, because 2ν is not an integer (see above). Hence

$$\varphi(\zeta) = \frac{\zeta^{2\nu}}{\varkappa + 1}\int_{\infty}^{\zeta} [A(\zeta) - \zeta A'(\zeta)]\zeta^{-2\nu} d\zeta. \qquad (128.18a)$$

The function $\psi(\zeta)$ may now be found from (128.16*a*) which gives

$$\psi(\zeta) = \frac{\varkappa}{\zeta^2}\,\varphi(\zeta) - \frac{1}{\zeta}\,\varphi'(\zeta) + \frac{1}{\zeta}\,A(\zeta) + \frac{b}{\zeta}, \qquad (128.19a)$$

where b is given by (128.17*a*).

Hitherto it has been assumed that the resultant vector of the forces (pressures), applied to the plate from the sides of the disc, is equal to zero. If it is finite (still assuming the stresses and rotation to vanish at infinity), then, by the same method as in the analogous cases of the first and second fundamental problems (cf. § 78), it is readily seen that the solution is given by

$$\varphi(\zeta) + \varphi_0(\zeta), \quad \psi(\zeta) + \psi_0(\zeta),$$

where $\varphi(\zeta)$, $\psi(\zeta)$ are the same functions as above, while

$$\varphi_0(\zeta) = -\frac{X + iY}{2\pi(1 + \varkappa)}\log\zeta - \frac{X + iY}{2\pi\varkappa}, \qquad (128.20a)$$

$$\psi_0(\zeta) = \frac{\varkappa(X - iY)}{2\pi(1 + \varkappa)}\log\zeta - \frac{(\varkappa - 1)\,(X + iY)}{4\pi(1 + \varkappa)\zeta^2}. \qquad (128.21a)$$

It is likewise easily verified directly that $\varphi_0(\zeta)$, $\psi_0(\zeta)$ solve the present boundary problem for $f = 0$ and for a given resultant vector (X, Y).

The solution $\varphi + \varphi_0$, $\psi + \psi_0$ corresponds to the case, where external forces, the resultant of which is equivalent to the force (X, Y) applied to its centre, act on the rigid disc.

If this force did not act at the centre, equilibrium of the disc would be impossible, because, since $T = 0$, the resultant moment (about the centre) of the forces, applied to the boundary of the disc, is equal to zero.

When the stresses do not vanish at infinity, but have given (finite) values there, the corresponding problem is likewise easily solved.

3°. Infinite plane with elliptic hole

As in the case of the first and second fundamental problems, one might use here the transformation on to the region $|\zeta| > 1$. However, use of the transformation on to the circle $|\zeta| < 1$ simplifies the calculations somewhat.

Thus, let

$$z = \omega(\zeta) = R\left(\frac{1}{\zeta} + m\zeta\right), \quad R > 0, \quad 0 < m < 1. \qquad (128.22a)$$

Then

$$\begin{gathered}\omega'(\zeta) = -\frac{R}{\zeta^2}(1 - m\zeta^2), \quad \bar{\omega}\left(\frac{1}{\zeta}\right) = R\left(\zeta + \frac{m}{\zeta}\right), \\ \bar{\omega}'\left(\frac{1}{\zeta}\right) \quad R(m - \zeta^2).\end{gathered} \qquad (128.22'a)$$

It will again be assumed that the stresses and rotation vanish at infinity and, in addition, that the resultant vector of the forces, applied to the boundary of the hole, is equal to zero (the general case may be reduced to this case.)

Under these conditions $\varphi(\zeta)$ and $\psi(\zeta)$ will be holomorphic inside γ and, in addition, near the origin

$$\Phi(\zeta) = \frac{\varphi'(\zeta)}{\omega'(\zeta)} = O(\zeta^2), \quad \Psi(\zeta) = \frac{\psi'(\zeta)}{\omega'(\zeta)} = O(\zeta^2). \qquad (128.23a)$$

On the basis of these formulae it is readily verified that the function $\Omega_1(\zeta)$, defined by (128.4), is holomorphic inside as well as outside γ (including the point $\zeta = \infty$).

However, the function $\Omega_2(\zeta)$, defined by (128.5), may have at $\zeta = 0$ a first order pole with the principal part

$$\frac{R}{\zeta}\{\varkappa m\varphi(0) + \psi(0)\}.$$

The following observation will now be made which considerably

simplifies the further analysis. It is known that addition of an arbitrary complex constant α to $\varphi(\zeta)$ together with addition of $\varkappa\bar{\alpha}$ to $\psi(\zeta)$ will not alter the displacements (and, in consequence, not the stresses). It is easily seen that this constant may always be chosen such that [remembering that $m \neq 1$ $(m < 1)$]

$$\varkappa m\varphi(0) + \psi(0) = 0. \tag{128.24a}$$

Thus, without affecting generality, it may be assumed that (128.24a) is fulfilled.

Then $\Omega_2(\zeta)$ will be holomorphic for $|\zeta| < 1$ and, as is easily seen, also for $|\zeta| > 1$, including the point at infinity. Therefore the functions $R_1(\zeta)$ and $R_2(\zeta)$ in (128.8), (128.9) are simply constants which will be denoted by a and b respectively and (128.12) and (128.13) may be written

$$\zeta^2\omega'(\zeta)\left\{\bar{\omega}\left(\frac{1}{\zeta}\right)\Phi'(\zeta) + \omega'(\zeta)\Psi'(\zeta)\right\} = a, \tag{128.25a}$$

$$-\zeta\bar{\omega}\left(\frac{1}{\zeta}\right)\varphi'(\zeta) - \zeta\omega'(\zeta)\psi(\zeta) + \frac{\varkappa}{\zeta}\bar{\omega}'\left(\frac{1}{\zeta}\right)\varphi(\zeta) = A(\zeta) + b \tag{128.26a}$$

(for $|\zeta| > 1$), where

$$A(\zeta) = \frac{2\mu}{\pi i}\int \frac{f(\sigma)\,|\,\omega'(\sigma)\,|\,d\sigma}{\sigma - \zeta}. \tag{128.27a}$$

The conditions (128.10) and (128.11) give

$$a = \bar{a}, \quad b + \bar{b} = -\frac{2\mu}{\pi i}\int f(\sigma)\,|\,\omega'(\sigma)\,\cdot\, d\sigma$$

$$= -\frac{2\mu}{\pi}\int^{2\pi} f(\sigma)\,|\,\omega'(\sigma)\,|\,d\vartheta. \tag{128.28a}$$

Comparing (128.25a) and (128.26a) with (128.14) and (128.15), it is seen that, in the notation of § 128,

$$G(\zeta) = \frac{a}{\zeta^2\omega'(\zeta)}, \quad H(\zeta) = \frac{A(\zeta)}{\zeta\omega'(\zeta)} + \frac{b}{\zeta\omega'(\zeta)}.$$

Finally, noting that in the present case

$$\Omega(\zeta) = \frac{\bar{\omega}'\left(\frac{1}{\zeta}\right)}{\zeta^2\omega'(\zeta)} \qquad \frac{m - \zeta^2}{1 - m\zeta^2},$$

one obtains from (128.19)

$$F(\zeta) = \frac{B(\zeta)}{R(\varkappa+1)} + \frac{2b}{R(\varkappa+1)(m-\zeta^2)(1-m\zeta^2)} + \\ + \frac{a-b}{R(\varkappa+1)(m-\zeta^2)} \qquad (128.29a)$$

where

$$B(\zeta) = \frac{1}{m-\zeta^2}\left\{\zeta A'(\zeta) + \frac{1+m\zeta^2}{1-m\zeta^2}A(\zeta)\right\}. \qquad (128.30a)$$

The function $\varphi(\zeta)$ is given by (128.21) which now becomes

$$\varphi(\zeta) = K\left(\frac{1-m\zeta^2}{m-\zeta^2}\right)^\nu + \left(\frac{1-m\zeta^2}{m-\zeta^2}\right)^\nu \int_{-\sqrt{m}}^{\zeta} F(\zeta)\left(\frac{m-\zeta^2}{1-m\zeta^2}\right)^\nu d\zeta, \quad (128.31a)$$

where K is a constant. The integrand has inside γ only two singular points $\zeta = \pm\sqrt{m}$, because $m < 1$, and it is easily seen that the integral converges (remembering that $\nu > 0$). Further, clearly the second term on the right-hand side of (128.31a) remains finite as $\zeta \to -\sqrt{m}$. Thus, for $\varphi(\zeta)$ to be holomorphic near $\zeta = -\sqrt{m}$, it is necessary that $K = 0$. Hence

$$\varphi(\zeta) = \left(\frac{1-m\zeta^2}{m-\zeta^2}\right)^\nu \int_{-\sqrt{m}}^{\zeta} F(\zeta)\left(\frac{m-\zeta^2}{1-m\zeta^2}\right)^\nu d\zeta. \qquad (128.32a)$$

Further, for $\varphi(\zeta)$ to remain finite for $\zeta \to +\sqrt{m}$, it is obviously necessary that

$$\int_{-\sqrt{m}}^{+\sqrt{m}} F(\zeta)\left(\frac{m-\zeta^2}{1-m\zeta^2}\right)^\nu d\zeta = 0. \qquad (128.33a)$$

If the condition (128.33a) is satisfied, the right-hand side of (128.32a) is easily seen to be holomorphic inside γ. Substituting from (128.32a) in (128.26a), assuming (128.33a) to be satisfied, an expression is found for $\psi(\zeta)$ which will clearly also be holomorphic inside γ. It is likewise readily verified that (128.24a) is fulfilled.

There only remains to determine the constants a and b in the expressions for $\varphi(\zeta)$ and $\psi(\zeta)$. For this purpose one has the relations (128.28a)

and (128.33*a*). It may be assumed that the integral in (128.33*a*) is taken over the segment $-\sqrt{m}$, $+\sqrt{m}$ of the real axis and that on the path of integration

$$\left(\frac{m-\zeta^2}{1-m\zeta^2}\right)^{\nu}$$

is positive.

The condition (128.33*a*) may be written

$$J + (2K_1 - K_2)b + K_2 a = 0, \tag{128.34a}$$

where

$$J = \int_{-\sqrt{m}}^{+\sqrt{m}} B(\zeta)\left(\frac{m-\zeta^2}{1-m\zeta^2}\right)^{\nu} d\zeta, \tag{128.35a}$$

$$K_1 = \int_{-\sqrt{m}}^{+\sqrt{m}} \frac{(m-\zeta^2)^{\nu-1}}{(1-m\zeta^2)^{\nu+1}}\, d\zeta, \quad K_2 = \int_{-\sqrt{m}}^{+\sqrt{m}} \frac{(m-\zeta^2)^{\nu-1}}{(1-m\zeta^2)^{\nu}}\, d\zeta. \tag{128.36a}$$

The constants K_1 and K_2 are real and it may be assumed that they have been calculated once and for all for ellipses of any given eccentricity (determined by m). It is easily seen that $K_2 < K_1$. The quantity J may likewise be assumed known, since $f(\sigma)$ will be given.

Equation (128.34*a*), together with (128.28*a*), determines a and b. In fact, subtracting from (128.34*a*) its conjugate complex equation, one finds

$$b - b = \frac{\bar{J} - J}{2K_1 - K_2} \tag{128.37a}$$

which, together with the second condition (128.28*a*), determines b. After this the constant a may be found from (128.34*a*).

Thus the problem is solved. The present solution is easily generalized to the case, where the stresses at infinity have given finite values and the resultant vector (X, Y) is different from zero.

PART VII

EXTENSION, TORSION AND BENDING OF HOMOGENEOUS AND COMPOUND BARS

The first three chapters (22—24) of this Part are reproduced here in the same form as in the first (1933) and second (1935) editions of this book, apart from minor editorial modifications. In Chapter 25 of the present edition the study of the solutions of the problems of extension and bending by couples of bars, consisting of different materials with different Poisson ratios (§§ 146, 147, 149), has been greatly extended. Further, a section (§ 150) has been added which deals with a solution, due to A. K. Rukhadze, of the problem of bending of such bars by transverse forces. Finally, certain papers, published since the appearance of the preceding edition and giving applications of the Author's methods, will be mentioned.

As there is no space even to touch upon the interesting results of A. Ya. Gorgidze and A. K. Rukhadze, referring to the (approximate) solution of the problem of extension, bending and torsion of almost prismatic compound bars, as well as to the calculation of "secondary effects" for prismatic compound bars, the relevant papers will only be listed here: A. Ya. Gorgidze [3—7], A. K. Rukhadze [4], A. Ya. Gorgidze and A. K. Rukhadze [2, 3].

In the present Part, the problems of extension, bending and torsion of cylindrical (prismatic) bars will be considered, since they are of great practical importance.

Chapter 22 is devoted to the classical results, referring to the problems of torsion and bending of homogeneous bars (the solution of the problem of extension being trivial in this case), which are, in principle, due to Saint-Venant. Since these results are studied with sufficient completeness in almost all text books on the theory of elasticity, only the basic theory will be presented here; certain results which are due to the Author and represent applications of complex function theory will be studied in greater detail with examples.

The remaining chapters of this Part give results, referring to the problems of extension, torsion and bending of compound bars which arise in connection with certain problems of civil engineering, such as those of reinforced concrete. In principle these results are due to the Author.

Chapter 22

TORSION AND BENDING OF HOMOGENEOUS BARS (PROBLEM OF SAINT-VENANT)

§ 129. Statement of the problem. Consider a homogeneous isotropic bar, bounded by a cylindrical (prismatic) surface ("side surface") and two planes ("ends"), normal to the side surface. It will be assumed that there are no body forces present, that the side surface of the bar is free from external stresses and that given forces (satisfying, of course, the equilibrium conditions of the body as a whole) are applied to its ends.

The Oz axis will be directed parallel to the generators of the side surface and the plane Oxy chosen to coincide with the "lower" of the ends of the bar. The "upper" end of the bar will then have the coordinate $z = l$, where l is the length of the bar.

The complete problem of the elastic equilibrium of such a bar under the stated conditions then leads to the following mathematical problem (§ 20): To find the quantities X_x, Y_y, Z_z, Y_z, Z_x, X_y, u, v, w which satisfy in the region V, occupied by the bar, the equations

$$\begin{aligned} \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} &= 0, \\ \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} &= 0, \\ \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} &= 0, \end{aligned} \tag{129.1}$$

$$\begin{gathered} X_x = \lambda\theta + 2\mu\frac{\partial u}{\partial x}, \quad Y_y = \lambda\theta + 2\mu\frac{\partial v}{\partial y}, \quad Z_z = \lambda\theta + 2\mu\frac{\partial w}{\partial z}, \\ Y_z = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \quad Z_x = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad X_y = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \end{gathered} \tag{129.2}$$

where

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

and, in addition, the following boundary conditions:

$$\left.\begin{aligned} X_x \cos(n, x) + X_y \cos(n, y) = 0 \\ Y_x \cos(n, x) + Y_y \cos(n, y) = 0 \\ Z_x \cos(n, x) + Z_y \cos(n, y) = 0 \end{aligned}\right\} \text{ on the side surface;} \tag{129.3}$$

finally,

$$X_z,\ Y_z,\ Z_z \text{ equal to given functions at the ends,} \tag{129.4}$$

i.e., for $z = 0$, $z = l$.

The problem, when formulated in the above manner, presents considerable mathematical difficulties, particularly, if one is not only interested in its theoretical solution, but also in a solution which permits effective calculations.

Fortunately, it has been found that in the majority of practical cases it is unnecessary (and even senseless) to consider the problem in such completeness. In fact, the *actual* distribution of the external stresses at the ends of the bar is rarely known; the resultant vector and moment of these stresses will be known more or less exactly. In other words, the *union of forces and couples*, statically equivalent to the resultant of the forces applied to the relevant end, will be given.

On the other hand, by Saint-Venant's principle (cf. § 23), if one is dealing with bars which are of great length in comparison with the dimensions of the ends, one only needs to ensure that the resultant vector and moment of the forces, applied to the ends, will have given values; the actual stress distribution at the ends, however, will have negligible influence on those parts of the beam which are not close to the ends.

Thus there appears to be a rather wide choice of solutions. This arbitrariness may be used to simplify the problem in the following manner: one may prescribe beforehand the form of the solution which must, however, be sufficiently general so that one can obtain on the ends of the beam a stress resultant, statically equivalent to that given ("semi-direct method" of Saint-Venant).

In this connection it is only necessary to consider one of the ends. In fact, having given the resultant vector and moment of the forces, acting on one of the ends, these quantities will also be determined for the other end, since the sum of the forces, applied to the ends, must be statically equivalent to zero (i.e., it must satisfy the equilibrium conditions of the body as a whole). On the other hand, each solution of the equations

(129.1) always gives a stress distribution on the surface of the body which is statically equivalent to zero (cf. end of § 20).

The complete theoretical solution of the problem with the above simplification and its application to a number of technically important cases is mainly due to Saint-Venant.

Saint-Venant's results are studied in his two extensive memoirs [1, 2] and in a number of other publications, in particular, in the lengthy notes in the French translation of A. Clebsch's book [2].

A. Clebsch (1833—1872), who was considerably younger and died earlier than his contemporary Saint-Venant, gave a very strict solution of a problem which is of interest here (A. Clebsch [1, 2]); he showed that, if one introduces beforehand the condition

$$X_x = Y_y = X_y = 0 \text{ in the region } V, \tag{129.5}$$

there remains just sufficient arbitrariness to satisfy the conditions at the ends and on the side surface, and that this condition leads to the solution, obtained by Saint-Venant by another lengthier method. Clebsch called the problem of the determination of the elastic equilibrium of a cylinder (with unstressed side surface) under the supplementary conditions (129.5) the "problem of Saint-Venant".

The condition (129.5) obviously has the following physical meaning: if one imagines the given cylinder to consist of a number of longitudinal "fibres" (i.e., thin longitudinal prisms), these fibres exert neither direct nor shear forces on each other in *transverse* directions (i.e., the fibres may only exert on each other cohesive forces in the longitudinal direction).

If (129.5) is satisfied, the conditions (129.3) on the side surface obviously reduce to

$$Z_x \cos(n, x) + Z_y \cos(n, y) = 0, \tag{129.3'}$$

because the first two conditions of (129.3) are automatically satisfied.

The method of Clebsch will not be studied here (it can be found, for example, in A. G. Webster [1] and also in I. Todhunter and K. Pearson [1]); a less strict, but simpler method will be applied instead which agrees, in essence, with that used by A. E. H. Love [1], Chaps. XIV and XV.

It should still be noted that the results of Saint-Venant may be obtained by beginning from the following formulation of the problem which is due to W. Voigt (cf. A. E. H. Love [1], Chap. XVI): To find the elastic equilibrium of the cylinder under consideration on the basis of the supposition that the stress components are linear functions of z.

Consider, for definiteness, the forces applied to the upper end. The

union of these forces is statically equivalent to a force, applied to some (arbitrary) point O', and to a couple. The point of intersection of the Oz axis with the upper face will be chosen as the point O'. The force may be divided into two components: one in the Oz direction and the other at right angles to the Oz axis. The couple may be decomposed in the same manner: the moment of one of its components will be taken parallel to the axis Oz ("twisting couple"), while the moment of the other will be in the plane of the end ("bending couple").

Correspondingly, the problem may now be divided into the following four component problems:

a) torsion by couples, acting in the plane of the ends;

b) extension (or compression) by longitudinal forces, applied to the ends;

c) bending by couples the plane of which is perpendicular to that of the ends;

d) bending by transverse forces, applied to one of the ends and acting in its plane (one must, of course, in consequence apply to the other end a force which is equal in magnitude and opposite in direction to the above; the same is true with regard to the couple, in order that the entire system of forces be in equilibrium).

It must be remembered that in the subsequent work one is not dealing with concentrated forces and with pairs of concentrated forces, but with forces and couples which are statically equivalent to certain stresses distributed over the ends.

§ 130. Certain formulae. In order to facilitate future reference, it will be recalled that the equations (129.2) may be replaced by the following which are equivalent to them (§ 19):

$$\frac{\partial u}{\partial x} = \frac{1}{E}[X_x - \sigma(Y_y + Z_z)], \frac{\partial v}{\partial y} = \frac{1}{E}[Y_y - \sigma(Z_z + X_x)],$$

$$\frac{\partial w}{\partial z} = \frac{1}{E}[Z_z - \sigma(X_x + Y_y)],$$

$$\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{2(1+\sigma)}{E} Y_z, \quad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{2(1+\sigma)}{E} Z_x, \tag{130.1}$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{2(1+\sigma)}{E} X_y,$$

where E is Young's modulus and σ Poisson's ratio which are related to λ and μ by the formulae (§ 19)

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \qquad \sigma = \frac{\lambda}{2(\lambda + \mu)}, \tag{130.2}$$

$$\lambda = \frac{E\sigma}{(1 + \sigma)(1 - 2\sigma)}, \quad \mu = \frac{E}{2(1 + \sigma)}. \tag{130.3}$$

Finally, the compatibility equations of Beltrami-Michell (§ 22) will be reproduced which in the present case, with absence of body forces, have the form

$$\begin{aligned}
&\Delta X_x + \frac{1}{1 + \sigma} \cdot \frac{\partial^2 \Theta}{\partial x^2} = 0, \quad \Delta Y_z + \frac{1}{1 + \sigma} \cdot \frac{\partial^2 \Theta}{\partial y \, \partial z} = 0, \\
&\Delta Z_z + \frac{1}{1 + \sigma} \cdot \frac{\partial^2 \Theta}{\partial z^2} = 0, \quad \Delta Z_x + \frac{1}{1 + \sigma} \cdot \frac{\partial^2 \Theta}{\partial z \, \partial x} = 0, \\
&\Delta Y_y + \frac{1}{1 + \sigma} \cdot \frac{\partial^2 \Theta}{\partial y^2} = 0, \quad \Delta X_y + \frac{1}{1 + \sigma} \cdot \frac{\partial^2 \Theta}{\partial x \, \partial y} = 0,
\end{aligned} \tag{130.4}$$

where

$$\Theta = X_x + Y_y + Z_z. \tag{130.5}$$

Each set of functions $X_x, \ldots, X_y$, satisfying these conditions (which will, in future, be simply called compatibility conditions) and the equations (129.1), gives a certain possible stress distribution in the body (under the requirement of single-valued displacements).

In the sequel, the set of equations (129.1) and (129.2) will be called *static equations of an elastic body*, while the equations (129.1) will, as before, be called *equilibrium equations*.

§ 131. General solution of the torsion problem. The solution of the component problems will now be considered and a beginning will be made with the problem of torsion.

Let the coordinate axes be as in § 129. The coordinate system will be assumed to be *right-handed*. Let the forces, applied to the ends, be statically equivalent to a twisting couple, i.e., to a couple with moment vector at right angle to the plane of the ends. Let M be the (scalar) moment of the couple, acting on the upper end ($M > 0$, when the couple tends to twist counter-clockwise, looking upwards, since, by supposition, the coordinate system is to be right-handed).

The first idea to enter one's mind is that all transverse sections of the cylinder will remain plane and that they will twist (in their own planes) about the Oz axis by some angle ε. If the lower end is restrained from moving, it is natural to assume that the angle ε is proportional to the distance z of the section under consideration from the lower end, i.e.,

$$\varepsilon = \tau z \tag{131.1}$$

where τ is a constant which measures the angle of relative twist of cross-sections, unit length apart. For this reason, τ is called the *relative twist*.

Under the present suppositions the displacement components will be given by

$$u = -\varepsilon y = -\tau z y, \quad v = \tau z x, \quad w = 0$$

(since an infinitely small rigid rotation through an angle ε in the Oxy plane about the origin leads to $u = -\varepsilon y$, $v = \varepsilon x$). Calculating the stress components from these displacements, it is easily seen that the equations (129.1) will be satisfied; however, it is also readily verified that the conditions (129.3) may not be fulfilled, unless one is dealing with a circular cylinder (this will become obvious on the basis of the later work). It is therefore clear that a too restrictive hypothesis has been introduced.

The following investigation will now be based on the assumption (which will be found to be successful) that the cross-sections do not remain plane, but that they warp (and that all cross-sections warp in an identical manner).

This supposition obviously leads to the following expressions for the displacement components:

$$u = -\tau z y, \quad v = \tau z x, \quad w = \tau \varphi(x, y), \tag{131.2}$$

where τ is a constant (relative twist) and $\varphi(x, y)$ is some function of x, y, to be determined later (the factor τ has been introduced into the expression for w simply for the sake of convenience).

The formulae (129.2) give for the stress components, corresponding to the displacements (131.2),

$$X_z = \mu\tau\left(\frac{\partial\varphi}{\partial x} - y\right), \quad Y_z = \mu\tau\left(\frac{\partial\varphi}{\partial y} + x\right), \tag{131.3}$$

and

$$X_x = Y_y = Z_z = X_y = 0. \tag{131.4}$$

Substituting these values in the equations (129.1), one sees that they will be satisfied, provided

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0. \tag{131.5}$$

In other words, φ must be a harmonic function of the two variables x, y in the region, occupied by the body; since φ does not depend on z, it is, of course, sufficient to consider any cross-section S of the cylinder.

Further, the condition (129.3′) (expressing absence of external stresses on the side surface) takes the form

$$\left(\frac{\partial \varphi}{\partial x} - y\right) \cos (n, x) + \left(\frac{\partial \varphi}{\partial x} + x\right) \cos (n, y) = 0 \text{ on } L,$$

where L denotes the boundary of the region S and n the outward normal to L (i.e., the normal, directed outwards from S). Noting that

$$\frac{\partial \varphi}{\partial x} \cos (n, x) + \frac{\partial \varphi}{\partial y} \cos (n, y) = \frac{d\varphi}{\partial n},$$

one obtains finally the boundary condition in the following form:

$$\frac{d\varphi}{dn} = y \cos (n, x) - x \cos (n, y) \text{ on } L. \tag{131.6}$$

Thus the function φ which is called the *torsion function* must satisfy the following conditions: it has to be single-valued (because otherwise w would be multi-valued and such multi-valued displacements will not be considered here) and harmonic in S, and on the boundary of this region its normal derivative must take a previously given value, in fact, the value

$$y \cos (n, x) - x \cos (n, y).$$

The problem of finding φ is thus a particular case of one of the fundamental problems of potential theory — "the Neumann problem" — which has already been mentioned in § 77.

It may be shown that φ is determined under the stated conditions, apart from an arbitrary constant. This constant is unimportant, because substitution of $\varphi + c$ for φ will not affect the state of stress, as is clear from (131.3), but only cause a rigid translation of the bar in the direction Oz; this last fact follows from (131.2).

The known condition for existence of a solution of the Neumann problem (that

the integral of the given value of the normal derivative, taken around the entire boundary, must vanish) is satisfied here [cf. (132.3)], because

$$\int_L [y \cos(n, x) - x \cos(n, y)]ds = \int_L (y\,dy + x\,dx) = \int_L d\tfrac{1}{2}(x^2 + y^2)$$

is equal to zero.

The complete solution of the more general case of a compound bar will be given in § 140.

Further, (131.3) and (131.4) show that the ends of the bar are only subject to tangential stresses.

It is easily shown that, if φ satisfies the above conditions, the resultant vector of the stresses is equal to zero, i.e.,

$$\iint_S X_z\,dx\,dy = 0, \quad \iint_S Y_z\,dx\,dy = 0. \tag{131.7}$$

This result will be proved in § 144 for a more general case.

The resultant moment of the external stresses, applied to the upper end, will be given by

$$M = \iint_L (xY_z - yX_z)dx\,dy = \mu\tau \iint_L \left(x^2 + y^2 + x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right)dx\,dy,$$

i.e.,

$$M = \tau D, \tag{131.8}$$

where

$$D = \mu \iint \left(x^2 + y^2 + x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right)dx\,dy. \tag{131.9}$$

The formula (131.8) shows that the torsion moment is proportional to the relative twist τ. The coefficient of proportionality D is called the *torsional rigidity*. It is seen to be the product of the shear modulus μ and a quantity which only depends on the shape of the cross-section, and not on the material.

Once the torsion function φ has been determined, the constant D may be calculated. It will be shown below that D is essentially positive. Therefore the constant τ will be given by (131.8) for a given couple, i.e., for a given value of M, and the problem is solved.

It only remains to prove that $D > 0$. This result is most simply de-

duced by an investigation of the potential energy, stored in the twisted bar. In fact, it is known that this energy is given by (cf. Note at the end of § 24)

$$U = \tfrac{1}{2} \iint (X_n u + Y_n v + Z_n w) dS,$$

where the integral must be taken over the entire surface of the bar. However, in the present case, the integrand vanishes on the side surface and on the lower end, so that there only remains the integral over the upper end, where (since $z = l$)

$$u = -\tau l y, \quad v = \tau l x, \quad X_n = X_z, \quad Y_n = Y_z, \quad Z_n = Z_z = 0;$$

hence

$$U = \tfrac{1}{2} \iint_S (uX_z + vY_z) dx\, dy = \frac{\tau l}{2} \iint_S (xY_z - yX_z) dx\, dy = \frac{\tau^2 l D}{2}$$

and, since in the presence of deformation $U > 0$, one finds $D > 0$ and the above assertion is proved.

This result may also be proved directly. In fact, applying Green's formula and using (131.6), one obtains

$$\iint \left(x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x} \right) dx\, dy = \iint \left(\frac{\partial (x\varphi)}{\partial y} - \frac{\partial (y\varphi)}{\partial x} \right) dx\, dy$$

$$= - \int_L \varphi \{ y \cos (n, x) - x \cos (n, y) \} ds = - \int_L \varphi \frac{d\varphi}{dn} ds.$$

However, one has by a known formula for every harmonic function φ

$$\int_L \varphi \frac{d\varphi}{dn} ds = \iint_S \left\{ \left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 \right\} dx\, dy.$$

Thus one finds from above

$$0 = \iint_S \left\{ x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x} + \left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 \right\} dx\, dy.$$

Multiplying this equation by μ and adding it to (131.9), one obtains

$$D = \mu \iint \left\{ \left[\frac{\partial \varphi}{\partial x} - y \right]^2 + \left[\frac{\partial \varphi}{\partial y} + x \right]^2 \right\} dx\, dy, \qquad (131.9')$$

and hence the assertion follows.

If one had $D = 0$, this would imply

$$\frac{\partial \varphi}{\partial x} = y, \qquad \frac{\partial \varphi}{\partial y} = -x$$

throughout S; however, this is impossible, because $y\,dx - x\,dy$ is not a perfect differential.

NOTE. 1. One may obviously add to u, v, w, respectively, terms of the form

$$\alpha + qz - ry, \quad \beta + rx - pz, \quad \gamma + py - qx,$$

expressing rigid body motion, without affecting the state of stress.

NOTE. 2. Since the above work was based on the formulae (131.2) the first two of which express rigid rotation of the cross-section about the axis Oz, it may be shown that a new solution of the problem is obtained by replacing this axis by another one, parallel to it. In fact, let $O_1(a, b)$ be the point of intersection of the new axis with the plane Oxy; then

$$u_1 = -\tau z(y - b), \quad v_1 = \tau z(x - a), \quad w_1 = \tau\varphi_1(x, y), \qquad (131.2')$$

where u_1, v_1, w_1 are the displacement components and φ_1 is the torsion function, corresponding to the new position of the axis. The corresponding stresses will be given by

$$X_z = \mu\tau\left(\frac{\partial \varphi_1}{\partial x} - y + b\right), \quad Y_z = \mu\tau\left(\frac{\partial \varphi_1}{\partial y} + x - a\right). \qquad (131.3')$$

As above, it may be shown that φ_1 is harmonic and that it satisfies on L the condition

$$\frac{d\varphi_1}{dn} \quad (y - b)\cos(n, x) - (x - a)\cos(n, y) =$$
$$= y\cos(n, x) - x\cos(n, y) - b\cos(n, x) + a\cos(n, y)$$

which may obviously be rewritten

$$\frac{d}{dn}(\varphi_1 + bx - ay) = y\cos(n, x) - x\cos(n, y).$$

Thus, the harmonic function $\varphi_1 + bx - ay$ must satisfy the same conditions as the function φ, whence it follows that these two functions may only differ by a constant, i.e.,

$$\varphi_1(x, y) = \varphi(x, y) + ay - bx + \text{const.} \qquad (131.10)$$

Thus, by (131.2) and (131.2′),

$$u_1 = u + \tau bz, \quad v_1 = v - \tau az, \quad w_1 = w + \tau ay - \tau bx + \text{const.} \tag{131.11}$$

The terms by which (u, v, w) differs from (u_1, v_1, w_1) only express rigid body motion and therefore do not affect the stresses; this is likewise readily verified directly from (131.3′) which give the same values as (131.3).

§ 132. Complex torsion function. Stress functions. It is at times more convenient to introduce instead of the torsion function $\varphi(x, y)$ its conjugate harmonic function $\psi(x, y)$ which is related to $\varphi(x, y)$ by the Cauchy-Riemann equations

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}. \tag{132.1}$$

The boundary condition (131.6) is easily expressed in terms of the function ψ.

For greater generality, it will be assumed that the bar under consideration may contain longitudinal (likewise cylindrical) cavities so that the boundary L of the region can consist of several simple contours $L_1, L_2, \ldots, L_{m+1}$ the last of which surrounds all the others (cf. Fig. 14, § 35).

Let t denote the tangent to one of the contours L_k in the positive direction (i.e., leaving S on the left). Then

$$\cos(n, x) = \cos(t, y) = \frac{dy}{ds}, \quad \cos(n, y) = -\cos(t, x) = -\frac{dx}{ds},$$

where s is the arc of L_k; it thus follows, by (132.1), that

$$\frac{d\varphi}{dn} = \frac{\partial\varphi}{\partial x}\cos(n, x) + \frac{\partial\varphi}{\partial y}\cos(n, y) = \frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds} = \frac{d\psi}{ds},$$

i.e.,

$$\frac{d\varphi}{dn} = \frac{d\psi}{ds}. \tag{132.2}$$

In addition,

$$y\cos(n, y) - x\cos(n, y) = x\frac{dx}{ds} + y\frac{dy}{ds} = \frac{d}{ds}\,\tfrac{1}{2}(x^2 + y^2). \tag{132.3}$$

Hence (131.6) takes the form

$$\frac{d\psi}{ds} = \frac{d}{ds}\frac{1}{2}(x^2 + y^2),$$

whence it follows that

$$\psi = \tfrac{1}{2}(x^2 + y^2) + C_k \text{ on } L_k, \tag{132.4}$$

where C_k are constants which may have different values on the different L_k.

A function ψ, conjugate to a given single-valued harmonic function, may, in general, be multi-valued (cf. Appendix 3.). However, in the present case, this cannot happen, because by (132.4) the function ψ reverts to its original value for a circuit of any of the contours L_k.

It is known that the function φ is determined, apart from an arbitrary constant, and hence its derivatives will be fully determined; it follows from this that ψ is defined by (132.1), apart from an arbitrary constant.

It is thus seen that the constants $C_1, \ldots, C_{m+1}$ in the boundary conditions (132.4) may not be fixed arbitrarily. Only one of these constants may be fixed in an arbitrary manner, i.e., one may, for example, put $C_{m+1} = 0$; all the remaining constants must then have completely determined (initially unknown) values.

As one may dispose freely of one of the constants C_k, it is clear that one will be justified in adding any arbitrary constant to the function ψ.

It will now be assumed that the constants C_k have been given some definite values. In that case the problem of determining ψ coincides with the problem of finding a harmonic function for given values on the boundary, i.e., the "Dirichlet problem" which has already been discussed in § 62 (Note) and in § 77 and which is known to have always a unique solution. Having found ψ, the function φ may be determined from (132.1). However, if the constants C_k are chosen in a haphazard manner, the function φ may be found to be multi-valued. Thus, *the constants C_k must be determined from the condition of single-valuedness of the function $\varphi(x, y)$*; as stated earlier, one of these constants may be fixed arbitrarily.

In the case of multiply connected regions it is therefore, generally speaking, more convenient to operate directly with the function φ rather that with ψ.

In the case of simply connected regions, bounded by one simple contour L, single-valuedness of the function φ will be automatically

ensured; only one constant which may be fixed arbitrarily will enter into the boundary condition. In this case it is often more convenient to operate with the function ψ.

It is often also very convenient to consider the function $F(z)$ of the complex variable $z = x + iy$, defined by

$$F(z) = \varphi + i\psi, \tag{132.5}$$

where φ is the torsion function and ψ its conjugate function. The function $F(z)$ will be called the *complex torsion function*. It is obviously holomorphic in the region S.

By (131.3)

$$X_z - iY_z = \pi\tau\left(\frac{\partial\varphi}{\partial x} - i\frac{\partial\varphi}{\partial y} - y - ix\right) = \pi\tau\left[\frac{\partial\varphi}{\partial x} + i\frac{\partial\psi}{\partial x} - i(x - iy)\right],$$

whence, in the customary notation used earlier,

$$X_z - iY_z = \mu\tau\{F'(z) - i\bar{z}\}. \tag{132.6}$$

It is also convenient to use the so-called *stress function*, defined by

$$\Psi(x, y) = \psi(x, y) - \tfrac{1}{2}(x^2 + y^2); \tag{132.7}$$

in terms of this function, the stress components are given by

$$X_z = \mu\tau\frac{\partial\Psi}{\partial y}, \quad Y_z = -\mu\tau\frac{\partial\Psi}{\partial x}. \tag{132.8}$$

The function Ψ is not harmonic, since it obviously satisfies the equation

$$\Delta\Psi = -2. \tag{132.9}$$

It satisfies on the boundary the conditions

$$\Psi = C_k \text{ on } L_k \; (k = 1, 2, \ldots, m), \tag{132.10}$$

where C_k are the same constants as in (132.4).

The curves, defined in the plane of the cross-section S by the equations

$$\Psi(x, y) = \text{const.}, \tag{132.11}$$

are the „lines of shear stress", i.e., the lines whose tangents have at every point the direction of the stress vector (X_z, Y_z), acting on the corresponding element of S. This is a direct consequence of (132.8). The boundaries of the region are always stress lines, as is, of course, obvious a priori.

From the practical point of view, it is of great interest to find those

points of the cross-section, where the magnitude of the resultant shear stress

$$T = \sqrt{X_z^2 + Y_z^2} \tag{132.12}$$

has a maximum, because it is at these points that failure of the material will begin.

It is easily shown that these points *lie on the boundary of the region.* In fact,

$$T^2 = \mu^2\tau^2 \left[\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(\frac{\partial\Psi}{\partial y}\right)^2\right].$$

Using (132.9), it is readily seen that

$$\Delta(T^2) = 2\mu^2\tau^2 \left\{\left(\frac{\partial^2\Psi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\Psi}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2\Psi}{\partial x\,\partial y}\right)^2\right\},$$

and hence that $\Delta(T^2) > 0$ throughout the region. On the basis of a well known theorem (cf. below), it then follows that the function T^2 may only attain its maximum value on the boundary, as was to be proved.

One may not have the equal sign in the above inequality, since, by (132.9), at least one of the quantities $(\partial^2\Psi/\partial x^2)$, $(\partial^2\Psi/\partial y^2)$ must have a modulus not less than unity. Incidentally, it may be shown by a simple reasoning that $\Delta(T^2) \geqslant 4\mu^2\tau^2$.

If some function U, having continuous second order derivatives in a region S, satisfies the inequality $\Delta U > 0$, this function can only attain its maximum value on the boundary. In fact, let it be supposed that U has its highest value at some internal point (x_0, y_0). Describe, with this point as centre, a circle γ of sufficiently small radius so that $dU/dn \leqslant 0$, where n is the outward normal to the circle. On the other hand, by Green's formula,

$$\int \frac{dU}{dn}\, ds = \int_\sigma \Delta U \, dx\, dy,$$

where σ is the area inside γ. Since $\Delta U > 0$, one is led to a contradiction, and hence the assertion is proved.

§ 133. On the solution of the torsion problem for certain particular cases. It has been seen that the torsion problem may either be reduced to the Neumann problem (for φ) or to the Dirichlet problem (for ψ; in the case of multiply-connected regions, the constants C_k must still be determined from the condition of single-valuedness of φ; cf. the preceding section). For this reason all the known, well developed methods of solution for the Neumann and Dirichlet problems may be applied.

In addition, in view of the special simplicity of the boundary values of ψ or $d\varphi/dn$, one may fall back successfully on particular methods, designed for the present problem.

Saint-Venant himself solved and studied in detail (by constructing tables and graphs) the torsion problem for a large number of cross-sections of different shapes which are of practical interest. He obtained solutions for many cases (ellipse, equilateral triangle, etc.) by very simple means. For the case of the rectangular section, he gave a solution in the form of a rapidly converging series.

The reader should consult Saint-Venant [1] and also the small book on the theory of elasticity by A. N. Dinnik [1] which is specially devoted to the torsion problem, where the solutions for a large number of cross-sections may be found. See also the book by I. Todhunter and K. Pearson [1] and B. G. Galerkin [1, 2]; in the first of his papers, Galerkin gives the solution for a section, represented by an isosceles right-angled triangle. Finally, one may also find remarks on approximate and experimental solutions in the book by A. N. Dinnik [1].

Only the (almost obvious) solution for the case of a circular or annular circular section will be quoted here. If the origin is placed at the centre, one has clearly: $y \cos(n, x) - x \cos(n, x) = 0$ on the boundary. Therefore

$$\frac{d\varphi}{dn} = 0$$

on the entire boundary. Hence $\varphi =$ const., and one may take $\varphi = 0$. The displacements and stresses are then given by

$$-\tau z y, \quad v = \tau z x, \quad w = 0, \tag{133.1}$$

$$X_z \quad \mu\tau y, \quad Y_z = \mu\tau x \tag{133.2}$$

(the remaining stresses being zero). It is thus seen that in this case transverse sections remain plane, unlike in the other cases.

By (131.9), the torsional rigidity is given by

$$D = \mu \iint_{S} (x^2 + y^2) dx\, dy = \mu I, \tag{133.3}$$

where I is the polar moment of inertia of the cross-section about the centre. In the case of a circular section with radius R, one has thus

$$I = \frac{\pi R^4}{2} \tag{133.4}$$

while in the case of the circular ring

$$I = \frac{\pi}{2} (R_2^4 - R_1^4), \tag{133.5}$$

where R_1 and R_2 are the inner and the outer radii.

§ 134. Application of conformal mapping.

The results of this section were given in the Author's papers [12, 13]. A detailed study of these results with several new applications is given in the book by I. S. Sokolnikoff [1].

The torsion problem may be considered solved, if one has been able to map the region S on to the circle (where, of course, S must be a simply connected region). In fact, let

$$z = x + iy = \omega(\zeta) \tag{134.1}$$

be the function, mapping S on to the circle $|\zeta| < 1$ whose boundary, as always, will be denoted by γ.

If the complex torsion function $F(z)$, expressed in terms of ζ, is given by

$$\varphi + i\psi = F(z) = f(\zeta), \tag{134.2}$$

the function $f(\zeta)$ will be holomorphic inside γ. The real part ψ of the function

$$\frac{1}{i} f(\zeta) = \psi - i\varphi \tag{134.3}$$

will satisfy on γ the following boundary condition [cf. (132.4)]:

$$\psi = \tfrac{1}{2}(x^2 + y^2) + \text{const.} = \lambda z\bar{z} + \text{const.},$$

or, by (134.1),

$$\psi = \tfrac{1}{2}\omega(\sigma)\overline{\omega(\sigma)} \text{ on } \gamma,$$

where $\sigma = e^{i\theta}$ denotes points of γ; the arbitrary constant has been justifiedly omitted.

However, in an earlier Part of this book, a formula has been deduced which permits to find a function, holomorphic inside γ, when the boundary value of its real part is known. In fact, by (77.5),

$$\frac{1}{i} f(\zeta) = \frac{1}{\pi i} \int_{\gamma} \frac{\omega(\sigma)\overline{\omega(\sigma)}}{2(\sigma - \zeta)}\, d\sigma + \text{const.},$$

whence, finally,

$$f(\zeta) = \frac{1}{2\pi} \int_{\gamma} \frac{\omega(\sigma)\overline{\omega(\sigma)}d\sigma}{\sigma - \zeta} + \text{const.}, \tag{134.5}$$

and the problem is solved.

If $\omega(\zeta)$ is a *rational function*, then $\omega(\sigma)\overline{\omega(\sigma)} = \omega(\sigma)\bar{\omega}(1/\sigma)$ will likewise be a rational function of σ. The integral on the right-hand side of (134.5) is, in this case, easily calculated by use of the residue theorem and it obviously leads to a rational function in ζ, so that the solution *will be expressed in terms of elementary functions.*

In general, if the expression $\omega(\sigma)\bar{\omega}(1/\sigma)$, considered as a function of σ, is an analytic function inside (or outside) γ, continuous up to γ and has inside (or outside) γ a finite number of poles, then the integral on the right-hand side of (134.5) may be directly evaluated by means of the residue theorem.

The torsional rigidity D may be calculated from a simple formula which will now be deduced (see the Author's papers [12,13]). One has (§ 131)

$$D = \mu \iint_S (x^2 + y^2)dx\,dy + \mu \iint_S x\left(\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right)dxdy = \mu I + \mu D_0, \tag{134.6}$$

where I is the polar moment of inertia of S about O and

$$D_0 = \iint_S \left(x\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial x}\right)dx\,dy = \iint_S \left\{\frac{\partial}{\partial y}(x\varphi) - \frac{\partial}{\partial x}(y\varphi)\right\}dx\,dy. \tag{134.7}$$

Applying Green's formula, one obtains

$$D_0 = -\int_L \varphi.(x\,dx + y\,dy) = -\int_L \varphi.d(\tfrac{1}{2}r^2), \tag{134.8}$$

where L is the boundary of the region.

Noting that on the boundary $r^2 = z\bar{z} = \omega(\sigma)\overline{\omega(\sigma)}$ and that

$$\varphi = \tfrac{1}{2}[f(\sigma) + \overline{f(\sigma)}],$$

one may rewrite (134.8)

$$D_0 = -\tfrac{1}{4}\int_{\gamma} \{f(\sigma) + \overline{f(\sigma)}\}d\,\{\omega(\sigma)\overline{\omega(\sigma)}\}. \tag{134.9}$$

If $\omega(\zeta)$ is a rational function, $f(\zeta)$ will likewise be a rational function (see above) and hence also $\bar{f}(1/\sigma)$, $\bar{\omega}(1/\sigma)$ will be rational, so that the preceding integral is easily evaluated in closed form, using the theorem of residues.

In this case it is sometimes convenient to transform also the expression for

$$I = \iint_S (x^2 + y^2) dx\, dy = \iint_S \left\{ \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{dx}(xy^2) \right\} dx\, dy =$$

$$= - \int_L xy(x\, dx - y\, dy).$$

Noting that

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i},$$

it is readily seen that

$$I = -\frac{1}{8i} \int_L (z^2 - \bar{z}^2)(z dz + \bar{z} d\bar{z}).$$

However,

$$\int_L z^3 dz = \int_L \bar{z}^3 d\bar{z} = 0, \quad \int_L z^2 \bar{z} d\bar{z} = \int_L z^2 d(\tfrac{1}{2}\bar{z}^2) = -\int_L \bar{z}^2 z dz$$

(where the last relation is obtained by an integration by parts). Thus

$$I = \frac{1}{4i} \int_L \bar{z}^2 z dz = \frac{1}{4i} \int_\gamma \overline{\omega^2(\sigma)}\, \omega(\sigma) d\omega(\sigma). \tag{134.10}$$

If $\omega(\zeta)$ is a rational function, formula (134.10) permits calculation of I in closed form by elementary means.

In the case of doubly connected regions, the torsion problem is likewise easily solved, provided the function $\omega(\zeta)$ is known which maps this region on to the circular ring. In fact, in this case the problem is reduced to the determination of a function $f(\zeta)$, holomorphic inside the ring and satisfying the following boundary conditions:

$$\begin{aligned} \Re \frac{1}{i} f(\zeta) &= \tfrac{1}{2}\omega(\zeta)\overline{\omega(\zeta)} + C_1 \text{ on } \gamma_1, \\ \Re \frac{1}{i} f(\zeta) &= \tfrac{1}{2}\omega(\zeta)\overline{\omega(\zeta)} + C_2 \text{ on } \gamma_2, \end{aligned} \tag{134.11}$$

where γ_1, γ_2 are the circles, bounding the ring, and C_1, C_2 are two real constants one of which may be fixed arbitrarily. One thus arrives at the problem solved in § 62 (Note).

In the present case, the function $(1/i)\, f(\zeta)$ plays the role of $F(\zeta)$ and in the expansion (62.7) (where z must be replaced by ζ) one has to put $A = 0$, because $f(\zeta)$ would otherwise be multi-valued. For the functions $f_1(\vartheta)$ and $f_2(\vartheta)$ of § 62 one has now

$$\tfrac{1}{2}\omega(\rho_1 e^{i\vartheta})\overline{\omega(\rho_1 e^{i\vartheta})} + C_1, \quad \tfrac{1}{2}\omega(\rho_2 e^{i\vartheta})\overline{\omega(\rho_2 e^{i\vartheta})} + C_2; \qquad (134.12)$$

ρ_1 and ρ_2 denote here the same quantities as R_1 and R_2 in § 62. If one writes $C_2 = 0$, then the constant C_1 will be determined by (62.9).

Having found $f(\zeta)$, the stresses may be calculated either in terms of the old coordinates x, y or in terms of the curvilinear coordinates of § 49 which are related to the conformal transformation.

Let $\vec{T}$ denote the vector of the shear stress, acting at some point of the cross-section S, and X_z, Y_z its components in the Ox, Oy directions. The projections of this vector on the axes (ρ), (ϑ) of the curvilinear coordinates will be given by (49.4) which has the conjugate complex form

$$T_\rho - iT_\vartheta = \frac{\zeta}{\rho} \frac{\omega'(\zeta)}{|\omega'(\zeta)|} (X_z - iY_z)$$

or, substituting for $X_z - iY_z$ from (132.6) and noting that

$$F'(z) = \frac{dF}{dz} = \frac{df}{dz} \cdot \frac{1}{\omega'(z)},$$

one deduces finally the very simple and convenient formula

$$T_\rho - iT_\vartheta = \frac{\mu\tau\zeta}{\rho\,|\,\omega'(\zeta)\,|} \{f'(\zeta) - i\,\overline{\omega(\zeta)}\,\omega'(\zeta)\}. \qquad (134.13)$$

On the boundary of the region $T_\rho = 0$, so that the preceding formula determines directly the boundary value of the shear stress T_ϑ and, in particular, the maximum value of this quantity.

§ 134a. Examples. The method of the preceding section will now be applied to certain particular cases.

1°. Epitrochoidal section. Let the section S be bounded by the epitrochoids, considered in § 48,3° (Fig. 23). Then

$$z = \omega(\zeta) = b(\zeta^n + a\zeta) \quad (n \text{ an integer} > 1,\ b > 0,\ a \geqslant n), \qquad (134.1a)$$

where, in the notation of § 48, $b = Rm$, $m = 1/a$.

Formula (134.5) gives (on replacing $\bar{\sigma}$ by $1/\sigma$)

$$f(\zeta) = \frac{1}{2\pi}\int_{\gamma} b^2(\sigma^n + a\sigma)\left(\frac{1}{\sigma^n} + \frac{a}{\sigma}\right)\frac{d\sigma}{\sigma - \zeta} + \text{const.} =$$

$$= \frac{ib^2}{2\pi i}\int_{\gamma}\left(1 + a^2 + a\sigma^{n-1} + \frac{a}{\sigma^{n-1}}\right)\frac{d\sigma}{\sigma - \zeta} + \text{const.},$$

whence, by (70.3),

$$f(\zeta) = ib^2a\zeta^{n-1} \tag{134.2a}$$

(the arbitrary constant having been omitted here), and the problem is solved.

By (134.13) one obtains for the stress components T_ρ, T_ϑ

$$T_\rho - iT_\vartheta = \mu\tau\zeta\,\frac{iab^2(n-1)\zeta^{n-2} - ib^2(\bar{\zeta}^n + a\bar{\zeta})\,(n\zeta^{n-1} - a)}{\rho\,|\,\omega'(\zeta)\,|}$$

or, putting $\zeta = \rho e^{i\vartheta}$ and separating real and imaginary parts,

$$T_\rho = -\frac{\mu\tau ab^2(n-1)\rho^{n-2}\,(1-\rho^2)\sin(n-1)\vartheta}{|\,\omega'(\zeta)\,|},$$

$$T_\vartheta = -\mu\tau b^2\,\frac{a\rho^{n-2}[n - 1 - (n+1)\rho^2]\cos(n-1)\vartheta - n\rho^{2n-1} - a^2\rho}{|\,\omega'(\zeta)\,|},$$

where

$$|\,\omega'(\zeta)\,| = \sqrt{\omega'(\zeta)\overline{\omega'(\zeta)}} = b\sqrt{n^2\rho^{2n-2} + 2an\rho^{n-1}\cos(n-1)\vartheta + a^2}.$$

On the boundary (i.e., for $\rho = 1$) one has $T_\rho = 0$ and

$$T = T_\vartheta = \mu\tau b\,\frac{n + 2a\cos(n-1)\vartheta + a^2}{\sqrt{n^2 + 2an\cos(n-1)\vartheta + a^2}}.$$

If $n < a$, i.e., if the contour has no angular points, the maximum value of T occurs at those points of the boundary where $\cos(n-1)\vartheta = -1$; these points are closest to the centre. The maximum value of T there is given by

$$T_{max} = \mu\tau b\,\frac{a^2 - 2a + n}{a - n}.$$

If $a \to n$, $T_{max} \to \infty$, i.e., in the case of the boundary with angular points, as shown in Fig. 23, the stress T_{max} becomes infinite at those points.

The torsional rigidity is easily obtained from (134.9) and (134.10) which give

$$D = \frac{\mu\pi b^4}{2}(a^4 + 4a^2 + n).$$

2°. Booth's Lemniscate.

The solution of this problem (as well as of all the others, presented in this section) were published by the Author in 1929 in the papers, quoted earlier, and reproduced in the first edition of this book. Recently (1942), T. J. Higgins published the solution of the same problem which he obtained by a more complicated method (cf. I. S. Sokolnikoff [1], p. 184.).

The transformation function for the region, bounded by this curve, was stated in § 46,6° (Fig. 27). Changing the notation slightly, one may write

$$\omega(\zeta) = \frac{k\zeta}{\zeta^2 + a} \quad (a > 1, \quad k > 0); \tag{134.3a}$$

one then obtains for $f(\zeta)$ the formula

$$f(\zeta) = \frac{1}{2\pi}\int_\gamma \frac{k^2\sigma^2 d\sigma}{(\sigma^2 + a)(1 + a\sigma^2)(\sigma - \zeta)}.$$

The integrand, considered as a function of σ, has outside γ two simple poles: $\sigma_1 = i\sqrt{a}$ and $\sigma_2 = -i\sqrt{a}$; for large $|\sigma|$, it is of order $1/\sigma^3$. Therefore, by the residue theorem,

$$f(\zeta) = -i(A_1 + A_2),$$

where A_1, A_2 are the residues, corresponding to the points σ_1, σ_2 respectively. One has

$$A_1 = \left[(\sigma - i\sqrt{a})\frac{k^2\sigma^2}{(\sigma^2 + a)(1 + a\sigma^2)(\sigma - \zeta)}\right]_{\sigma = i\sqrt{a}}$$
$$= -\frac{k^2\sqrt{a}}{2i(1 - a^2)(i\sqrt{a} - \zeta)}$$

similarly,

$$A_2 = -\frac{k^2\sqrt{a}}{2i(1 - a^2)(i\sqrt{a} + \zeta)},$$

whence, finally,

$$f(\zeta) = \frac{iak^2}{(a^2 - 1)(\zeta^2 + a)} \tag{134.4a}$$

The stress components may be calculated as in the preceding example. Only the value of $T = T_\vartheta$ on the boundary will be given here:

$$T = \frac{\mu\tau k(1 + a^2)}{(a^2 - 1)\sqrt{1 - 2a\cos 2\vartheta + a^2}} .$$

The maximum value of T occurs for $\cos 2\vartheta \quad 1$, i.e., at the ends of the minor axis, and it is given by

$$T_{max} = \frac{\mu\tau k(a^2 + 1)}{(a + 1)\,(a - 1)^2} .$$

One easily obtains for the torsional rigidity

$$D = \frac{\mu\pi k^4(a^4 + 1)}{2(a^2 - 1)^4} .$$

3°. The loop of Bernoulli's Lemniscate. An example will now be given of a simply connected region for which $\omega(\zeta)$ is not a rational function.

Assuming $|\zeta| < 1$, let

$$\mathfrak{z} = \omega(\zeta) = a\sqrt{1 + \zeta} \quad (a > 0), \tag{134.5a}$$

and select that branch of the multi-valued function $\sqrt{1 + \zeta}$ which is equal to unity for $\zeta = 0$. In other words (Fig. 57a),

$$\omega(\zeta) = \sqrt{r}\, e^{i\frac{\varphi}{2}} \qquad \left(-\frac{\pi}{2} \leqslant \varphi \leqslant \frac{\pi}{2}\right).$$

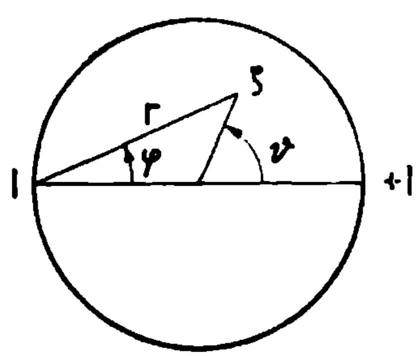


Fig. 57a. ζ plane.

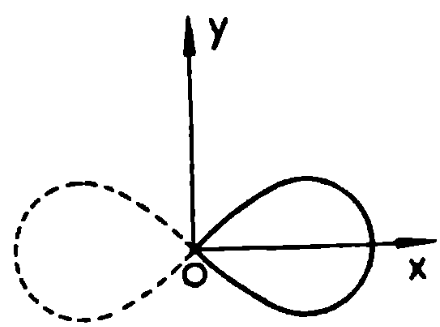


Fig. 57b. $\mathfrak{z}$ plane.

As ζ describes the unit circle γ,

$$\varphi \quad \frac{\vartheta}{2} \qquad (-\pi < \vartheta < \pi)$$

and

$$r = 2\cos\frac{\vartheta}{2}.$$

Hence

$$z = a\sqrt{2\cos\frac{\vartheta}{2}}\, e^{i\frac{\vartheta}{4}}.$$

If R and ψ denote the modulus and argument of z, then, by the preceding formulae,

$$R = a\sqrt{2\cos\frac{\vartheta}{2}}, \quad \psi = \frac{\vartheta}{4},$$

whence

$$R = a\sqrt{2\cos 2\psi}. \tag{134.6a}$$

Thus z describes one loop of Bernoulli's lemniscate [Fig. 57*b*] and (134.5) maps the region inside this loop on to the circle $|\zeta| < 1$.

One finds for $f(\zeta)$

$$f(\zeta) = \frac{a^2}{2\pi}\int_\gamma \sqrt{1+\sigma}\sqrt{1+\frac{1}{\sigma}}\,\frac{d\sigma}{\sigma-\zeta}$$
$$= \frac{a^2}{2\pi}\int \frac{1+\sigma}{\sqrt{\sigma}}\,\frac{d\sigma}{\sigma-\zeta} \tag{134.7a}$$

where one has to take that branch of the function $(1+\sigma)/\sqrt{\sigma}$ which is positive on γ, i.e., one must take $\sqrt{\sigma} = e^{i\frac{\vartheta}{2}}$.

The integrand will be single-valued in the region, bounded by γ and cut as shown in Fig. 58. Therefore (in the notation shown in this figure, where, in particular, γ_1 denotes an infinitely small circle)

$$\frac{1}{2\pi i}\left[\int_\gamma + \int_\alpha + \int_\beta + \int_{\gamma_1}\right] = A,$$

where A is the residue at $\sigma = \zeta$ which is obviously equal to

$$\frac{1+\zeta}{\sqrt{\zeta}}.$$

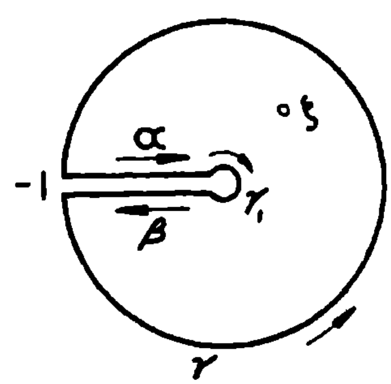


Fig. 58.

By means of a simple transformation of the integrals, taken along α and β (the integral over γ_1 being infinitely small), one finds

$$\frac{1}{2\pi i}\int_{\gamma}\frac{1+\sigma}{\sqrt{\sigma}}\,\frac{d\sigma}{\sigma-\zeta} \quad \frac{1+\zeta}{\sqrt{\zeta}} \quad \frac{1}{\pi}\int_{0}\frac{1-t}{\sqrt{t}}\,\frac{dt}{t+\zeta} \tag{134.8a}$$

whence, omitting an arbitrary constant,

$$f(\zeta) = \frac{2ia^2}{\pi}\cdot\frac{1+\zeta}{\sqrt{\zeta}}\,\mathrm{artan}\,\sqrt{\zeta};$$

in this formula one must take for

$$\mathrm{artan}\,\sqrt{\zeta} = \frac{1}{2i}\log\frac{1+i\sqrt{\zeta}}{1-i\sqrt{\zeta}}$$

the branch which is defined by the series

$$\mathrm{artan}\,\sqrt{\zeta} = \sqrt{\zeta} - \frac{(\sqrt{\zeta})^3}{3} + \frac{(\sqrt{\zeta})}{5} - \ldots = \sqrt{\zeta}\left(1 - \frac{\zeta}{3} + \zeta^2 \quad \ldots\right).$$

The problem is thus solved.

4°. Confocal ellipses. Eccentric circles. When the cross-section of the (complete) cylinder is bounded by two confocal ellipses or two (eccentric) circles, the solution is likewise easily obtained by transformation on to the circular ring. In particular, the solution of the last case may be deduced directly from example 1° of § 140*a*.

§ 135. Extension by longitudinal forces. The solution of this problem is quite elementary and has, in essence, already been deduced in § 19. In fact, if

$$Z_z = \frac{F}{S} \text{ (and all other stress components are equal to zero)}, \tag{135.1}$$

where F is the magnitude of the given force, assumed positive in the case of tension, and S is the area of the transverse section of the bar, then all the required conditions will be satisfied. This solution corresponds to normal stresses, distributed uniformly over the ends. The resultant of the stresses, applied to the upper end, will be equivalent to a force F, applied at its centre of area.

If the given force is not applied to the centre of area of the end, it may be transferred to that point by adding a couple the plane of which is perpendicular to the end (i.e., a bending couple). Thus, the solution of the problem of bending by a couple, stated in the next section, has to be added in this case.

The displacements, corresponding to the stresses (135.1), are easily verified to be given by

$$u = -\frac{\sigma F}{SE}x, \quad v = -\frac{\sigma F}{SE}y, \quad w = \frac{F}{SE}z, \tag{135.2}$$

where any rigid displacement of the bar as a whole may be added.

The quantity SE, which is the coefficient of proportionality between the extending force F and the corresponding extension of the bar, may be called the *rigidity of extension* (*compression*).

§ 136. Bending by couples, applied to the ends. In this case the solution is also quite elementary.

In accordance with established custom, the bar will be placed horizontally with the Oz axis running from the left to the right and the Ox axis vertically downward, as shown in Fig. 59 (where the Oy axis is not shown; it is directed away from the reader, because the coordinate system is to be right-handed).

The ends which were formerly called "lower" and "upper" will now be called "left" and "right" respectively. In addition, it will be assumed that the point O *lies at the centre of area of the left end*, so that Oz is the "axis of centroids", i.e., the locus of the centroids of the cross-sections.

An attempt will now be made to satisfy the conditions of the problem by writing

$$Z_z = ax, \quad X_x = Y_y = Y_z = Z_x = X_y = 0. \tag{a}$$

These values obviously satisfy the equations of equilibrium and of compatibility (§ 130). It will now be investigated whether the stresses, applied to any transverse section (from the right), are statically equivalent to a bending couple.

Clearly, if the forces which are applied, say, to the right end are equivalent to a couple, then the forces, applied from the right to any section, must be equivalent to the same couple.

The resultant vector of these forces is equal to zero, since

$$\iint_S Z_z\, dx\, dy = a \iint_S x\, dx\, dy = 0;$$

the last integral vanishes, because the origin lies at the centroid of the section S.

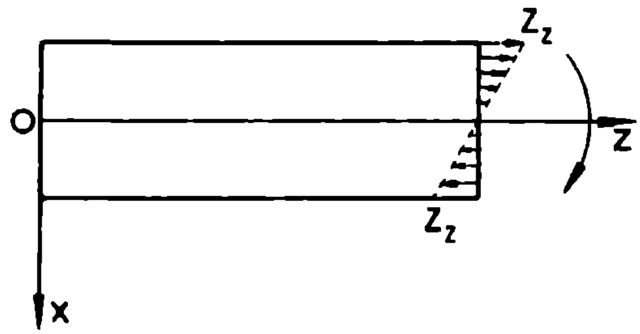


Fig. 59.

The resultant moment of the above-mentioned forces about an axis through the centre of area of the section and parallel to the axis Oy is

$$M = -\iint Z_z x\, dx\, dy = -a \iint x^2\, dx\, dy = -aI, \tag{b}$$

where I is the second moment of area of the section S about the axis Oy.

Finally, the resultant moment of the forces about the axis passing through the centroid of the section and parallel to Ox is equal to

$$\iint_S Z_z y\, dx\, dy = a \iint_S xy\, dx\, dy. \tag{c}$$

If

$$\iint_S xy\, dx\, dy = 0,$$

i.e., if the coordinate axes Ox, Oy are principal axes of inertia of the section S (with regard to its centroid), then the moment (c) is equal to zero and the forces are equivalent to a couple with moment vector parallel to Oy and determined by (b). For a given value of M the constant a is determined by

$$a = -\frac{M}{I}.$$

Let it be assumed that the coordinate axes have been chosen in the stated manner. In that case the solution has been obtained of the problem of bending of a bar by couples, applied to the ends, whose moments are parallel to one of the principal axes of inertia of the section with regard to its centre of area.

The above results will now be summarized. Let a couple with vector moment, parallel to one of the principal axes of inertia of the end with regard to its centroid, act on the right end of the bar. If one takes as axes Ox, Oy the principal axes of inertia of any cross-section, e.g. of the left end, and directs the axis Oy parallel to the moment of the couple, then the solution of the problem of bending is given by

$$Z_z = -\frac{M}{I}x, \quad X_x = Y_y = Y_z = Z_x = X_y = 0. \qquad (136.1)$$

In these formulae I denotes the moment of inertia of the end-section about the axis Oy and M is the magnitude of the moment of the couple (which is positive, if the moment is directed along the axis Oy).

It is readily verified by direct substitution that the displacements, corresponding to these stresses are:

$$u = \frac{M}{2EI}(z^2 + \sigma x^2 - \sigma y^2), \quad v = \frac{M}{EI}\sigma xy, \quad w = -\frac{M}{EI}xz; \qquad (136.2)$$

terms, expressing rigid body displacements, may be added to these formulae.

The plane $x = 0$ is a "neutral plane": fibres, lying in this plane, will neither be stretched nor compressed. Fibres, lying to one side of this plane, will be extended, while those on the other side will be compressed.

The normal stresses are distributed over the cross-section according to the linear law, expressed by (136.1); cf. also Fig. 59.

The points of the "central fibre", having before deformation the coordinates $(0, 0, z)$, will move to points with coordinates (ξ, η, ζ), where

$$\xi = \frac{M}{2EI}z^2, \quad \eta = 0, \quad \zeta = z; \qquad (136.3)$$

this is easily seen from (136.2).

Thus the central fibre remains in the plane Oxz which is therefore called the ***plane of bending***. In the present case it is parallel to the plane of the bending couple. The radius of curvature R of this line (after

deformation) is determined (apart from small, higher order terms) by

$$\frac{1}{R} = \frac{d^2\xi}{dz^2}$$

(where it will be assumed that $R < 0$, if the curve is convex downward); hence one obtains the important relation.

$$\frac{1}{R} = \frac{M}{EI} \tag{136.4}$$

which expresses the so-called law of Bernoulli-Euler: *the curvature of the central fibre is proportional to the bending moment.* The quantity EI is called the *flexural rigidity.* Since a constant value had been obtained for R, the central fibre in its deformed state will represent a circular arc of radius R which must be assumed very large in view of the assumed smallness of the deformations; in fact, the quantity $1/R$ must be of the same order of smallness to which the deformations are restricted.

The relation $1/R = d^2\xi/dz^2$ used above is based on the following reasoning; by a well known formula,

$$\frac{1}{R} = \frac{\xi''}{(1+\xi'^2)^{3/2}} = \xi''(1 + \xi'^2)^{-3/2} = \xi''\left(1 - \frac{3}{2}\xi'^2 + \ldots\right),$$

where accents denote differentiation with respect to z. In view of the smallness of the deformations, all but the first term of this expansion may be omitted and the stated result is obtained.

In actual fact, the equation of the curve of the central fibre is given by

$$\xi = \frac{z^2}{2R}$$

which is a parabola; however, the difference between this curve and the circle with radius R is a second order quantity.

Points, lying before deformation on the normal to the section $z = c$, move as the result of the deformation to points (ξ, η, ζ), where, in particular, by the last of the formulae (136.2)

$$\zeta = c + w = c - \frac{M}{EI}xc = c\left(1 - \frac{x}{R}\right).$$

Replacing on the right-hand side x by ξ which is justified in view of the smallness of $1/R$, one finds

$$= c\left(1 - \frac{\xi}{R}\right);$$

this is the equation of the plane, perpendicular to the plane of bending. Thus, *normal sections remain plane.*

If the moment of the bending couple is not directed along one of the principal axes of the cross-section, this couple may always be decomposed into two each of which satisfy this condition, and the solution of the problem will be obtained by superimposing two solutions of the stated form. In this general case the plane of bending does not coincide with the plane of the couple; however, in this case also it is perpendicular to the neutral plane which will again exist. It will be left to the reader to prove this simple property.

§ 137. Bending by transverse forces. Let the coordinate axes be the same as in the preceding section, i.e., select as origin the centroid of one of the ends ("left end"), and let the Ox, Oy axes be parallel to the principal axes of inertia of this end about its centroid. Let the forces, acting on the right end, be equivalent to a force W, applied at its centre of area and directed parallel to the axis Ox (Figs. 60*a*, 60*b*). The

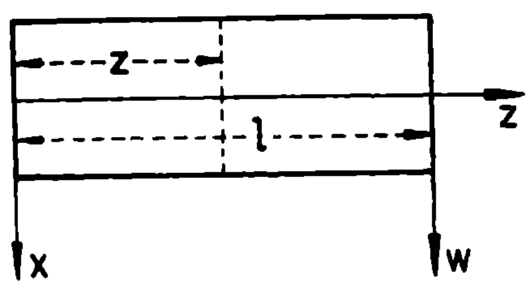


Fig. 60*a*.

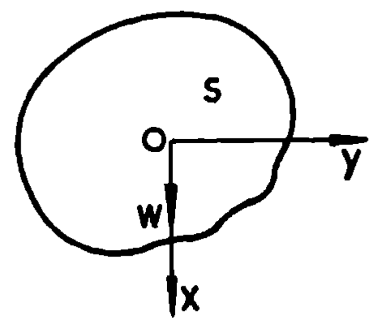


Fig. 60*b*.

resultant moment M of the forces, applied (from the right) to any section which is a distance z from the left end, about the axis through the centroid of this section and parallel to Oy is obviously given by

$$M = W(l - z), \tag{137.1}$$

where l is the length of the bar.

If only a couple with moment M were acting on the section under consideration, one might, on the basis of the results of § 136, write

$$Z_z = -\frac{M}{I}x,$$

where I is the moment of inertia of the section about the axis, parallel to Oy and passing through its centroid.

An attempt will now be made to satisfy the conditions of the problem by writing

$$Z_z = -\frac{M}{I}x = \frac{W(l-z)x}{I}. \tag{137.2}$$

However, it is clear that one may not now assume that all the remaining stress components vanish, because in that case the forces, acting at the cross-section, might not give the resultant vector W, acting in the plane of the section. However, let it nevertheless be assumed that

$$X_x = Y_y = X_y = 0. \tag{137.3}$$

Substituting these values in the equations (129.1), one obtains

$$\frac{\partial X_z}{\partial z} = 0, \quad \frac{\partial Y_z}{\partial z} = 0$$

and

$$\frac{\partial X_z}{\partial x} + \frac{\partial Y_z}{\partial y} + \frac{Wx}{I} = 0. \tag{137.4}$$

It follows from the first two of these equations that X_z, Y_z do not depend on z. Equation (134.4) may be rewritten

$$\frac{\partial X_z}{\partial x} + \frac{\partial}{\partial y}\left(Y_z + \frac{Wxy}{I}\right) = 0,$$

whence it follows that

$$X_z = \frac{\partial \Omega}{\partial y}, \qquad \frac{\partial \Omega}{\partial x} \quad \frac{Wxy}{I} \tag{137.5}$$

where Ω is some function of x and y. Substituting the expressions for the stress components in the compatibility equations (130.4), one sees that the first, second, third and sixth of these equations are satisfied identically, while the two remaining ones give

$$\frac{\partial \Delta\Omega}{\partial x} = 0, \quad \frac{\partial \Delta\Omega}{\partial y} \quad \frac{W}{(1+\sigma)I}$$

whence

$$\Delta\Omega = -\frac{Wy}{(1+\sigma)I} - 2\mu\tau, \tag{137.6}$$

where $-2\mu\tau$ denotes some constant.

Noting that

$$\Delta\left\{-\frac{W}{2(1+\sigma)I}\left[\tfrac{1}{2}\sigma x^2 y+\left(1-\frac{\sigma}{2}\right)\frac{y^3}{3}\right]-\frac{\mu\tau}{2}(x^2+y^2)\right\}=$$

$$-\frac{Wy}{(1+\sigma)I}-2\mu\tau,$$

one may write

$$\Omega=\psi_1-\frac{W}{2(1+\sigma)I}\left\{\tfrac{1}{2}\sigma x^2 y+\left(1-\frac{\sigma}{2}\right)\frac{y^3}{3}\right\}-\frac{\mu\tau}{2}(x^2+y^2), \qquad (137.7)$$

where ψ_1 is a harmonic function.

If φ_1 denotes the harmonic function, conjugate to ψ_1, i.e., the function for which

$$\frac{\partial\varphi_1}{\partial x}=\frac{\partial\psi_1}{\partial y}, \qquad \frac{\partial\varphi_1}{\partial y}=-\frac{\partial\psi_1}{\partial x}$$

then obviously (137.5) may be written

$$\begin{aligned} X_z&=\frac{\partial\varphi_1}{\partial x}-\mu\tau y-\frac{W}{2(1+\sigma)I}\left\{\tfrac{1}{2}\sigma x^2+(\right.\\ Y_z&=\frac{\partial\varphi_1}{\partial y}+\mu\tau x-\frac{W(2+\sigma)}{2(1+\sigma)I}xy. \end{aligned} \qquad (137.8)$$

Finally, one can always write

$$\varphi_1=\mu\tau\varphi-\frac{W}{2(1+\sigma)I}\chi, \qquad (137.9)$$

where φ is the torsion function, defined earlier (§ 131), and χ is some new harmonic function. Hence

$$\begin{aligned} X_z&=\mu\tau\left(\frac{\partial\varphi}{\partial x}-y\right)-\frac{W}{2(1+\sigma)I}\left\{\frac{\partial\chi}{\partial x}+\tfrac{1}{2}\sigma x^2+(1-\tfrac{1}{2}\sigma)y^2\right\},\\ Y_z&=\mu\tau\left(\frac{\partial\varphi}{\partial y}+x\right)-\frac{W}{2(1+\sigma)I}\left\{\frac{\partial\chi}{\partial y}+(2+\sigma)xy\right\}, \qquad (137.10)\\ Z_z&=-\frac{W(l-z)}{I}x. \end{aligned}$$

The displacements corresponding to these stresses are easily calculated (by the general formulae of § 15 or by simple elementary means, as

used e.g. in A. E. H. Love [1], Chap. XV); the reader will readily verify that the following expressions satisfy (129.2):

$$u = -\tau zy + \frac{W}{EI}\{\tfrac{1}{2}\sigma(l-z)(x^2-y^2) + \tfrac{1}{2}lz^2 - \tfrac{1}{6}z^3\},$$

$$v = \tau zx + \frac{W}{EI}\sigma(l-z)xy, \tag{137.11}$$

$$w = \tau\varphi - \frac{W}{EI}\{x(lz - \tfrac{1}{2}z^2) + \chi + xy^2\}.$$

A rigid body displacement may again be added to these expressions.

Substituting from (137.10) into the boundary condition on the side surface

$$X_z \cos(n, x) + Y_z \cos(n, y) = 0$$

and taking into consideration (131.6), to be satisfied by the torsion function φ, one obtains

$$\frac{d\chi}{dn} = -[\tfrac{1}{2}\sigma x^2 + (1 - \tfrac{1}{2}\sigma)y^2]\cos(n, x) - (2+\sigma)xy\cos(n, y) \tag{137.12}$$

on the boundary L of the region S.

Hence one has to solve the Neumann problem, just as in the case of the torsion function, in order to find the function χ.

It is easily seen that the condition for existence of the solution of the Neumann problem, i.e.,

$$\int_L \{[\tfrac{1}{2}\sigma x^2 + (1 - \tfrac{1}{2}\sigma)y^2]\cos(n, x) + (2+\sigma)xy\cos(n, y)\}ds = 0,$$

is satisfied in the present case. In fact, applying Green's formula, this condition becomes

$$2(1+\sigma)\iint x\,dx\,dy = 0,$$

where the integral vanishes, since, by supposition, the centroid of S lies at the origin.

The resultant vector of the external stresses, applied to the right end, is easily seen to be parallel to Ox and equal in magnitude to W (cf. § 144 for the proof in the more general case of a compound bar). However, if τ remains arbitrary, the forces applied to this end still give

a twisting couple. In fact, the terms containing τ give a couple with a moment determined by (131.8), while the terms with W give a couple with moment

$$\frac{W}{2(1+\sigma)I}\iint\left\{y\frac{\partial\chi}{\partial x} - x\frac{\partial\chi}{\partial y}+(1-\tfrac{1}{2}\sigma)y^3 - (2+\tfrac{1}{2}\sigma)x^2y\right\}dx\,dy. \qquad (137.13)$$

In order to remove the twisting couples, it is sufficient to give τ a suitable value so that the sum of the stated moments vanishes.

The terms involving W determine the *bending* of the bar. The plane $x = 0$ is here the neutral plane and Oxz the plane of bending. The central line (i.e., the line $x = 0$, $y = 0$, which passes through the centroids of the cross-sections) becomes a curve in the Oxz plane, its radius of curvature (at a given point z) being determined by the Bernoulli-Euler relation

$$\frac{1}{R} = \frac{M}{EI}, \qquad (137.14)$$

where

$$M = W(l - z)$$

is the moment of the forces, applied to the transverse section (from the right) at the given point, about an axis which lies in the plane of the section in the direction of Oy.

In addition, the terms involving τ cause torsion of the beam about the axis Oz. Clearly, in the case of sections symmetrical with regard to the Ox axis, one will have $\tau = 0$ and no torsion will occur.

Finally, if the force W is not parallel to one of the principal axes of inertia of the cross-section and does not pass through its centre of area, then its point of application may be moved to the centroid by introducing a suitable couple and the force may be decomposed into two parts, parallel to the principal axes of inertia. The unknown solution of the problem is then obtained by solving the problem of torsion and two problems of bending by forces, parallel to the principal axes.

Consider now again the above case. Instead of χ, its conjugate function χ' will be introduced so that

$$\frac{\partial\chi}{\partial x} = \frac{\partial\chi'}{\partial y}, \quad \frac{\partial\chi}{\partial y} = -\frac{\partial\chi'}{\partial x}$$

Then, using the relation [cf. (132.2)]

$$\frac{d\chi}{dn} = \frac{d\chi'}{ds},$$

one obtains for χ' the boundary condition

$$\chi' = F_k(s) + C_k \text{ on } L_1, L_2, \ldots, L_{m+1}, \tag{137.15}$$

where L_k denotes the contours forming the boundary L of the region, C_k are constants and

$$\begin{aligned} F_k(s) &= -\int [\tfrac{1}{2}\sigma x^2 + (1 - \tfrac{1}{2}\sigma)y^2] \cos(n, x)ds - (2 + \sigma)\int xy \cos(n,y)\, ds \\ &= -\int [\tfrac{1}{2}\sigma x^2 + (1 - \tfrac{1}{2}\sigma)y^2]dy + (2 + \sigma)\int xy\, dx \\ &= -\tfrac{1}{3}(1 - \tfrac{1}{2}\sigma)y^3 + \int\{(2 + \sigma)xy\, dx - \tfrac{1}{2}\sigma x^2\, dy\} + \text{const.}, \end{aligned}$$

where the integral is taken along L_k from an arbitrary point of this contour to the variable point (x, y). Noting that

$$\int x^2\, dy = x^2 y - 2\int xy\, dx + \text{const.},$$

the preceding formula may be written

$$F_k(s) = -(1 - \tfrac{1}{2}\sigma)\frac{y^3}{3} - \sigma\frac{x^2 y}{2} + 2(1 + \sigma)\int xy\, dx + \text{const.} \tag{137.16}$$

Since the last integral, if taken along the entire contour, will not, in general, vanish, the function χ' will be multi-valued. However, in the case of simply connected sections bounded by one contour L, the integral, if taken around L, vanishes and χ' will be single-valued, as was to be expected.

By Green's formula

$$\int_{L_k} xy\, dx = \mp \iint x\, dx\, dy,$$

where S_k is the part of the plane surrounded by L_k. The upper sign must be chosen for L_{m+1}, the lower for the remaining contours. The integral will only vanish in the case, where the centroid of S_k lies on the Oy axis.

As in the case of torsion, it is sometimes convenient to consider the complex function

$$G(z) = \chi + i\chi'. \tag{137.17}$$

§ 138. On the solution of problems of bending for different cross-sections. In his major work on bending [2] as well as in other papers Saint-Venant gave the solutions of bending problems for a number of cross-sections, in particular for a rectangle. As in the case of the torsion problem, Saint-Venant illustrated his solutions by detailed explanations, numerical examples and graphs. The reader should consult his original work as well as the book by I. Todhunter and K. Pearson [1] (cf. also the interesting paper by B. G. Galerkin [3]).

Also in the problems of bending, conformal mapping may be of great help, just as it was for torsion. In particular, the results of § 134 (with obvious modifications) are easily applied to the present problem and the problem of bending may thus be solved for all the cases, considered in § 134*a*. But not much space will be devoted to it here and consideration will be limited to the simple example, presented in the next section.

In a recently published paper, S. Ghosh [1] applied the method of conformal transformation to bending problems; evidently he was only conversant with the Author's paper [12] and with the study of the problem, contained in the book by I. S. Sokolnikoff [1].

§ 138a. Example. Bending of a circular cylinder or tube*.

Consider a cross-section of the shape of a circular ring, bounded by concentric circles L_1 and L_2 with radii R_1 and R_2 $(R_1 < R_2)$. For the ring (cf. § 62, Note)

$$G(z) = \chi + i\chi' = A \log z + \sum^{+\infty} (a_k + ib_k) z^k, \tag{138.1a}$$

whence, putting $z = re^{i\vartheta}$,

$$\chi = A \log r + \sum^{+\infty} (a_k \cos k\vartheta - b_k \sin k\vartheta) r^n. \tag{138.2a}$$

In (137.12) take for n the normal, directed away from the centre; then obviously

$$\cos(n, x) = \cos\vartheta, \quad \cos(n, y) = \sin\vartheta.$$

*) Cf. A. E. H. Love [1], Chap. XV.

Further, noting that

$$[\tfrac{1}{2}\sigma x^2 + (1 - \tfrac{1}{2}\sigma)y^2] \cos \vartheta + (2 + \sigma)xy \sin \vartheta =$$
$$= (\tfrac{3}{4} + \tfrac{1}{2}\sigma)r^2 \cos \vartheta - \tfrac{1}{4}r^2 \cos 3\vartheta \tag{138.3a}$$

and that

$$\frac{d\chi}{dn} = \frac{\partial\chi}{\partial r}$$

one obtains the boundary conditions in the form

$$\frac{A}{r} + \sum_{-\infty}^{+\infty} k(a_k \cos k\vartheta - b_k \sin k\vartheta)r^{k-1} =$$
$$= -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)r^2 \cos \vartheta + \tfrac{1}{4}r^2 \cos 3\vartheta \text{ for } r = R_1, R_2,$$

whence, comparing coefficients of $\cos k\vartheta$ and $\sin k\vartheta$,

$$A = 0, \quad b_k = 0 \; (k = \pm 1, \pm 2, \ldots), \quad a_k = 0 \quad (k \neq 0, \pm 1, \pm 3),$$
$$a_1 - R_1^{-2}a_{-1} = -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)R_1^2, \quad a_1 - R_2^{-2}a_{-1} = -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)R_2^2,$$
$$3a_3R_1^2 - 3a_{-3}R_1^{-4} = \tfrac{1}{4}R_1^2, \quad 3a_3 - 3a_{-3}R_2^{-4} = \tfrac{1}{4}R_2^2.$$

These equations give

$$a_1 = -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)(R_1^2 + R_2^2), \quad a_{-1} = -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)R_1^2R_2^2,$$
$$a_3 = \tfrac{1}{4}, \quad a_{-3} = 0.$$

The constants a_0, b_0 remain arbitrary, as was to be expected.

Finally, one finds for $\chi(x, y)$

$$\chi = -(\tfrac{3}{4} + \tfrac{1}{2}\sigma)\left\{(R_1^2 + R_2^2)r + \frac{R_1^2R_2^2}{r}\right\} \cos \vartheta + \tfrac{1}{4}r^3 \cos 3\vartheta + \text{const.} \tag{138.4a}$$

and the problem is solved.

For $R_1 = 0$, one obtains the solution for the solid bar with circular cross-section.

CHAPTER 23

TORSION OF BARS CONSISTING OF DIFFERENT MATERIALS *)

§ 139. General Formulae. 1°. The problem of the torsion of bars consisting of prismatic (cylindrical) components made of different materials and joined along their side surfaces will be studied next. Each component will be assumed homogeneous and isotropic.

The cross-section S of the bar will consist of several regions S_0, S_1, $S_2, \ldots, S_m$, corresponding to different materials and bounded by curves to be called dividing lines. In the sequel, when speaking of a part of the bar, this will refer to such a region S_j.

Although the majority of the results to be stated below are true in the most general case, the reasoning will be given only for the special *basic* case, when the bar under consideration consists of a series of parallel solid bars which do not touch each other and which are surrounded by elastic material filling the space between them and the surrounding cylinder which is parallel to the component bars.

The cross-section S of such a bar will consist of a set of different regions S_1, S_2, $\ldots$, S_m, corresponding to the component bars, and a region S_0 corresponding to the surrounding material. Let the boundaries of the regions S_j be denoted by L_j $(j = 1, 2, \ldots, m)$; the boundary of S_0 will then consist of the closed contours L_1, $L_2, \ldots, L_m$, L_{m+1}, where the last contains all the preceding ones (Fig. 61).

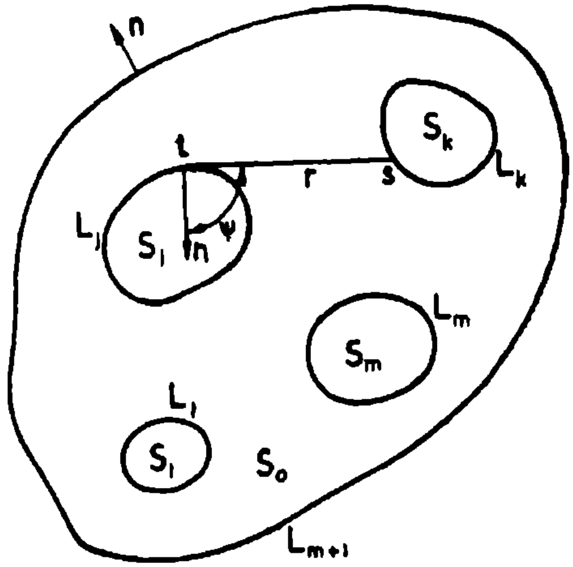


Fig. 61.

2°. It will now be attempted to satisfy the conditions of the torsion problem, by writing, as for the case

*) The results of the present chapter were first given in the Author's papers [14, 15].

of a homogeneous bar,

$$u = -\tau zy, \quad v = \tau zx, \quad w = \tau\varphi(x, y), \tag{139.1}$$

where the constant τ and the function $\varphi(x, y)$ are subject to definition; the latter function will be called the *torsion function*.

On the basis of (129.2), one finds, as in the case of the homogeneous bar, that in each region $S_j (j = 0, \ldots, m)$

$$X_z = \tau\mu_j\left(\frac{\partial\varphi}{\partial x} - y\right), \quad Y_z = \tau\mu_j\left(\frac{\partial\varphi}{\partial y} + x\right), \tag{139.2}$$

where μ_j denotes the shear modulus corresponding to the region S_j; the remaining stress components are zero.

By substituting these expressions in (129.1), it is easily seen that these equations, as in the case of the homogeneous bar, reduce to the Laplace equation

$$\Delta\varphi = 0.$$

Thus, in the present case, the function φ must also be harmonic in each of the regions S_j. The difference from the case of the homogeneous bar manifests itself only in the boundary conditions. These conditions express that:

a) the external surface of the bar is free from external forces,

b) the forces acting on elements of the surfaces, separating the different materials, are equal in magnitude and opposite in direction,

c) the displacements u, v, w remain continuous across the dividing surfaces (because, by supposition, the various parts of the bar are joined together).

The condition *a*) obviously leads to

$$X_z \cos(n, x) + Y_z \cos(n, y) = 0 \text{ on } L_{m+1} \tag{139.3}$$

and *b*) to

$$[X_z \cos(n, x) + Y_z \cos(n, y)]_j = [X_z \cos(n, x) + Y_z \cos(n, y)]_0 \tag{139.4}$$

on $L_1, L_2, \ldots, L_m$; here and later on, n denotes the normal to one of the contours L_j and is directed *outwards* with regard to S_0. The subscripts 0 and j indicate that the expressions in the brackets are to be calculated for the material lying in S_0 or S_j respectively.

Substituting for X_z and Y_z from (139.2), the conditions (139.3) and (139.4) may be expressed by the single formula

$$\mu_0\left(\frac{d\varphi}{dn}\right)_0 - \mu_j\left(\frac{d\varphi}{dn}\right)_j = (\mu_0 - \mu_j)\,[y\cos(n, x) - x\cos(n, y)] \tag{139.5}$$

on L_j, $j = 1, \ldots, m + 1$, *assuming* $\mu_{m+1} = 0$.

The condition *c*) leads to the requirement that *the function* φ *is to remain continuous* for the transition from one material to another. In other words, *the function* φ *must be continuous in the entire region*

$$S = S_0 + S_1 + \ldots + S_m,$$

including the dividing lines.

It will be shown in § 144 that, if φ satisfies the preceding condition, the resultant force vectors acting on either of the ends of the bar are zero. Consequently, the forces acting on the "ends" produce pure couples. The moment M of the couple acting on the "upper" end is obtained by calculating the resultant moment of the above forces about the axis Oz. Obviously

$$M = \tau D, \tag{139.6}$$

where

$$D = \sum_{j=0}^{m} \iint\limits_{S_j} \mu_j \left(x^2 + y^2 + x \frac{\partial \varphi}{\partial y} - y \frac{\partial \varphi}{\partial x} \right) dx\, dy. \tag{139.6'}$$

The constant D is the torsional rigidity. As in § 131, it is quite easily shown that $D > 0$. For a given moment M, the constant τ is obtained from (139.6). Thus, in the end, the problem is reduced to the determination of the harmonic function φ which is continuous throughout S and the normal derivative of which has on the contours S_j discontinuities, subject to the conditions (139.5); on the external contour, the normal derivative is also determined by (139.5) with $j = m + 1$.

In the next section the theoretical solutions of several more general problems will be given, in which the conditions (139.5) are replaced by

$$\mu_0 \left(\frac{d\varphi}{dn} \right)_0 - \mu_j \left(\frac{d\varphi}{dn} \right)_j = f_j \text{ on } L_j,\ j = 1, 2, \ldots, m + 1, \tag{139.7}$$

where the f_j are functions given on the contours L_j.

For the problem of torsion

$$f_j = (\mu_0 - \mu_j)\,[y \cos (n, x) - x \cos (n, y)]. \tag{139.8}$$

It is easily shown that the condition (139.7), apart from an arbitrary constant, determines the unknown function φ uniquely. In fact,

$$\sum_{j=1}^{m+1} \int\limits_{L_j} \varphi f_j ds = \sum_{j=1}^{m+1} \int\limits_{L_j} \varphi \left[\mu_0 \left(\frac{d\varphi}{dn} \right)_0 - \mu_j \left(\frac{d\varphi}{dn} \right)_j \right] ds =$$

$$= \mu_0 \int\limits_{L} \varphi \left(\frac{d\varphi}{dn} \right)_0 ds - \sum_{j=1}^{m} \mu_j \int\limits_{L_j} \varphi \left(\frac{d\varphi}{dn} \right)_j ds,$$

where L denotes the union of the contours $L_1, L_2, \ldots, L_{m+1}$. But by a well known formula

$$\int_L \varphi \frac{d\varphi}{dn} ds = \iint_S \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] dx\, dy,$$

where φ is a function, harmonic in some region S bounded by a contour L and $d\varphi/dn$ is the derivative in the direction of the *outward* normal. Taking into consideration that for $S_1, \ldots, S_m$, in the present notation, n represents the inward normal, one finds

$$\sum_{j=1}^{m+1} \int_{L_j} \varphi f_j\, ds = \sum_{j=0}^{m} \mu_j \iint_{S_j} \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] dx\, dy, \qquad (139.9)$$

whence it follows that, if on all contours L_j

$$f_j = 0,$$

then

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial y} = 0$$

in the entire region S, and hence $\varphi = \text{const.}$

If now φ_1 and φ_2 are two solutions of the problem, then

$$\varphi = \varphi_1 - \varphi_2$$

will likewise be a solution, corresponding to the case $f_j = 0$ on all contours. It follows from this that

$$\varphi_1 - \varphi_2 = \text{const.},$$

which proves the above statement.

Next consider the question of the existence of a solution and of its determination. First of all, it is easily seen from (139.7) that

$$\sum_{j=1}^{m+1} \int_{L_j} f_j\, ds = \mu_0 \int_L \left(\frac{d\varphi}{dn}\right)_0 ds - \sum_{j=1}^{m} \mu_j \int_{L_j} \left(\frac{d\varphi}{dn}\right)_j ds,$$

whence, using the fact that the integral over the normal derivative of a function, harmonic in a region, taken over the boundary of the region is zero, it follows that

$$\sum_{j=1}^{m+1} \int_{L_j} f_j\, ds = 0. \qquad (139.10)$$

Consequently, the condition (139.10) is necessary for the existence of a solution. It can also be shown to be sufficient, as will be seen later. This condition is always satisfied in the torsion problem, because, if f_j has the form (139.8),

$$\int f_j \, ds = 0$$

for every L_j separately (cf. § 131 p. 574). Thus condition (139.10) is certainly fulfilled in this case.

§ 140. Solution by means of integral equations. Taking into consideration the continuity of the unknown function φ in the entire region S and the discontinuity of its normal derivative on the contours L_j, it is natural to try to represent this function in the form of a potential of a *simple* layer, distributed over these contours, since the potential of a simple layer has just the required properties. In this way one is led to the generalization of the known problem of Robin—Poincaré (as it has been called in J. Plemelj's book [2]).

Thus, let

$$\varphi(x, y) = \int\limits_L \rho(s) \log \frac{1}{r} \, ds = \sum_{j=1}^{m+1} \int\limits_{L_j} \rho(s) \log \frac{1}{r} \, ds, \tag{140.1}$$

where r denotes the distance of the point (x, y) from the point s lying on one of the contours L_j and $\rho(s)$ [the density of the layer] is an unknown continuous function of s. The symbol s will refer at the same time to any of the contours L_j over which the integration is extended.

On the basis of the known properties of the potential function of a simple layer, the function φ, defined in (140.1), will be continuous throughout the region. Its normal derivative will be discontinuous for a passage through the L_j. In fact, the following relations will hold:

$$\begin{aligned} \left(\frac{d\varphi}{dn}\right)_j & \qquad \pi\rho(t) + \int \rho(s) \frac{\cos\psi}{r} \, ds \quad (j = 1, 2, \ldots, m), \\ \left(\frac{d\varphi}{dn}\right)_0 &= \quad \pi\rho(t) + \int \rho(s) \frac{\cos\psi}{r} \, ds, \end{aligned} \tag{140.2}$$

where $(d\varphi/dn)_0$ and $(d\varphi/dn)_j$ relate to a point t on one of the contours L_j, r denotes the distance between the points s and t and ψ is the

angle between the vector ts and the normal n at t (remembering that n is always the normal outward with regard to the region S_0; see Fig. 61).

Using (140.2), the boundary condition (139.7) now becomes

$$\pi(\mu_0 + \mu_j)\rho(t) + (\mu_0 - \mu_j)\int \rho(s)\frac{\cos\psi}{r}\,ds = f_j(t), \tag{140.3}$$

where t denotes a point on L_j $(j = 1, \ldots, m+1)$. In this way a system of Fredholm equations has been obtained which may be reduced to one single equation

$$\rho(t) + \int K(t, s)\rho(s)ds = f(t), \tag{140.4}$$

where

$$K(t,s) \quad \frac{\mu_0 - \mu_j}{\pi(\mu_0 + \mu_j)}\,\frac{\cos\psi}{r} \qquad f(t) \quad \frac{f_j(t)}{\pi(\mu_0 + \mu_j)} \text{ for } t \text{ on } L_j. \tag{140.5}$$

Next examine for what conditions (140.4) has a solution. The homogeneous equation

$$\rho(t) + \int K(t, s)\rho(s)ds = 0, \tag{140.6}$$

obtained from (140.4) by putting $f(t) = 0$, i.e.,

$$f_j(t) = 0, \quad (j = 1, \ldots, m+1),$$

has only one linearly independent solution. In fact, the function φ, defined by (140.1) where $\rho(s)$ is the solution of the homogeneous equation (140.6), will satisfy the boundary conditions (139.7) for $f_j = 0$. As proved in § 139, such a function φ is constant throughout the region S. But for $\varphi = \text{const.}$, (140.2) gives

$$2\pi\rho(t) = \left(\frac{d\varphi}{dn}\right)_0 - \left(\frac{d\varphi}{dn}\right)_j = 0 \text{ on } L_j \ (j = 1, 2, \ldots, m).$$

Thus, the solution ρ of the homogeneous equation (140.6) is the density of a layer, distributed over the outside boundary L_{m+1} of the region S and giving a constant potential in this region; for example, this is the two-dimensional analogue of the "natural distribution" of electricity on a conductor. As is known (cf. e.g. J. Plemelj [2], p. 63), the density of such a distribution is determined uniquely apart from a constant multiplier, and this proves the supposition.

By a known theorem of Fredholm, the adjoint homogeneous equation, i.e.,

$$\rho(s) + \int K(t, s)\rho(t)dt = 0, \tag{140.7}$$

will likewise have a unique (linearly independent) solution.

In fact, it is easily verified that this solution is

$$\rho^*(t) = \mu_0 + \mu_j \text{ (when } t \text{ is on } L_j,\ j = 1, \ldots, m + 1). \tag{140.8}$$

Actually, if the point s is taken on L_j $(j < m + 1)$,

$$\int_{L_j} K(t, s)dt = \frac{\mu_0 - \mu_j}{\pi(\mu_0 + \mu_j)} \int_{L_j} \frac{\cos\psi}{r}\, dt = \frac{\mu_0 - \mu_j}{\mu_0 + \mu_j}$$

$$\int_{L_i} K(t, s)dt \quad \frac{\mu_0 - \mu_j}{\pi(\mu_0 + \mu_j)} \int_{L_i} \frac{\cos\psi}{r}\, dt = 0, \quad i \neq j, \quad i \neq m + 1\ ,$$

$$\int_{L_{m+1}} K(t, s)dt = \frac{\mu_0}{\pi\mu_0} \int_{L_{m+1}} \frac{\cos\psi}{r}\, dt = \quad 2.$$

The above leads to the well known formula

$$\int_{L_j} \frac{\cos\psi}{r}\, dt = \begin{cases} 0, & \text{if } s \text{ is outside } L_j, \\ \pi, & \text{if } s \text{ is on } L_j, \\ 2\pi, & \text{if } s \text{ is inside } L_j. \end{cases}$$

(For L_{m+1} the sign has to be changed, since in that case, by the adopted notation, n will be the outward normal with regard to that contour, and not inward as it is with respect to all other L_j.)

Using these formulae, one immediately establishes that ρ^*, defined by (140.8), satisfies (140.7), if the point s is taken on $L_j(j < m + 1)$. Finally, if s is on L_{m+1}, then, similarly as above,

$$\int_{L_j} K(t, s)dt = 0 \ \ (j < m + 1), \int_{L_{m+1}} K(t, s)dt = -1,$$

whence it follows that (140.7) is also satisfied in this case.

Thus ρ^*, defined by (140.8), is one of the solutions of (140.7); the remaining solutions may differ from it only by a constant factor.

According to a known theorem of Fredholm, the original integral

equation will have a solution if, and only if, the condition

$$\int\limits_{L} \rho^* f(s)\,ds = 0$$

is satified, i.e., by (140.8) and (140.5), if

$$\sum_{j=1}^{m+1} \int\limits_{L_j} f_j\,ds = 0. \tag{140.9}$$

With the fulfillment of this condition the original integral equation (140.4) has a solution, determined apart from an additive term of the form $K\rho^{**}$, where K is an arbitrary constant and ρ^{**} is the solution of the homogeneous equation (140.6). The potential, corresponding to this term, is a constant. Therefore the function φ is uniquely determined apart from an arbitrary constant.

In the case of the torsion problem, the condition for the existence of a solution, i.e., (140.9), is always satisfied, as has already been shown above. Consequently, the torsion problem always has a solution of the stated form; the torsion function φ is determined uniquely apart from an arbitrary constant which has no influence on the stresses and deformations.

The problem of the torsion of a bar, consisting of a number of hollow cylinders, inserted one into another and joined together along the side surfaces so that the curves, dividing the cross-section S of the bar into regions corresponding to different materials, are themselves closed contours, may be solved in quite an analogous manner (i.e., the problem of "composite tubes"). The case when the various component bars have longitudinal cylindrical cavities does not present any particular difficulties.

In the preceding work, it has, of course, been assumed that the closed contours L_1, L_2, etc. satisfy a definite condition of regularity. For the preceding work to be valid, it is sufficient to assume, for example, that each of the considered contours has at every point a continuously changing tangent and a bounded curvature.

§ 140a. Applications. In several particular cases it is, of course, possible to obtain a solution of the problem without using integral equations. Thus, use of conformal transformation may sometimes be preferable, as will be shown in the first of the examples to be treated below.

1°. **Torsion of a circular cylinder, reinforced by a longitudinal round bar of a different material.**

Solution of this problem was obtained by I. N. Vekua and A. K. Rukhadze and published in their paper [1]; part of this paper has been reproduced here almost without any change. Solution of the case when there are cavities instead of the reinforcing rods was given by H. M. Macdonald [1]. Another solution of the same problem was obtained by E. Weinel [1]. See also a recent paper by R. C. F. Bartels [1].

Let the cross-section S of the bar consist of the region S_1, bounded by the circle L_1, and the region S_2, bounded by the same circle L_1 and a circle L_2 enclosing the former. Let μ_1 and μ_2 be the moduli of rigidity of S_1 and S_2 respectively.

If is easily seen that, if L_1 and L_2 are concentric circles and if the origin of the coordinates is taken at their centre, the torsion function will be constant, so that the ends of the inner rod and the surrounding cylinder move as if they were not connected with one another and the torsional rigidity of the composite bar were equal to the sum of the rigidities of the component parts.

The case when L_1 and L_2 are not concentric is more complicated. The notation of the first part of § 48 will be used with the exception that $z = x + iy$ is replaced by $\mathfrak{z}$. Let

$$x + iy = \mathfrak{z} = \frac{\zeta}{1 - a\zeta} = \omega(\zeta) \tag{140.1a}$$

be the relation mapping the $\mathfrak{z}$ plane on to the ζ plane. The circles L_1 and L_2 will correspond to circles γ_1 and γ_2 in the ζ plane the radii of the latter being ρ_1 and ρ_2 $(\rho_1 < \rho_2)$. These radii and the constant a are related to the radii r_1 and r_2 of the circles L_1 and L_2 and the distance l between their centres by the formulae (48.7) and (48.8). It should be remembered (§ 48) that

$$0 < \rho_1 < \rho_2 < \frac{1}{a}. \tag{140.2a}$$

By (140.1a), the region S_1 will correspond to the circle $|\zeta| < \rho_1$ and S_2 to the circular ring $\rho_1 < |\zeta| < \rho_2$. Let φ be the torsion function; its values in S_1 and S_2 will be denoted by φ_1 and φ_2 respectively. Let ψ be the function, conjugate to φ and defined separately in S_1 and S_2; its values in S_1 and S_2 will be denoted by ψ_1 and ψ_2. The functions φ_1, φ_2, ψ_1 and ψ_2 are harmonic in the respective regions.

The boundary conditions satisfied by φ_1 and φ_2 are (cf. § 139):

$$\begin{aligned} \frac{d\varphi_2}{dn} &= y\cos(n, x) - x\cos(n, y) \quad \text{on } L_2, \\ \varphi_1 &= \varphi_2 \qquad\qquad\qquad\qquad\qquad \text{on } L_1, \end{aligned} \tag{140.3a}$$

$$\mu_2\frac{d\varphi_2}{dn} - \mu_1\frac{d\varphi_1}{dn} = (\mu_2 - \mu_1)\,[y\cos(n, x) - x\cos(n, y)] \quad \text{on } L_1,$$

where n is now the normal directed away from the centre of the respective circle. Also let s be the arc measured anti-clockwise. It will be assumed that the relations

$$\frac{d\varphi_1}{dn} = \frac{d\psi_1}{ds}, \quad \frac{d\varphi_2}{dn} = \frac{d\psi_2}{ds} \tag{140.4a}$$

hold on the contour L_1 (and the second of them also on L_2). This will be confirmed, once the final solution has been found.

With the above, (140.3a) gives for ψ_1, ψ_2 the conditions

$$\begin{aligned} \psi_2 &= \tfrac{1}{2}(x^2 + y^2) + \text{const. on } L_2, \\ \frac{d\psi_1}{dn} &= \frac{d\psi_2}{dn} \text{ on } L_1, \\ \mu_2\psi_2 - \mu_1\psi_1 &= \tfrac{1}{2}(\mu_2 - \mu_1)(x^2 + y^2) + \text{const. on } L_1. \end{aligned} \tag{140.3a'}$$

Let $F(z) = \varphi + i\psi$ be the complex torsion function and let

$$f(\zeta) = \varphi + i\psi \tag{140.5a}$$

be the same function, expressed in terms of ζ. Let f_1 and f_2 be the values of this function in σ_1 and σ_2, where σ_1 is the circle $|\zeta| < \rho_1$ and σ_2 the ring

$$\rho_1 < |\zeta| < \rho_2.$$

Then one will have

$$f_1(\zeta) = \sum_{k=0} (a_k' + ib_k')\zeta^k \text{ in } \sigma_1, \tag{140.6a}$$

$$f_2(\zeta) = \sum_{k=-\infty}^{+\infty} (a_k'' + ib_k'')\zeta^k \text{ in } \sigma_2, \tag{140.7a}$$

whence, putting $\zeta = \rho e^{i\vartheta}$,

$$\psi_1 = b_0' + \sum_{k=1}^{\infty} (a_k' \sin k\vartheta + b_k' \cos k\vartheta)\rho^k, \tag{140.6a'}$$

$$\psi_2 = b_0'' + \sum_{k=1}^{\infty} [(\rho^k a_k'' - \rho^{-k}a_{-k}'') \sin k\vartheta + (\rho^k b_k + \rho^{-k}b_{-k}'') \cos k\vartheta]. \tag{140.7a'}$$

Further, note that

$$\tfrac{1}{2}(x^2+y^2) = \tfrac{1}{2}z\bar{z} = \tfrac{1}{2}\cdot\frac{\zeta\bar{\zeta}}{(1-a\zeta)(1-a\bar{\zeta})} = \tfrac{1}{2}\cdot\frac{\rho^2}{(1-a\zeta)(1-a\bar{\zeta})}.$$

But

$$\frac{1-a^2\rho^2}{(1-a\zeta)(1-a\bar{\zeta})} = 1 + \frac{a\zeta}{1-a\zeta} + \frac{a\bar{\zeta}}{1-a\bar{\zeta}} = 1 + (a\zeta + a^2\zeta^2 + \ldots) +$$
$$+ (a\bar{\zeta} + a^2\bar{\zeta}^2 + \ldots) = 1 + 2a\rho\cos\vartheta + 2a^2\rho^2\cos 2\vartheta + \ldots.$$

This series converges absolutely for $\rho < 1/a$. Thus

$$\tfrac{1}{2}(x^2+y^2) = \frac{\rho^2}{1-a^2\rho^2}\left\{\tfrac{1}{2} + \sum_{k=1}^{\infty} a^k\rho^k\cos k\vartheta\right\}. \tag{140.8a}$$

Substitute (140.6a'), (140.7a') and (140.8a) in (140.3a') [where the middle condition may be replaced by $\partial\psi_1/\partial\rho = \partial\psi_2/\partial\rho$ (for $\rho = \rho_1$), since $d\psi_1/dn$ and $d\psi_2/dn$ differ from these only by a factor] and compare coefficients of $\cos k\vartheta$, $\sin k\vartheta$. One easily finds for $k \geqslant 1$:

$$\begin{aligned} a_k' = a_k'' &= a_{-k}'' = 0, \\ \rho_2^{2k}b_k'' + b_{-k}'' &= c_2 a^{k-1}\rho_2^{2k}, \\ \nu\rho_1^{2k}b_k'' + b_{-k}'' &= \nu c_1 a^{k-1}\rho_1^{2k}, \\ b_k' &= b_k'' - \rho_1^{-2k}b_{-k}'', \end{aligned} \tag{140.9a}$$

where

$$\nu = \frac{\mu_2-\mu_1}{\mu_2+\mu_1}, \quad c_1 = \frac{a\rho_1^2}{1-a^2\rho_1^2}, \quad c_2 = \frac{a\rho_2^2}{1-a^2\rho_2^2} \tag{140.10a}$$

here c_1 and c_2 are the distances of the centres of L_1 and L_2 from the origin [cf. (48.6)].

The constants b_0'', b_0' remain quite arbitrary, as was to be expected. The condition of continuity of φ gives, as is easily seen, $a_0' = a_0''$, where the value of a_0' and a_0'' is arbitrary. Therefore put

$$b_0' = b_0'' = a_0' = 0.$$

Then, by (140.9a),

$$\begin{aligned} b_k'' &= c_2 a^{k-1} + \frac{\nu l\alpha^k}{1-\nu\alpha^k}a^{k-1}, \\ b_{-k}'' &= -l\nu\frac{\rho_1^{2k}}{1-\nu\alpha^k}a^{k-1} \\ b_k' &= c_2 a^{k-1} + l\nu\frac{1+\alpha^k}{1-\nu\alpha^k}a^{k-1} \end{aligned} \tag{140.11a}$$

where

$$\alpha = \rho_1^2/\rho_2^2, \quad l = c_2 - c_1. \tag{140a.12}$$

Substituting these values in (140.6*a*) and (140.7*a*), one finally obtains

$$\begin{aligned} f_1(\zeta) &= \frac{ic_2\zeta}{1 - a\zeta} + il\nu \sum_{k=1}^{\infty} \frac{1-\alpha^k}{1-\nu\alpha^k} a^{k-1}\zeta^k, \\ f_2(\zeta) &= \frac{ic_2\zeta}{1 - a\zeta} + il\nu \sum_{k=1}^{\infty} \frac{\alpha^k a^{k-1}\zeta^k}{1-\nu\alpha^k} - il\nu \sum_{k=1}^{\infty} \frac{\rho_1^{2k} a^{k-1}}{1-\nu\alpha^k} \cdot \frac{1}{\zeta^k}. \end{aligned} \tag{140.13a}$$

These series and their derivatives are easily seen to converge absolutely and uniformly in the relevant regions including the boundaries.

If $\nu = 0$, i.e., $\mu_1 = \mu_2$, one finds for f_1 and f_2 one and the same expression

$$f(\zeta) = \frac{ic_2\zeta}{1 - a\zeta},$$

i.e.,

$$F(z) = ic_2 z = ic_2(x + iy).$$

This is the complex torsion function for the homogeneous cylinder. If the origin of coordinates is taken at the centre of the cylinder, one obtains

$$F(z) = \text{const.}$$

(cf. § 131, Note 2).

Thus it may be said that the function $f_2(\zeta)$, determined by the second of the formulae (140.13*a*), consists of two parts: one, corresponding to the case when one deals with a continuous, homogeneous bar (first term), and the other, expressing the "indignation" aroused by the presence of the component bar.

Once the functions $f_1(\zeta)$ and $f_2(\zeta)$ have been found, the stress components can be calculated by the formulae of § 134 (cf. I. N. Vekua and A. K. Rukhadze [1]). The torsional rigidity D is likewise easily calculated from a formula of § 134. One finds

$$\begin{aligned} D = \mu_2 I + (\mu_1 - \mu_2) I' &- \frac{\pi l^2 r_1^2 (\mu_1 - \mu_2)^2}{\mu_1 + \mu_2} \\ &- 2\mu_2 \pi l^2 \nu \rho_1^2 \sum_{k=1} \frac{\alpha^k \nu^k}{(1 - a^2\rho_1^2\alpha^k)^2}, \end{aligned} \tag{140a.14}$$

where

$$I = \frac{\pi r_2^4}{2}, \quad I' = \frac{\pi r_1^4}{2} + \pi r_1^2 l^2; \tag{140.15a}$$

here I is the polar moment of inertia of the solid bar of radius r_2 referred to its centre, and I' is the polar moment of inertia of the bar with radius r_1 referred to the centre of the first bar.

If D' and D'' denote the torsional rigidities of the component bars with the moduli of rigidity μ_1 and μ_2, taken separately, then

$$D' = \mu_1(I' - \pi l^2 r_1^2),$$

$$D'' = \mu_2(I - I') \quad \mu_2 \pi l^2 r_1^2 - 2\mu_2 \pi l^2 \rho_1^2 \sum_{k=1}^{\infty} \frac{\alpha^k}{(1 - a^2\rho_1^2\alpha^k)^2} \tag{140.16a}$$

[where the latter follows from (140.14a) by putting $\mu_1 = 0$].
By (140.14a) and (140.16a)

$$D \quad (D' + D'') = \frac{4\pi\mu_1\mu_2 l^2 r_1^2}{\mu_1 + \mu_2} + 2\mu_2 \pi l^2 \rho_1^2 \sum_{k=1}^{\infty} \frac{\alpha^k(1 - \nu^{k+1})}{(1 - a^2\rho_1^2\alpha^k)^2}, \tag{140.17a}$$

whence it follows that

$$D' + D'' < D, \tag{140.18a}$$

as may have been expected beforehand.

For the homogeneous cylinder ($\mu_1 = \mu_2$), one has instead of D

$$D_0 = \mu_2 I.$$

In the general case when r_1 is small, one has approximately, neglecting fourth and higher order terms in ρ_1,

$$D = \mu_2 I + \frac{2\mu_2(\mu_1 - \mu_2)}{\mu_1 + \mu_2} I',$$

whence follows the approximate formula

$$\frac{D}{D_0} = 1 + \frac{2(\mu_1 - \mu_2)}{\mu_1 + \mu_2} \cdot \frac{I'}{I}. \tag{140.19a}$$

This formula reduces for $\mu_1 = 0$ to that obtained by H. M. Macdonald [1] for the hollow cylinder.

If the cylinder is reinforced not only by one but by several longitudinal bars of the same material and if these rods are so thin and removed from one another that the regions which are "affected" practically do not overlap, then obviously the approximate formula (140.19a) may also be used in this case, provided one interprets I' as the sum of the

moments of inertia of the component rods with regard to the centre of the circle L_2.

The torsion problem has also been solved in the case when L_1 and L_2 are confocal ellipses (cf. I. N. Vekua and Rukhadze [2]) and when the boundaries are epitrochoids disposed in a definite manner.

2°. Torsion of a rectangular bar, consisting of two different rectangular parts. The problem of the torsion of a rectangular homogeneous bar was solved by Saint-Venant (cf. e.g. A. E. H. Love [1] § 221). It is often possible to obtain a solution of the torsion problem in cases which are excluded from consideration by the reduction of the problem to integral equations. For example, such cases occur when the boundaries have corners, as in the problem to be treated here.

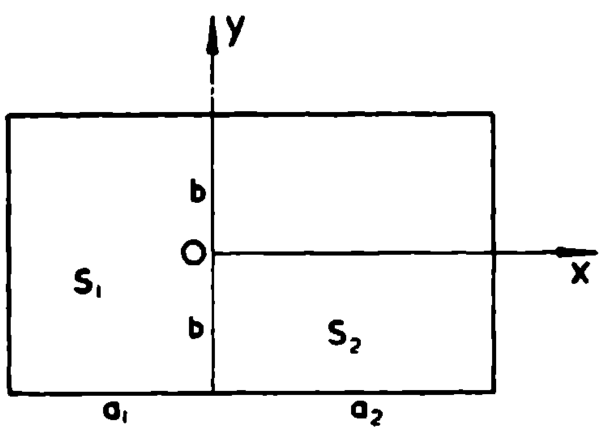


Fig. 62.

Consider a bar, consisting of two parts the cross-sections of which are rectangles with sides a_1, $2b$ and a_2 and which meet along the boundary of width $2b$ (Fig. 62). Let the moduli of rigidity of the component bars be μ_1 and μ_2 respectively. Take the axis Oy along the dividing line of the regions S_1 and S_2, corresponding to the different materials, and let the origin be at the midpoint of this line; denote by φ_1 and φ_2 the values of the torsion function φ in the regions S_1 and S_2.

Further, introduce the harmonic function $\Phi = \varphi + xy$ and denote its values in S_1 and S_2 by Φ' and Φ''. It is immediately seen that the boundary conditions are as follows:

$$\frac{\partial \Phi'}{\partial x} = 2y \, (x = -a_1, -b \leqslant y \leqslant b), \quad \frac{\partial \Phi''}{\partial x} = 2y \, (x = a_2, \ -b \leqslant y \leqslant b), \tag{a}$$

$$\mu_1 \frac{\partial \Phi'}{\partial x} - \mu_2 \frac{\partial \Phi''}{\partial x} = 2(\mu_1 - \mu_2)y \quad (x = 0, \ -b < y < b), \tag{b}$$

$$\Phi' = \Phi'' \qquad (x = 0, \ -b < y < b), \tag{c}$$

$$\frac{\partial \Phi'}{\partial y} = 0 \, (y = \pm b, \ -a_1 \leqslant x < 0), \quad \frac{\partial \Phi''}{\partial y} = 0 \, (y = \pm b, \ 0 < x \leqslant a_2). \tag{d}$$

The harmonic functions Φ' and Φ'' will be determined in the form of the series

$$\begin{aligned} \Phi' &= \sum_{n=0}^{\infty} (A'_{2n+1} \sinh mx + B_{2n+1} \cosh mx) \sin my, \\ \Phi'' &= \sum_{n=0}^{\infty} (A''_{2n+1} \sinh mx + B_{2n+1} \cosh mx) \sin my, \end{aligned} \tag{e}$$

where

$$m = \frac{(2n+1)\pi}{2b} \tag{f}$$

Each term of the two preceding series is obviously a harmonic function. The coefficient m has been chosen so that the conditions (d) are satisfied; clearly also the condition (c) is fulfilled.

There remains to satisfy (a) and (b). For this purpose it will be remembered that in the interval $(-b, +b)$ the function $2y$ may be represented in the form of a series:

$$2y = \sum_{n=0}^{\infty} mA_{2n+1} \sin my, \tag{g}$$

where

$$mA_{2n+1} = 4b\left(\frac{2}{\pi}\right)^2 \frac{(-1)^n}{(2n+1)^2}, \tag{h}$$

i.e.,

$$A_{2n+1} = 4b^2\left(\frac{2}{\pi}\right)^3 \frac{(-1)^n}{(2n+1)^3}.$$

The series (g) is a Fourier series for the function f, defined in the interval $(-2b, +2b)$ in the following manner:

$$\begin{aligned} f &= 2y && \text{in the interval} && (-b, +b), \\ f &= 4b - 2y && \quad ,, \quad ,, && (b, 2b), \\ f &= -4b + 2y && \quad ,, \quad ,, && (-b, -2b). \end{aligned}$$

On the basis of (g), the conditions (a) will be satisfied, if

$$\begin{aligned} A'_{2n+1} \cosh ma_1 - B_{2n+1} \sinh ma_1 &= A_{2n+1}, \\ A''_{2n+1} \cosh ma_2 + B_{2n+1} \sinh ma_2 &= A_{2n+1}, \end{aligned}$$

and the condition (b), if

$$\mu_1 A'_{2n+1} - \mu_2 A''_{2n+1} = (\mu_1 - \mu_2) A_{2n+1}.$$

Solving the three preceding equations for A'_{2n+1}, A''_{2n+1} and B_{2n+1} and substituting the values thus found in (e), one obtains after some obvious transformations

$$\Phi' = 4b^2\left(\frac{2}{\pi}\right)^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cdot$$

$$\cdot \frac{[\mu_2 + (\mu_1-\mu_2)\cosh ma_2]\cosh m(x+a_1) + \mu_2 \sinh ma_2 \sinh mx - \mu_1 \cosh ma_2 \cosh mx}{\mu_1 \cosh ma_2 \sinh ma_1 + \mu_2 \cosh ma_1 \sinh ma_2} \sin my,$$

$$\Phi'' = 4b^2\left(\frac{2}{\pi}\right)^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cdot$$

$$\cdot \frac{[-\mu_1 + (\mu_1-\mu_2)\cosh ma_1]\cosh m(x-a_2) + \mu_1 \sinh ma_1 \sinh mx + \mu_2 \cosh ma_1 \cosh mx}{\mu_1 \cosh ma_2 \sinh ma_1 + \mu_2 \cosh ma_1 \sinh ma_2} \sin my.$$

The form of the coefficients shows that the series obtained converge rapidly (uniformly and absolutely). Also, the use of differentiationduring the process of deduction is justified.

The torsion functions are given by

$$\varphi_1 = \Phi' - xy \text{ in } S_1, \quad \varphi_2 = \Phi'' - xy \text{ in } S_2.$$

The torsional rigidity is obtained from (139.6′) which in the present case has the form

$$D = \mu_1 \iint_{S_1} \left(x^2 + y^2 + x\frac{\partial \varphi_1}{\partial y} - y\frac{\partial \varphi_1}{\partial x}\right) dx\, dy +$$

$$+ \mu_2 \iint_{S_2} \left(x^2 + y^2 + x\frac{\partial \varphi_2}{\partial y} - y\frac{\partial \varphi_2}{\partial x}\right) dx\, dy.$$

Substituting in this formula the expressions for φ_1 and φ_2, one finds, using the result that $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$ (compare with the case of the homogeneous bar, A. E. H. Love [1], § 225),

$$D = \frac{8}{3}(\mu_1 a_1 + \mu_2 a_2)b^3 +$$

$$+ \left(\frac{4}{\pi}\right)^5 b^4 \sum_{n=0}^{\infty} \frac{\mu_1^2 \cosh ma_2 + \mu_2^2 \cosh ma_1 - (\mu_1^2 + \mu_2^2)\cosh ma_1 \cosh ma_2}{(2n+1)^5 (\mu_1 \cosh ma_2 \sinh ma_1 + \mu_2 \cosh ma_1 \sinh ma_2)} -$$

$$- \left(\frac{4}{\pi}\right)^5 b^4 \mu_1 \mu_2 \sum_{n=0}^{\infty} \frac{\cosh ma_1 + \cosh ma_2 - \cosh m(a_1 - a_2) - 1}{(2n+1)^5 (\mu_1 \cosh ma_2 \sinh ma_1 + \mu_2 \cosh ma_1 \sinh ma_2)}.$$

If a_1 and a_2 are large compared with b (in fact, if $a_1, a_2 > 5b$), one may with sufficient accuracy put

$$\frac{\sinh ma_1}{\cosh ma_1} = 1, \quad \frac{\sinh ma_2}{\cosh ma_2} = 1, \quad \frac{1}{\sinh ma_1} = \frac{1}{\cosh ma_1} = \frac{1}{\sinh ma_2} = \frac{1}{\cosh ma_2} = 0$$

and obtain for D the approximate formula (compare with the analogous expression for the case of the homogeneous bar, A. E. H. Love [1], § 225)

$$D = \frac{8}{3}(\mu_1 a_1 + \mu_2 a_2)b^3 - \frac{\mu_1^2 + \mu_2^2}{\mu_1 + \mu_2} b^4 \left(\frac{4}{\pi}\right)^5 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^5} =$$

$$= \frac{8}{3}(\mu_1 a_1 + \mu_2 a_2)b^3 - 3.361\, b^4 \frac{\mu_1^2 + \mu_2^2}{\mu_1 + \mu_2}.$$

EXTENSION AND BENDING OF BARS, CONSISTING OF DIFFERENT MATERIALS WITH UNIFORM POISSON'S RATIO*

The various cases of elastic equilibrium of bars, which were stated in § 129, will now be considered. It will be assumed that the different materials, constituting the bar, *have the same Poisson's ratio* σ, but, generally speaking, different Young's moduli.

Remembering that σ has almost the same value for many materials, it becomes clear that this restriction is not very severe. On the other hand, it considerably simplifies the solution.

According to Poisson's original theory, the quantity σ was the same for all materials and its value was equal to $\frac{1}{4}$. However, this circumstance is not confirmed by experiment. The variations in the values of σ for different materials are considerably less than those for E. For example,
for copper: $1/\sigma = 2.87$, $E = 1.25 \times 10^6$ [kg/cm²],
while for aluminium: $1/\sigma = 2.92$, $E = 740\,000$ [kg/cm²]
(cf. Note 2 at the end of § 146).

In particular, for problems of extension (compression) and of bending by couples, the present case is almost as simple as that for homogeneous bars; this will be shown in §§ 142, 143.

§ 141. Notation.

The concepts of this section as well as the corresponding formulae also apply to the case, where the values of Poisson's ratios of the various materials are different.

Consider the quantity

$$S_E = \iint E\,dx\,dy = \sum_j S_j E_j, \tag{141.1}$$

where E denotes the modulus of elasticity at a given point of the cross-section which takes constant values E_j in different parts S_j, corresponding

*) The contents of this chapter were taken from the Author's paper [15].

to the materials constituting the elastic body; the areas of these parts of the cross-section will likewise be denoted by S_j.

Further, the "reduced centre of gravity" of the cross-section will be understood to be the centre of gravity which is obtained by ascribing the various parts of the cross-section surface densities which are equal to the corresponding moduli of elasticity; thus, if the origin of the coordinate system is placed at the reduced centre of gravity,

$$\iint Ex\,dx\,dy = \iint Ey\,dx\,dy = 0. \tag{141.2}$$

The "reduced moment of inertia" will now be defined as the moment of inertia, calculated under the same supposition with regard to the densities of the different parts of the cross-section. In particular, the reduced moment of inertia I_E about the axis Oy in the plane of the cross-section will then be given by

$$I_E = \iint\limits_S Ex^2\,dx\,dy = \sum_j S_j I_j, \tag{141.3}$$

where I_j is the customary moment of inertia of the area S_j about the same axis.

Finally, the principal axis of inertia of the cross-section, under the same assumption with regard to the densities, will be called the "reduced principal axes".

If the axes Ox, Oy coincide with the reduced principal axes of inertia, one will have

$$\iint\limits_S Exy\,dx\,dy = \sum_j E_j \iint\limits_{S_j} xy\,dx\,dy = 0. \tag{141.4}$$

Here, as well as in § 142, 143 and at the beginning of § 144, it is unneccessary to assume that one is dealing with "basic" cases (§ 139,1°); it is sufficient to suppose that the bar consists of a number of homogeneous, isotropic, cylindrical bodies (fibres or strips), welded along their side surfaces.

§ 142. Extension. It is easily seen that in the above notation the problem of extension of bars by longitudinal forces, applied to the reduced centre of gravity of the cross-section, is solved by the following formulae

[cf. (135.1), (135.2)]:

$$Z_z = \frac{E_j F}{S_E} \text{ in the region } S_j,$$

$$u \quad \frac{\sigma F}{S_E} x, \quad v = \quad \frac{\sigma F}{S_E} y, \quad w = \frac{F}{S_E} z \tag{142.1}$$

(the remaining stress components being zero); F denotes here the total tensile force ($F < 0$ will correspond to compression).

The rigidity of the bar for extension (compression) is equal to S_E (see § 135).

§ 143. Bending by a couple. The problem of bending by a couple whose moment lies in the plane of the ends is likewise very little different from the same problem for the homogeneous bar (§ 136).

Let the origin lie at the reduced centre of gravity of the "left" end and let the axes Ox, Oy coincide with the reduced principal axes of inertia.

If the moment of the couple, acting on the "right" end, is parallel to the axis Oy and if its magnitude is M, the solution is given by

$$Z_z = \quad \frac{ME_j}{I_E} x \text{ in } S_j \tag{143.1}$$

(the remaining stress components being equal to zero) and

$$u = \frac{M}{2I_E} (z^2 + \sigma x^2 \quad \sigma y^2), \quad v = \frac{M}{I_E} \sigma xy, \quad w = - \frac{M}{I_E} xz. \tag{143.2}$$

Substitution of these expressions in the static equations of the elastic body show that all the equations are satisfied; the boundary conditions are obviously fulfilled.

The resultant vector of the external stresses applied, say, to the right end is equal to zero, since, by (141.2),

$$\iint Z_z \, dx \, dy = 0.$$

The moment of these stresses about the axis Oy is, by (141.3),

$$- \iint xZ_z \, dx \, dy = \frac{M}{I_E} \qquad Ex^2 dx \, dy = M;$$

finally, the moment about the axis Ox is, by (141.4), equal to

$$\iint_S yZ_z dx\,dy = \frac{M}{I_E} \iint_S Exy dx\,dy = 0.$$

The above solution thus satisfies all the imposed conditions. It is easily seen that in the case under consideration the Bernoulli-Euler law is valid; it is now expressed by

$$\frac{1}{R} = \frac{M}{I_E}. \tag{143.3}$$

The flexural rigidity is equal to I_E.

§ 144. Bending by a transverse force. The solution of the problem of bending by a transverse force will now be considered. Let the origin O be at the reduced centre of gravity of the "left" end and let the axes Ox, Oy coincide with the reduced principal axes of inertia.

This problem may always be reduced to the case, where the transverse force, applied to the "right" end, acts through its reduced centre of gravity and parallel to the axis Ox (cf. § 137).

Guided by the form of (137.10), (137.11) which refer to the homogeneous bar, it will be assumed that the conditions of the problem may be satisfied by expressions of the following form:

$$\begin{aligned} u &= -\tau yz + A[\tfrac{1}{2}\sigma(l-z)(x^2-y^2) + \tfrac{1}{2}lz^2 - \tfrac{1}{6}z^3], \\ v &= \tau xz + A\sigma(l-z)xy, \\ w &= \tau\varphi - A[x(lz - \tfrac{1}{2}z^2) + \chi + xy^2], \end{aligned} \tag{144.1}$$

where φ is the torsion function of Chapter 23 and $\chi = \chi(x, y)$ is some function which has still to be defined; l is the length of the bar and τ, A are constants.

Calculating the stress components, corresponding to these displacements, one finds $X_x = Y_y = X_y = 0$ (as in the case of the homogeneous bar) and, in the regions $S_j (j = 0, 1, \ldots, m)$,

$$\begin{aligned} X_z &= \mu_j\tau\left(\frac{\partial\varphi}{\partial x} - y\right) - B_j\left\{\frac{\partial\chi}{\partial x} + \tfrac{1}{2}\sigma x^2 + (1 - \tfrac{1}{2}\sigma)y^2\right\}, \\ Y_z &= \mu_j\tau\left(\frac{\partial\varphi}{\partial y} + x\right) - B_j\left\{\frac{\partial\chi}{\partial y} + (2+\sigma)xy\right\}, \\ Z_z &= -K_j(l - z)x, \end{aligned} \tag{144.2}$$

where B_j, K_j are constants which may have different values in the different regions S_j; in fact,

$$B_j = A\mu_j = \frac{AE_j}{2(1+\sigma)}, \quad K_j = AE_j. \tag{144.3}$$

Substituting the expressions (144.2) in the equilibrium equations, i.e., in (129.1), it is readily verified that the function χ, as well as the function φ, must satisfy the Laplace equation in each of the subregions S_j; conversely, the above equations will be satisfied under these conditions.

Now consider the boundary conditions. To ensure that the displacements u, v, w will be continuous throughout the body, the function χ must obviously be continuous throughout the entire cross-section S (since the torsion function φ is, by definition, continuous throughout S).

The boundary conditions with regard to the stresses lead, as in the case of torsion, to the requirement that the expression

$$X_z \cos(n, x) + Y_z \cos(n, y) \tag{a}$$

must vanish on the free side surface and that it must be continuous for a passage through the surfaces separating the different materials.

The resultant vector and moment of the stresses, acting on the right end, will now be calculated. First of all, it is clear that the component of the resultant vector in the Oz direction is equal to zero. Its component in the Ox direction is given by

$$X = \iint_S X_z dx\, dy.$$

Remembering that (144.2) satisfies the equilibrium equations and, in particular, the equation

$$\frac{\partial X_z}{\partial x} + \frac{\partial Y_z}{\partial y} + \frac{\partial Z_z}{\partial z} = 0,$$

one obtains, after substituting for Z_z from (144.2),

$$\frac{\partial X_z}{\partial x} + \frac{\partial Y_z}{\partial y} + AE_j x = 0.$$

On the basis of this identity, one may write

$$X = \sum_j \iint_{S_j} \left\{X_z + x\left(\frac{\partial X_z}{\partial x} + \frac{\partial Y_z}{\partial y}\right) + AE_j x^2\right\} dx\, dy =$$

$$= \sum_j \iint_{S_j} \left\{\frac{\partial}{\partial x}(xX_z) + \frac{\partial}{\partial y}(xY_z)\right\} dx\, dy + AI_E.$$

Finally, transforming the integrals by use of Green's formula, one obtains

$$X = \sum_j \int_{L_j} x[X_z \cos(\nu, x) + Y_z \cos(\nu, y)]ds + AI_E,$$

where L_j are the boundaries of S_j and ν is the normal, outwards with regard to S_j. Two integrations will occur along the lines of division of two parts S_j, since these lines belong to the boundaries of two regions. The expression $X_z \cos(\nu, x) + Y_z \cos(\nu, y)$ has opposite signs, but the same numerical values in these two integrations; hence the integrals along the lines of division will cancel. The integrals along the lines, corresponding to the free side surface, will likewise vanish, since one has there

$$X_z \cos(\nu, x) + Y_z \cos(\nu, y) = 0.$$

Note that it had been assumed in the expression (a) that the positive normal n is the same at points, belonging to either side of the line of division. In the preceding reasoning the normals ν have opposite directions at such points.

Thus one obtains

$$X = AI_E. \tag{144.4}$$

However, by supposition, one must have $X = W$, where W is the given force. This condition determines the constant A:

$$A = \frac{W}{I_E}.$$

In a similar manner one finds for the component of the resultant vector in the Oy direction

$$Y = \iint_S Y_z dx\,dy = \sum_j \int_{L_j} y[X_z \cos(\nu, x) + Y_z \cos(\nu, y)]ds + A \sum_j E_j \iint_{S_j} xy dx\,dy.$$

It thus follows, on the basis of the above results and of (141.4), that

$$Y = 0.$$

In particular, if $A = 0$, one finds $X = Y = 0$; hence the proof has been obtained that for $A = B_j = K_j = 0$, (144.1), (144.2) give a pure twisting couple on the right end.

Finally, since $Z_z = 0$ for $z = l$, no bending couple will act on the right end.

The moment of the twisting couple is given by

$$M = \tau D + \frac{W}{2(1+\sigma)I_E} \sum_{j=0}^{m} E_j \iint_{S_j} \Big\{ y \frac{\partial \chi}{\partial x} - x \frac{\partial \chi}{\partial y} + $$

$$+ (1 - \tfrac{1}{2}\sigma) y^3 - (2 + \tfrac{1}{2}\sigma) x^2 y \Big\} \, dx \, dy, \qquad (144.6)$$

where D is the torsional rigidity. The constant τ must be determined from the condition $M = 0$ which may always be done, once the functions φ and χ have been calculated.

The function φ can be determined, using the results of Chapter 23. The function χ has still to be found. Assuming, for definiteness, that one is dealing with the "basic" case (§ 139,1°; Fig. 61), it is readily verified on the basis of (144.2) and (144.3) that the boundary conditions reduce, in the notation of Chapter 23, to the following (remembering that, by supposition, $\mu_{m+1} = 0$):

$$\mu_0 \left(\frac{d\chi}{dn} \right)_0 - \mu_j \left(\frac{d\chi}{dn} \right)_j = f_j \text{ on } L_j \; (j = 1, 2, \ldots, m+1), \qquad (144.7)$$

where

$$f_j = -(\mu_0 - \mu_j) \left\{ \left[\tfrac{1}{2}\sigma x^2 + \left(1 - \frac{\sigma}{2}\right) y^2 \right] \cos(n, x) + \right.$$

$$\left. + (2 + \sigma)\, xy \cos(n, y) \right\}. \qquad (144.8)$$

Thus, one has arrived at exactly the same problem as in the case of torsion, except that the functions f_j, given on the contours, do not have the same values.

It will now be investigated whether the condition (140.9) for the existence of a solution is satisfied. One has

$$\sum_{j=1}^{n+1} \int_{L_j} f_j ds =$$

$$= -\mu_0 \int_{L} \{[\tfrac{1}{2}\sigma x^2 + (1 - \tfrac{1}{2}\sigma) y^2] \cos(n, x) + (2 + \sigma) xy \cos(n, y)\} ds +$$

$$+ \sum_{j=1}^{m} \mu_j \int_{L_j} \{[\tfrac{1}{2}\sigma x^2 + (1 - \tfrac{1}{2}\sigma) y^2] \cos(n, x) + (2 + \sigma) xy \cos(n, y)\} ds$$

or, transforming the integrals by use of Green's formula,

$$\sum_{j=1}^{m+1} \int_{L_j} f_j ds = -\iint_{S_0} 2(1+\sigma)\mu_0 x\, dx\, dy - \sum_{j=1}^{m} \iint_{S_j} 2(1+\sigma)\mu_j x\, dx\, dy =$$
$$= -\iint Ex\, dx\, dy;$$

however, the last integral vanishes, since, by supposition, the origin lies at the reduced centre of gravity.

Thus, the existence condition is fulfilled and the present problem will always have a definite solution which may be obtained by the help of the same integral equation as in the preceding chapter, except that the functions f_j are now determined by (144.8).

In particular, *the remark in Chapter 23 with regard to the applicability of the solutions for other shapes of cross-sections*, e.g., for the case of a compound tube, *still remains true.*

Finally, note that it follows from the formula for u, i.e., from the first formula of (144.1), that the curvature of the central line (which is the locus of the reduced centres of gravity) satisfies the relation

$$\frac{1}{R} = \frac{W}{I_E}(l - z);$$

in other words, *the Bernoulli-Euler law again holds true.*

§ 144a. Example. Bending of a compound circular tube by a transverse force, applied to one of its ends.

Let the cross-section of the bar consist of two concentric circular rings S_1, S_2 the first of which surrounds the second one, as shown in Fig. 63. The inner, middle and outer radii will be denoted by R_2, R_1, R_0 respectively, and the moduli of elasticity, corresponding to S_1 and S_2, by E_1, E_2.

Let the transverse force act through the centre of the circles in the direction of the Ox axis. In view of the complete symmetry, it is clear that $\tau = 0$, i.e., no torsion takes place. The function $\chi(x, y)$ will now be found

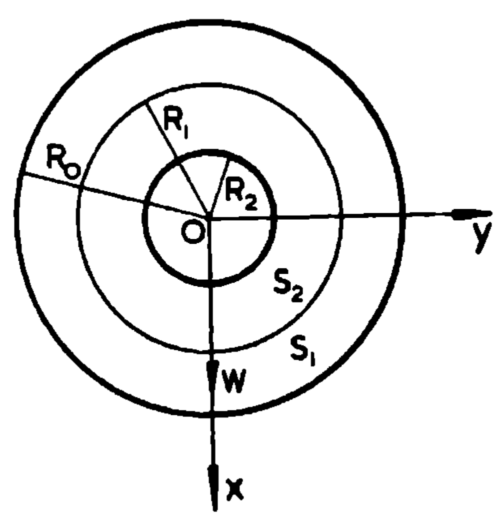


Fig. 63.

and its values in the regions S_1, S_2 will be denoted by χ_1 and χ_2, respectively.

Let r, ϑ denote the polar coordinates in the Oxy plane. By (138.3a)

$$[\tfrac{1}{2}\sigma x^2 + (1 - \tfrac{1}{2}\sigma)y^2] \cos\vartheta + (2+\sigma)xy \sin\vartheta =$$
$$= -\tfrac{3}{4}r^2 \cos 3\vartheta + (\tfrac{3}{4} + \tfrac{1}{2}\sigma)r^2 \cos\vartheta. \qquad (a)$$

Correspondingly, the boundary conditions have the form

$$\frac{\partial\chi_1}{\partial r} = kR_0^2 \cos\vartheta + \tfrac{3}{4}R^2 \cos 3\vartheta \ \text{ for } r = R_0,$$

$$\left.\begin{aligned} &\chi_1 = \chi_2 \\ &E_1\frac{\partial\chi_1}{\partial r} - E_2\frac{\partial\chi_2}{\partial r} = (E_1 - E_2)\,(-kR_1^2\cos\vartheta + \tfrac{3}{4}R_1^2\cos 3\vartheta)\} \end{aligned}\right\} \text{for } r = R_1, \qquad (b)$$

$$\frac{\partial\chi_2}{\partial r} = -kR_2^2 \cos\vartheta + \tfrac{3}{4}R_2^2 \cos 3\vartheta \ \text{ for } r = R_2,$$

where

$$\tfrac{3}{4} + \tfrac{1}{2}\sigma = k$$

and in the second condition the shear moduli μ_1, μ_2 have been replaced by the moduli of elasticity E_1, E_2 which are proportional to them.

Expanding the harmonic functions χ_1, χ_2 in series and substituting in the preceding formulae, these functions may be determined. However, it is easily shown that these conditions may be satisfied by writing (cf. solution for the hollow homogeneous circular cylinder in § 138a)

$$\chi_1 = \left(a_1 r + \frac{a_1'}{r}\right)\cos\vartheta + \tfrac{1}{4}r^3\cos 3\vartheta \quad (R_1 \leqslant r < R_0),$$

$$\chi_2 = \left(a_2 r + \frac{a_2'}{r}\right)\cos\vartheta + \tfrac{1}{4}r^3\cos 3\vartheta \quad (R_2 < r \leqslant R_1).$$

Substituting these expressions in (b), one immediately sees that all the conditions will be satisfied, provided

$$a_1R_0^2 - a_1' = -kR_0^4, \quad a_2R_2^2 - a_2' = -kR_2^4,$$
$$a_2R_1^2 + a_2' = a_1R_1^2 + a_1',$$
$$E_1(a_1R_1^2 - a_1') - E_2(a_2R_1^2 - a_2') = -k(E_1 - E_2)R_1^4.$$

Hence one finds

$$a_1 = -k\,\frac{E_1(R_0^4 - R_1^4)\,(R_1^2 + R_2^2) + E_2(R_1^2 - R_2^2)\,[(R_1^2 + R_2^2)^2 + R_0^4 - R_2^4]}{E_1(R_1^2 + R_2^2)\,(R_0^2 - R_1^2) + E_2(R_0^2 + R_1^2)\,(R_1^2 - R_2^2)},$$

$$a_2 = -k\frac{E_1(R_0^2-R_1^2)\,[(R_0^2+R_1^2)^2-R_0^4+R_2^4]+E_2(R_1^4-R_2^4)\,(R_0^2+R_1^2)}{E_1(R_1^2+R_2^2)\,(R_0^2-R_1^2)+E_2(R_0^2+R_1^2)\,(R_1^2-R_2^2)},$$

while

$$a_1' = a_1R_0^2 + kR_0^4, \quad a_2' = a_2R_2^2 + kR_2^4.$$

Thus the problem is solved. The solution for the case where the circles are not concentric was given by A. K. Rukhadze [1]. The solution for the case of confocal ellipses was given by I. N. Vekua and A. K. Rukhadze [2]. In the paper [2] by A. K. Rukhadze the solution is given for the case of epitrochoids.

The problem of bending of the rectangular bar, considered in § 140a, is likewise easily solved.

CHAPTER 25

EXTENSION AND BENDING FOR DIFFERENT POISSON'S RATIOS*

In the general case, when the Poisson's ratios of the various materials may also differ, the problems of extension and bending become considerably more complicated. In fact, it will be found that it is now impossible to assume $X_x = Y_y = X_y = 0$, as was done in the case of Saint-Venant's problem as well as in the case where the Poisson's ratios were uniform.

As a consequence, one has to give attention to a certain auxiliary problem of plane deformation which will now be introduced.

§ 145. An auxiliary problem of plane deformation. The auxiliary problem, mentioned in the introduction to this chapter, consists of the following. It is required to find the elastic equilibrium of a beam, consisting of different materials in the same manner as described at the beginning of § 139 and under the supposition that it is subject to *plane deformation* parallel to the plane Oxy (i.e., that $w = 0$ and u, v depend only on x, y and not on z), for the following conditions:

1. The external stresses, applied to the side of the bar, are equal to zero, i.e.,

$$X_n = 0, \quad Y_n = 0, \tag{145.1}$$

where, as always,

$$X_n = X_x \cos(n, x) + X_y \cos(n, y), \quad Y_n = Y_x \cos(n, x) + Y_y \cos(n, y)$$

and n denotes the normal to the side surface.

2. On the dividing surfaces of the different materials

$$(X_n)_j = (X_n)_k, \quad (Y_n)_j = (Y_n)_k, \tag{145.2}$$

where n is the normal to the (cylindrical) dividing surface in a definite direction and the subscripts j, k indicate the values for the materials

* The problem of extension and of bending by a couple was solved by the Author in his paper [15]. The present chapter presents a new, more detailed study of the solution.

occupying the regions j, k, adjoining the dividing surface. The conditions (145.2) express that the stresses applied to elements of the dividing surface from either side must balance each other.

3. The displacements undergo the following discontinuities on the dividing surface:

$$u_j - u_k = g, \quad v_j - v_k = h, \tag{145.3}$$

where (u_j, v_j), (u_k, v_k) are the values of the displacements on either side of the dividing surface and g, h are functions, given on these surfaces (and not depending on z).

Since one is dealing with plane deformations and all the functions under consideration are independent of z, one may restrict the investigation to any transverse cross-section of the bar, just as this was done in the preceding chapters.

It is easily shown by ordinary means that the solution, if it exists, will be unique (apart from rigid body displacement). Further, it may be assumed to be physically obvious that a solution exists. In fact, the present problem corresponds to the following physical problem which, for the sake of brevity, will be formulated for the case, where there are only two parts with transverse cross-sections S_1 and S_2, divided by the line L. Consider two bars which consist of the same materials as the given one, but which have cross-sections S_1', S_2', different from S_1, S_2. In fact, let it be assumed that the cross-section S_1' is obtained from S_1 by imposing on the points of the line L the displacement $(-u_1, -v_1)$, while S_2' results from S_2 by the displacement $(-u_2, -v_2)$ of the points of L; further, let

$$u_1 - u_2 = g, \quad v_1 - v_2 = h.$$

If the corresponding sides of these beams with cross-sections S_1', S_2' are now forced into contact, so that corresponding points touch each other, and if they are then welded together without disturbing the plane deformation, the compound bar, thus obtained, will exhibit exactly the same stresses and displacements as must be expected in the case of the above problem.

The existence of the solution (under certain ordinary suppositions of a general character) may also be proved mathematically. This was done most simply in the paper [20] by D. I. Sherman (mentioned already in § 103) who considered the case which has been called basic in § 139,1°. No space will be devoted here to this proof.

With a view to what follows, it will be recalled that for plane deform-

ation

$$X_z = Y_z = 0 \text{ throughout the bar,} \tag{145.4}$$

$$Z_z = \lambda_j\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \sigma_j(X_x + Y_y) \text{ on } S_j, \tag{145.5}$$

where λ_j, σ_j are the values of the Lamé constant and Poisson's ratio in the region S_j.

§ 146. The problem of extension and of bending by a couple. In the case of the compound bar for which, however, Poisson's ratio was constant, the problem of extension and of bending by a couple was solved very simply and it was found possible to investigate *separately* the problem of a tensile force with its line of action along the axis Oz and the problems of bending by couples in planes parallel to the planes Oxz and Oyz. The possibility of such an independent study depended on the particular choice of the system of axes Oxy in the plane of the "left" ("lower") end (and, in fact, the origin O was placed at the reduced centre of gravity, while the axes Ox, Oy were directed along the principal reduced axes of inertia of this end).

It will be seen later that in the case of different Poisson's ratios such a choice of coordinate axes does not, in general, offer the possibility of solving the above-mentioned problems separately.

However, as will be shown in § 148, one may also in the present case find a special system of axes which permits separate consideration of these problems; however, the determination of such a system requires the solution of several auxiliary problems of plane deformation.

Therefore, in the present section, the system of axes Oxy will refer to any (rectilinear) system in the plane of the left end S and it will not be assumed that the plane of the bending couple is parallel to one of the planes Oxz, Oyz.

1°. Let M_y and M_x denote the projections of the moment vector of the bending couple on to the axes Oy and Ox, and F the magnitude of the tensile force with line of action along the axis Oz.

Guided by the form of the solution for the case of constant Poisson's ratio, an attempt will be made to satisfy the conditions of the problem by linear combinations of the following three solutions:

$$Z_z = E_j x, \quad u = -\tfrac{1}{2}(z^2 + \sigma_j x^2 - \sigma_j y^2), \quad v = -\sigma_j xy, \quad w = xz, \tag{146.1}$$

$$Z_z = E_j y, \quad u = -\sigma_j xy, \quad v = -\tfrac{1}{2}(z^2 + \sigma_j y^2 - \sigma_j x^2), \quad w = yz, \tag{146.2}$$

$$Z_z = E_j, \quad u = -\sigma_j x, \quad v = -\sigma_j y, \quad w = z, \tag{146.3}$$

in the region S_j (the remaining stress components being equal to zero).

If all the Poisson's ratios were the same and if the coordinate axes were chosen as indicated at the beginning of this section, these solutions, multiplied by suitable constants (the same constants being used for stresses and displacements), would give those of the problems of bending by a couple in the plane Oxz, of bending by a couple in the plane Oyz and of extension by a force, directed along Oz.

In reality, however, solutions constructed in this manner do not satisfy the conditions of the above-mentioned problems for the reason that the corresponding displacements have discontinuities on the dividing lines between the sections S_j, S_k.

In order to remove these discontinuities, the solutions of three auxiliary problems of plane deformation will be constructed which represent particular cases of the problem, formulated in § 145; the functions g, h in the formulae (145.3) will now be given the following values:

$$g_1 = \tfrac{1}{2}(\sigma_j - \sigma_k)(x^2 - y^2), \qquad h_1 = (\sigma_j - \sigma_k)xy, \tag{146.1a}$$

$$g_2 = (\sigma_j - \sigma_k)xy, \qquad h_2 = \tfrac{1}{2}(\sigma_j - \sigma_k)(y^2 - x^2), \tag{146.2a}$$

$$g_3 = (\sigma_j - \sigma_k)x, \qquad h_3 = (\sigma_j - \sigma_k)y \tag{146.3a}$$

on the dividing lines between the regions S_j, S_k.

For the sake of brevity, these three problems will be denoted by (146.1a), (146.2a), (146.3a) respectively and it will be assumed that they have been solved.

The components of displacement and stress, corresponding to these three auxiliary problems, will be denoted by superscripts $^{(1)}$, $^{(2)}$, $^{(3)}$. In particular, one will have in the region S_j

$$Z_z^{(1)} = \sigma_j(X_x^{(1)} + Y_y^{(1)}), \tag{146.1b}$$

$$Z_z^{(2)} = \sigma_j(X_x^{(2)} + Y_y^{(2)}), \tag{146.2b}$$

$$Z_z^{(3)} = \sigma_j(X_x^{(3)} + Y_y^{(3)}). \tag{146.3b}$$

Superposition of the solutions (146.1), (146.2), (146.3), multiplied by certain constants a_1, a_2, a_3 respectively, and of the solutions of the problems (146.1a), (146.2a), (146.3a), multiplied by the same corresponding constants, is easily seen to give the solution of the problems of bending and extension of a bar for the following values of the moments M_x, M_y of

the bending couple and of the magnitude F of the tensile force:

$$
\begin{aligned}
-M_y &= (I_{11}+K_{11})a_1 + (I_{12}+K_{12})a_2 + (I_{13}+K_{13})a_3,\\
M_x &= (I_{21}+K_{21})a_1 + (I_{22}+K_{22})a_2 + (I_{23}+K_{23})a_3,\\
F &= (I_{31}+K_{31})a_1 + (I_{32}+K_{32})a_2 + (I_{33}+K_{33})a_3,
\end{aligned} \tag{146.4}
$$

where

$$I_{\alpha\beta} = \iint\limits_S E x^{(\alpha)} x^{(\beta)} dx\,dy = \sum_j E_j \iint\limits_{S_j} x^{(\alpha)} x^{(\beta)} dx\,dy, \tag{146.5}$$

$$K_{\alpha\beta} = \iint\limits_S E x^{(\alpha)} Z_z^{(\beta)} dx\,dy = \sum_j E_j \iint\limits_{S_j} x^{(\alpha)} Z_z^{(\beta)} dx\,dy, \tag{146.6}$$

$$\alpha, \beta = 1, 2, 3,$$

and $x^{(1)} = x$, $x^{(2)} = y$, $x^{(3)} = 1$; in more detail,

$$I_{11} = \int\!\!\int Ex^2 dx\,dy, \quad I_{22} = \int\!\!\int Ey^2 dx\,dy, \quad I_{12} = I_{21} \qquad Exy dx\,dy,$$

$$I_{33} = \int\!\!\int E dx\,dy = S_E, \quad I_{13} = I_{31} = \int\!\!\int Ex dx\,dy = S_E x_0, \quad \text{(146.5}'$$

$$I_{23} = I_{32} = \iint\limits_S Ey dx\,dy = S_E y_0,$$

where $S_E = \sum_j E_j S_j$ is the same as before and x_0, y_0 are the coordinates of the reduced centre of gravity of the end S; I_{11} and I_{22} are reduced moments of inertia of the end S with regard to the axes Oy and Ox, and $I_{12} = I_{21}$ is the reduced product of inertia. Further,

$$
\begin{aligned}
K_{11} &= \iint\limits_S xZ_z^{(1)} dx\,dy, & K_{12} &= \iint\limits_S xZ_z^{(2)} dx\,dy, & K_{13} &= \iint\limits_S xZ_z^{(3)} dx\,dy,\\
K_{21} &= \iint\limits_S yZ_z^{(1)} dx\,dy, & K_{22} &= \iint\limits_S yZ_z^{(2)} dx\,dy, & K_{23} &= \iint\limits_S yZ_z^{(3)} dx\,dy,\\
K_{31} &= \iint\limits_S Z_z^{(1)} dx\,dy, & K_{32} &= \iint\limits_S Z_z^{(2)} dx\,dy, & K_{33} &= \iint\limits_S Z_z^{(3)} dx\,dy;
\end{aligned} \tag{146.6$'$}
$$

it will be assumed that these constants have been calculated.

The problem will be solved, when the unknown constants a_1, a_2, a_3 have been determined from the system (146.4) for given M_y, M_x, F. *The determinant of this system*

$$\Delta = \begin{vmatrix} I_{11} + K_{11} & I_{12} + K_{12} & I_{13} + K_1 \\ I_{21} + K_{21} & I_{22} + K_{22} & I_{23} + K_2 \\ I_{31} + K_{31} & I_{32} + K_{32} & I_{33} + K_3 \end{vmatrix}$$

(as will be proved below) *is always different from zero; more exactly,* $\Delta > 0$. Hence the system (146.4) determines the constants a_1, a_2, a_3 uniquely and the problem may be considered solved.

2°. Before proceeding to the proof of the inequality $\Delta > 0$, certain formulae will be considered which are connected with the expression for the potential energy of deformation and which will be required later on.

It will be recalled that the following expression was introduced in § 20:

$$2W(e) = \lambda(e_{xx} + e_{yy} + e_{zz} \quad + 2\mu(e_{xx}^2 + e_{yy}^2 + e_{zz}^2 + \\ + 2e_{yz}^2 + 2e_{zx}^2 + 2e_{xy}^2) \qquad (146.7)$$

which represents twice the potential energy per unit volume, corresponding to the strain components $e_{xx}, \ldots, e_{xy}$; this deformation will be denoted by (e), and accordingly $W(e)$ has been written above instead of simply W, as was done in § 20.

The expression $W(e)$ represents a positive definite quadratic form of the components of deformation (e) and it only vanishes for $(e) = 0$ (i.e., for $e_{xx} = e_{yy} = e_{zz} = e_{yz} = e_{zx} = e_{xy} = 0$).

It will be remembered that a quadratic form $\Omega(x_1, x_2, \ldots, x_n)$ of the variables $x_1, x_2, \ldots, x_n$ is called *positive definite*, if $\Omega(x_1, x_2, \ldots, x_n) > 0$ for all (real) values of the variables, except when $x_1 = x_2 = \ldots = x_n = 0$. The form is called *positive semidefinite*, if for all values of the variables $\Omega(x_1, x_2, \ldots, x_n) \geqslant 0$, i.e., if there exist some (real) values of $x_1, x_2, \ldots, x_n$, not all zero, so that $\Omega(x_1, x_2, \ldots, x_n) = 0$.

The stress components, corresponding to the deformation (e), are given by the formulae

$$X_x = \lambda\theta + 2\mu e_{xx}, \quad Y_y = \lambda\theta + 2\mu e_{yy}, \quad Z_z = \lambda\theta + 2\mu e_{zz},$$
$$Y_z = 2\mu e_{yz}, \quad Z_x = 2\mu e_{zx}, \quad X_y = 2\mu e_{xy}, \qquad (146.8)$$
$$(\theta = e_{xx} + e_{yy} + e_{zz})$$

and, accordingly, (146.7) may be written

$$2W(e) = X_x e_{xx} + Y_y e_{yy} + Z_z e_{zz} + 2Y_z e_{yz} + 2Z_x e_{zx} + 2X_y e_{xy}. \tag{146.9}$$

Consider now two different deformations (e') and (e'') and indicate the corresponding strains by one or two accents. The following expression, analogous to (146.9), will now be introduced:

$$\begin{aligned} 2W(e', e'') &= X'_x e''_{xx} + Y'_y e''_{yy} + Z'_z e''_{zz} + 2Y'_z e''_{yz} + 2Z'_x e''_{zx} + 2X'_y e''_{xy} = \\ &= X''_x e'_{xx} + Y''_y e'_{yy} + Z''_z e'_{zz} + 2Y''_z e'_{yz} + 2Z''_x e'_{zx} + 2X''_y e'_{xy}. \end{aligned} \tag{146.10}$$

If one interprets the stress components X'_x, etc. and X''_x, etc. as their expressions in terms of the strain components e'_{xx}, etc. and e''_{xx}, etc., then $W(e', e'')$ reduces to a bilinear form in these last components. The equality between the two expressions for $W(e', e'')$, given in (146.10), may be verified directly; this proves that

$$W(e', e'') = W(e'', e'),$$

i.e., that the *bilinear form* $W(e', e'')$ *is symmetrical.*

If the deformations (e') and (e'') are identical, i.e., $(e') = (e'') = (e)$, then

$$W(e, e) = W(e), \tag{146.11}$$

where $W(e)$ is the same as in (146.7) or (146.9).

It has been proved in § 20 that

$$(X_n u + Y_n v + Z_n w) d\Sigma = 2 \qquad W(e) dx\, dy\, dz = 2U, \tag{146.12}$$

where Σ is the surface of the deformed body, n is the outward normal and V is the region, occupied by the body; U denotes the potential energy of strain of the entire body.

The reader will easily prove in an analogous manner the following formulae:

$$\begin{aligned} 2U_{12} &= \iint_{\Sigma} (X'_n u'' + Y'_n v'' + Z'_n w'') d\Sigma = 2 \iiint_{V} W(e', e'') dx\, dy\, dz, \\ 2U_{21} &= \iint (X''_n u' + Y''_n v' + Z''_n w') d\Sigma = 2 \iiint W(e'', e') dx\, dy\, dz. \end{aligned} \tag{146.13}$$

It follows from (146.10) that $U_{12} = U_{21}$, i.e.,

$$\iint\limits_{\Sigma} (X_n' u'' + Y_n' v'' + Z_n' w'')d\Sigma = \iint\limits_{\Sigma} (X_n'' u' + Y_n'' v' + Z_n'' w')d\Sigma, \quad (146.14)$$

which expresses the *Reciprocal Theorem* due to *Betti* (more correctly, the theorem of Betti has a somewhat more general form which also involves body forces).

The preceding formulae will only be applied here to the case of plane deformations of a bar. In this case: $X_z = Y_z = w = 0$, and all the functions under consideration are independent of z. Hence

$$\begin{aligned} W(e', e'') &= X_x' e_{xx}'' + Y_y' e_{yy}'' + 2X_y' e_{xy}'' = X_x'' e_{xx}' + Y_y'' e_{yy}' + 2X_y'' e_{xy}' = \\ &= \lambda(e_{xx}' + e_{yy}')(e_{xx}'' + e_{yy}'') + 2\mu(e_{xx}' e_{xx}'' + e_{yy}' e_{yy}'' + 2e_{xy}' e_{xy}''), \quad (146.15) \end{aligned}$$

$$W(e, e) \quad W(e) = \lambda(e_{xx} + e_{yy})^2 + 2\mu(e_{xx}^2 + e_{yy}^2 + 2e_{xy}^2). \quad (146.16)$$

In the case of plane strain, it is more convenient to apply the formulae (146.12)—(146.14) not to the entire bar, but to a segment of unit length included between two normal transverse cross-sections. Instead of (146.12), one then obviously obtains

$$\int\limits_{L} (X_n u + Y_n v)ds = 2\iint\limits_{S} W(e)dx\,dy = 2U, \quad (146.17)$$

where U is now the potential energy per unit length of the bar, and, instead of (146.13), (146.14),

$$\begin{aligned} 2U_{12} = 2U_{21} = \int (X_n' u'' + Y_n' v'')ds = \int\limits_{L} (X_n'' u' + Y_n'' v')ds = \\ = 2 \quad W(e', e'')dx\,dy. \quad (146.18) \end{aligned}$$

In these formulae S denotes the cross-section of the bar and L its boundary.

In the case where the displacement components, as in the auxiliary problem of plane deformation of § 145, have discontinuities at the dividing lines between the parts S_j, *one has to understand by L the union of the boundaries of these regions*, so that, if L_j is the boundary of S_j, the integral is to be taken along the whole of L_j, and those parts of L_j which are common to the regions S_j, S_l will be covered twice, once in the capacity of

boundary to S_j and a second time in the capacity of boundary to S_l; see (146.21) and (146.22) below.

3°. Consider now the proof of the inequality $\Delta > 0$. Let $(e^{(1)})$, $(e^{(2)})$, $(e^{(3)})$ denote the deformations, corresponding to the auxiliary problems of plane deformation (146.1*a*), (146.2*a*), (146.3*a*), and $U_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ the expressions (146.18) with $(e^{(\alpha)})$, $(e^{(\beta)})$ taking the places of (e'), (e''). It will be proved that

$$K_{\alpha\beta} = 2U_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3, \tag{146.19}$$

where $K_{\alpha\beta}$ are the constants, defined by (146.6); in particular, it will follow from this that $K_{\alpha\beta} = K_{\beta\alpha}$.

For this purpose transform the formula

$$2U_{\alpha\beta} = \int_L [X_n^{(\alpha)} u^{(\beta)} + Y_n^{(\alpha)} v^{(\beta)}] ds \tag{146.20}$$

in the following manner. As stated earlier, L denotes here the union of all contours bounding the parts S_j of the region S. Hence

$$2U_{\alpha\beta} = \sum_j \int_{L_j} [X_n^{(\alpha)} u_j^{(\beta)} + Y_n^{(\alpha)} v_j^{(\beta)}] ds, \tag{146.21}$$

where L_j is the boundary of S_j and $u_j^{(\beta)}$, $v_j^{(\beta)}$ are the boundary values of $u^{(\beta)}$, $v^{(\beta)}$ on L_j from the direction of the region S_j; n denotes the normal to L_j which is outward with respect to S_j.

Noting now that, by supposition, $X_n = Y_n = 0$ on the boundary of S and that during the integration the dividing line L_{kl} between S_k, S_l is covered twice, it is easily concluded that (146.21) may be rewritten

$$2U_{\alpha\beta} = \sum_{k,l} \int_{L_{kl}} \{X_\nu^{(\alpha)}(u_k^{(\beta)} - u_l^{(\beta)}) + Y_\nu^{(\alpha)}(v_k^{(\beta)} - v_l^{(\beta)})\} ds, \tag{146.22}$$

where now the lines L_{kl} are only covered once and ν is the normal, directed from S_k into S_l.

The truth of (146.19) is easily proved by use of (146.22). Consider, for example, the relation $K_{12} = K_{21} = 2U_{12}$. By (146.22),

$$2U_{12} = \sum_{k,l} \int_{L_{kl}} \{X_\nu^{(1)}(u_k^{(2)} - u_l^{(2)}) + Y_\nu^{(1)}(v_k^{(2)} - v_l^{(2)})\} ds.$$

Noting that by (146.2*a*)

$$u^{(2)} - u_l^{(2)} = (\sigma_k - \sigma_l)xy, \quad v_k^{(2)} - v_l^{(2)} = \tfrac{1}{2}(\sigma_k - \sigma_l)(y^2 - x^2)$$

and substituting these expressions in the preceding formula, one obtains

$$2U_{12} = \sum_{k,l} (\sigma_k - \sigma_l) \int_{L_{kl}} \{X_\nu^{(1)} xy + \tfrac{1}{2} Y_\nu^{(1)} (y^2 - x^2)\} ds.$$

Applying to this expression the same transformation which was used to deduce (146.22) from (146.21), but in the opposite direction, one finds

$$2U_{12} = \sum_j \sigma_j \int_{L_j} [X_n^{(1)} xy + \tfrac{1}{2} Y_n^{(1)} (y^2 - x^2)] ds,$$

where L_j and n are the same as in (146.21).

Further, noting that

$$X_n^{(1)} = X_x^{(1)} \cos (n, x) + X_y^{(1)} \cos (n, y),$$
$$Y_n^{(1)} = Y_x^{(1)} \cos (n, x) + Y_y^{(1)} \cos (n, y),$$

and transforming by use of Green's formula, one obtains

$$2U_{12} = \sum_j \sigma_j \iint_{S_j} y(X_x^{(1)} + Y_y^{(1)}) dx\, dy,$$

since

$$\frac{\partial X_x^{(1)}}{\partial x} + \frac{\partial X_y^{(1)}}{\partial y} = 0, \qquad \frac{\partial Y_x^{(1)}}{\partial x} + \frac{\partial Y_y^{(1)}}{\partial y} = 0;$$

hence, by (146.1b),

$$2U_{12} = \iint y Z_z^{(1)} dx\, dy = K_{21}.$$

In exactly the same manner, applying the formula

$$2U_{12} = \int_L [X_n^{(2)} u^{(1)} + Y_n^{(2)} v^{(1)}] ds$$

for the calculation of $2U_{12} = 2U_{21}$, one finds $2U_{12} = K_{12}$. Hence $2U_{12} = K_{12} = K_{21}$, as was to be proved.

The remaining formulae (146.19) may be proved in quite an analogous manner; this will be left to the reader.

On the basis of (146.19), the determinant Δ may now be considered as

the discriminant of the following quadratic form in terms of a_1, a_2, a_3:

$$2\Omega(a_1, a_2, a_3) = 2G_0(a_1, a_2, a_3) + 2G(a_1, a_2, a_3), \tag{146.23}$$

where

$$2G_0(a_1, a_2, a_3) = \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} I_{\alpha\beta}\, a_\alpha\, a_\beta, \tag{146.24}$$

$$2G(a_1, a_2, a_3) = \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} K_{\alpha\beta} a_\alpha b_\beta = 2 \sum_{\alpha=1}^{3} \sum_{\beta=1}^{3} U_{\alpha\beta} a_\alpha a_\beta. \tag{146.25}$$

It is easily seen that the quadratic form G_0 is positive definite, i.e., that $G_0(a_1, a_2, a_3) > 0$, unless $a_1 = a_2 = a_3 = 0$. In fact, it follows from the very definition of $I_{\alpha\beta}$ that

$$2G_0(a_1, a_2, a_3) = \iint E(a_1 x + a_2 y + a_3)^2 dx\, dy,$$

which proves the statement.

It should be pointed out that it is readily verified by the help of (146.12) that $G_0(a_1, a_2, a_3)$ itself represents the potential strain energy per unit length of the bar, arising from the superposition of the deformations corresponding to the solutions (146.1)—(146.3) after they have been multiplied by a_1, a_2, a_3 respectively (where it has been assumed that the component parts of the bar deform independently of each other, i.e., that they are not welded together).

It is likewise easily proved that the quadratic form $G(a_1, a_2, a_3)$ is positive definite, unless all the Poisson's ratios have the same value. (If this is the case, then obviously all $K_{\alpha\beta}$ vanish and the form $G(a_1, a_2, a_3)$ is identically zero.)

In fact, it may be shown that $G(a_1, a_2, a_3)$ represents the potential energy of deformation per unit length of the bar, arising from superposition of the solutions of the auxiliary problems (146.1a), (146.2a), (146.3a) after multiplying them by a_1, a_2, a_3 respectively.

Indeed, let, as before, $(e^{(1)})$, $(e^{(2)})$, $(e^{(3)})$ be the (plane) deformations corresponding to the problems (146.1a), (146.2a), (146.3a) and let (e) denote the deformation

$$(e) = a_1(e^{(1)}) + a_2(e^{(2)}) + a_3(e^{(3)}),$$

i.e., the deformation with strain components

$$e_{xx} = a_1 e_{xx}^{(1)} + a_2 e_{xx}^{(2)} + a_3 e_{xx}^{(3)}, \ldots, e_{xy} = a_1 e_{xy}^{(1)} + a_2 e_{xy}^{(2)} + a_3 e_{xy}^{(3)}.$$

By (146.17), the strain energy per unit length of the bar is given by

$$2U = \iint_S W(e)dx\,dy,$$

where $W(e)$ is defined by (146.16). However, it is readily seen that

$$W(e) = a_1^2 W(e^{(1)}) + a_2^2 W(e^{(2)}) + a_3^2 W(e^{(3)}) + 2a_2 a_3 W(e^{(2)}, e^{(3)}) + \\ + 2a_3 a_1 W(e^{(3)}, e^{(1)}) + 2a_1 a_2 W(e^{(1)}, e^{(2)}),$$

and hence, taking into consideration the definition of $U_{\alpha\beta}$, $2U = = 2G(a_1, a_2, a_3)$, as was to be proved.

If not all the Poisson's ratios have the same value and if at least one of the quantities a_1, a_2, a_3 is different from zero, deformation necessarily takes place and therefore $U > 0$. Thus the original proposition has been proved.

If there is no deformation, one obviously has on the dividing lines L_{jk} between the parts S_j, S_k

$$u_k = -\varepsilon_{jk} y + \alpha_{jk}, \qquad v_k = \varepsilon_{jk} x + \beta_{jk},$$

where ε_{jk}, α_{jk}, β_{jk} are constants; on the other hand, one must have on these lines, by (146.1a)—(146.3a),

$$u_j - u_k = (\sigma_j - \sigma_k)\,[\tfrac{1}{2}a_1(x^2 - y^2) + a_2 xy + a_3 x],$$
$$v_j - v_k = (\sigma_j - \sigma_k)\,[a_1 xy + \tfrac{1}{2}a_2(y^2 - x^2) + a_3 y];$$

it is easily verified by comparison of these expressions that, if $\sigma_j \neq \sigma_k$, one must have $a_1 = a_2 = a_3 = 0$, $\varepsilon_{jk} = \alpha_{jk} = \beta_{jk} = 0$.

The form $\Omega(a_1, a_2, a_3)$, being the sum of the two positive forms G_0, G of which the former is certainly positive definite, must also be positive definite. However, it is known that the discriminant of such a form is certainly positive; therefore the assertion made at the end of subsection 1° with regard to Δ is proved.

NOTE 1. The fact that $\Omega(a_1, a_2, a_3)$ is positive definite could have been proved more simply without splitting it up into the forms G_0 and G. Such a proof may be carried out, based on the fact that $\Omega = G_0 + G$ is the potential strain energy, corresponding to the earlier stated combination of the solutions (146.1)–(146.3) and (146.1a)–(146.3a); this statement is easily proved directly.

However, a different procedure has been followed here, because it was desired to calculate the additional coefficients $K_{\alpha\beta}$ which characterize

the influence of the different Poisson's ratios of the component materials.

NOTE 2. Generally speaking, the coefficients $K_{\alpha\beta}$ are very small, if the Poisson's ratios of the different materials do not differ much from each other; in other words, they are of the same order as the squares and products of the differences $\sigma_j - \sigma_k$. In fact, denoting temporarily by σ_{jk} the differences $\sigma_j - \sigma_k$ which occur on the right-hand sides of (146.1a)—(146.3*a*) and considering σ_{jk} as variables, it is readily verified that the solutions of the auxiliary problems (146.1*a*)—(146.3*a*) depend linearly on σ_{jk}. Further, by (146.22) and due to the fact that $X_\nu^{(\alpha)}$, $Y_\nu^{(\alpha)}$ depend linearly on σ_{jk}, it is seen that $K_{\alpha\beta}$ depends linearly on the squares and products of σ_{jk}, as was to be proved.

§ 147. Particular cases. 1°. Extension of a bar, having an axis of symmetry. It will be assumed that the axis Oz is an axis of symmetry of the bar, where the symmetry refers to geometrical as well as elastic properties.

In that case the origin O is obviously the reduced centre of gravity of the "left" end. Directing the axes Ox, Oy along the reduced axes of inertia of this end, one has $I_{12} = 0$. Further, on the basis of the symmetry and of the form of the functions g_3, h_3 in the formulae (146.3*a*), it is easily seen that the solution of the corresponding auxiliary problem will likewise be symmetrical about O and, in particular, that

$$Z_z^{(3)}(-x, -y) = Z_z^{(3)}(x, y).$$

It follows from this that

$$K_{31} = K_{13} = \iint xZ_z^{(3)}dx\,dy = 0, \quad K_{32} = K_{23} = \iint yZ_z^{(3)}dx\,dy = 0.$$

Hence the equations (146.4) have the form (remembering that $x_0 = y_0 = 0$)

$$\begin{aligned} -M_y &= (I_{11} + K_{11})a_1 + K_{12}a_2, \\ M_x &= K_{21}a_1 + (I_{22} + K_{22})a_2, \\ F &= (S_E + K_{33})a_3. \end{aligned}$$

If it is proposed to solve the problem of extension by a force of magnitude F, directed along the axis of symmetry Oz, one must put in these

equations $M_x = M_y = 0$, i.e., $a_1 = a_2 = 0$, and

$$a_3 = \frac{F}{S_E + K_{33}}. \tag{147.1}$$

If all the Poisson ratios of the different materials, constituting the bar, have the same value, then $K_{33} = 0$, and one obtains the result, deduced earlier. If not all the σ's are equal, then necessarily $K_{33} > 0$.

In the last case, the quadratic form $2G(a_1, a_2, a_3)$ is positive definite and therefore all the coefficients K_{11}, K_{22}, K_{33} are positive; this follows from the fact that $K_{11} = 2G(1, 0, 0)$, etc.

Since a_3 represents the relative lengthening of the bar as the result of F, $S_E + K_{33}$ is the rigidity of extension and the preceding formula shows that *the difference of Poisson's ratios (for constant S_E) increases the rigidity of extension*, independently *of the sign of the difference $\sigma_j - \sigma_k$*.

2°. B a r w i t h p l a n e o f s y m m e t r y, b e n t b y a c o u p l e. Let Oxz be the plane of symmetry of the bar (as regards its geometry as well as its elastic properties). It may then be assumed that O coincides with the reduced centre of gravity of the "left" end; the axes Ox, Oy are again to be principal reduced axes of inertia of this end with regard to O.

For this choice of axes, one has in (146.4): $I_{13} = I_{23} = I_{12} = 0$. Further, on the basis of the symmetry and of the form of the functions g_1, h_1 in (146.1a), it is easily concluded that the solution of the corresponding problem of plane deformation is likewise symmetrical with regard to Ox; in particular, $Z_z^{(1)}(x, -y) = Z_z^{(1)}(x, y)$. Similarly, one finds $Z_z^{(2)}(x, -y) = - Z_z^{(2)}(x, y)$. Hence

$$K_{12} = K_{21} = \iint_S y Z_z^{(1)} dx\, dy = 0, \quad K_{23} = K_{32} = \iint_S Z_z^{(2)} dx\, dy = 0$$

and the equations (146.4) take the form

$$\begin{aligned} -M_y &= (I_{11} + K_{11})a_1 + K_{13}a_3, \\ M_x &= (I_{22} + K_{22})a_2, \\ F &= K_{31}a_1 + (S_E + K_{33})a_3. \end{aligned} \tag{147.2}$$

If it is desired to solve *the problem of bending by a couple whose plane is perpendicular to the plane of symmetry*, one has $M_y = 0$, $F = 0$;

hence $a_1 = a_3 = 0$ and

$$a_2 = \frac{M_x}{I_{22} + K_{22}} \tag{147.3}$$

where $K_{22} > 0$, unless all the Poisson's ratios have the same value.

However, if it is proposed *to solve the problem of bending by a couple whose plane is parallel to the plane of symmetry*, one has $M_x = 0$, $F = 0$; hence

$$a_2 = 0, \quad a_3 = -\frac{a_1 K_{31}}{S_E + K_{33}}, \quad a_1 = \frac{M_y}{I_{11} + K}, \tag{147.4}$$

where

$$K = K_{11} - \frac{K_{13}^2}{S_E + K_{33}} = \frac{S_E K_{11} + K_{11}K_{13} - K_{13}^2}{S_E + K_{33}}. \tag{147.5}$$

If not all σ_k have the same value, then $K > 0$, because $K_{11}K_{33} - K_{13}^2 > 0$, as this is the discriminant of the positive definite quadratic form in the variables a_1, a_3

$$2G(a_1, 0, a_3) = K_{11}a_1^2 + 2K_{13}a_1a_3 + K_{33}a_3^2.$$

It is easily seen that in both the above cases the Bernoulli-Euler law holds true and that in the first case the flexural rigidity is equal to

$$I_{22} + K_{22}, \tag{147.6}$$

while in the second case it is given by

$$I_{11} + K; \tag{147.7}$$

it must not be forgotten that I_{22} and I_{11} are now reduced moments of inertia about the axes Ox, Oy.

It is seen that in both cases *the difference of the Poisson's ratios increases the flexural rigidity (for constant I_{11} and I_{22}), independently of the sign of $\sigma_j - \sigma_k$.*

Some simple examples will be presented in § 149.

§ 148. Principal axis of extension and principal planes of bending. The equations (146.4) may be considerably simplified, if the arbitrary systém of coordinate axes Oxy in the plane of the "left" ("lower") end is replaced by another system $O'x'y'$ in the same plane, where the new axis $O'z'$ is given the same direction as the old axis Oz. In fact, as will be seen below, this new coordinate system may be chosen

in such a way that on the right-hand sides of the equations (146.4) all but the coefficients on the main diagonal vanish.

Let $K'_{\alpha\beta}$ denote the constants for the system $O'x'y'$, corresponding to the constants $K_{\alpha\beta}$ for the system Oxy. The relations expressing $K'_{\alpha\beta}$ in terms of $K_{\alpha\beta}$ are easily found; this will be left to the reader (cf. Note at the end of this section), and so, in what follows, only those relations will be deduced which are required in the later reasoning.

For greater clarity, the transit to the new axes will be carried out in two steps, producing a translation of the origin and a rotation of the coordinate axes.

Let the new system $O'x'y'$ only differ from the old system Oxy by the position of the origin and let a, b be the coordinates of the new origin in the old system, so that

$$x' = x - a, \quad y' = y - b.$$

It is easily seen that in the present case $K'_{33} = K_{33}$. In fact, in the auxiliary problem corresponding to (146.3a), but in the new coordinates $O'x'y'$, one will have for the discontinuities in the displacements on the dividing lines

$$u_j - u_k = (\sigma_j - \sigma_k)x' = (\sigma_j - \sigma_k)(x - a) = (\sigma_j - \sigma_k)x + \text{const.},$$
$$v_j - v_k = (\sigma_j - \sigma_k)y' = (\sigma_j - \sigma_k)(y - b) = (\sigma_j - \sigma_k)y + \text{const.},$$

and clearly the solution of this problem leads to the same stress distribution as the solution of the problem for the following discontinuities:

$$u_j - u_k = (\sigma_j - \sigma_k)x, \quad v_j - v_k = (\sigma_j - \sigma_k)y,$$

because the constants in the previous formulae may be removed by rigid translations of some of the parts, constituting the bar. Thus, in particular, the stress component $Z_z^{(3)}$ will be the same for these auxiliary problems in the systems Oxy and $O'x'y'$. This means that

$$K_{33} \quad \iint Z_z^{(3)} dx\, dy$$

remains unchanged for the transit to the new system.

Next, the constants $K'_{13} = K'_{31}$ and $K'_{23} = K'_{32}$ will be calculated. Using (146.6′) and the above result regarding $Z_z^{(3)}$, one has

$$K'_{13} = \iint x' Z_z^{(3)} dx'\, dy' = \iint (x - a) Z_z^{(3)} dx\, dy,$$

whence

$$K'_{13} = K_{13} - aK, \tag{148.1}$$

similarly

$$K'_{23} = K_{23} - bK_3 \tag{148.2}$$

Denoting by $I'_{\alpha\beta}$ the quantities, defined for $O'x'y'$ in the same way as $I_{\alpha\beta}$ was defined for Oxy, the coordinates a, b will be chosen such that

$$I'_{13} + K'_{13} = S_E x'_0 + K'_{13} = 0, \quad I'_{23} + K'_{23} = S_E y'_0 + K'_{23} = 0,$$

or, since $x'_0 = x_0 - a$, $y'_0 = y_0 - b$, one obtains by the preceding formulae

$$a = \frac{S_E x_0 + K_{13}}{S_E + K_{33}}, \quad b = \frac{S_E y_0 + K_{23}}{S_E + K_{33}}. \tag{148.3}$$

With these values of a and b, the formulae, corresponding to (146.4), but for the new system of axes, acquire the simpler form

$$\begin{aligned} -M_y &= (I_{11} + K_{11})a_1 + (I_{12} + K_{12})a_2, \\ M_x &= (I_{21} + K_{21})a_1 + (I_{22} + K_{22})a_2, \\ F &= (S_E + K_{33})a_3, \end{aligned} \tag{148.4}$$

where, for simplicity, accents have been omitted, i.e., M_y, M_x, $I_{\alpha\beta}$, $K_{\alpha\beta}$ have been written instead of $M_{y'}$, $M_{x'}$, $I'_{\alpha\beta}$, $K'_{\alpha\beta}$. Accordingly, the new system of axes $O'x'y'$ will now again be denoted by Oxy.

The new axis Oz will be called *principal axis of extension* (*compression*).

The principal axis of extension may also be determined in the following manner. In (146.4), let $a_1 = a_2 = 0$, $a_3 \neq 0$. Then

$$M_y = -(S_E x_0 + K_{13})a_3, \quad M_x = (S_E y_0 + K_{23})a_3, \quad F = (S_E + K_{33})a_3.$$

Thus, in the present case, the forces applied to the "right" end are statically equivalent to a tensile force of magnitude $F \neq 0$, directed along the axis Oz, and to a couple with moment perpendicular to the line of action of the force. However, such a system of forces is known to be statically equivalent to a force of the same direction and magnitude. The line of action of this last force is easily found and it is the principal axis of extension, defined above.

This term is justified by the fact that, if tensile forces of magnitude F be applied to the ends of the bar for which the line of action is the principal axis of extension, the solution of the problem will be obtained by putting $a_1 = a_2 = 0$,

$$a_3 = \frac{F}{S_E + K_{33}},$$

so that the extension will not be accompanied by bending.

The preceding formulae show that the rigidity of extension is equal to

$$S_E + K_{33}. \tag{148.5}$$

Since for different Poisson's ratios $K_{33} > 0$, it is seen that the difference of Poisson's ratios (for constant S_E) *increases the rigidity of extension, independently of the sign of* $\sigma_j - \sigma_k$, a circumstance which had been observed above for a case with axes of symmetry.

The formulae (148.4) may still be further simplified by means of a rotation of the axes Oxy in their plane.

If the new system of axes $O'x'y'$ is obtained from the old system by rotation through an angle α, then

$$x = x' \cos\alpha - y' \sin\alpha, \quad y = x' \sin\alpha + y' \cos\alpha, \tag{148.6}$$

$$x' = x \cos\alpha + y \sin\alpha, \quad y' = -x \sin\alpha + y \cos\alpha. \tag{148.7}$$

The quantities K'_{11}, $K'_{12} = K'_{21}$, K'_{22} in the new system will now be expressed in terms of K_{11}, $K_{12} = K_{21}$, K_{22}. For this purpose, the auxiliary problems corresponding to (146.1a) and (146.2a) will be compared with the auxiliary problems in the new system.

The discontinuities of the displacements for the above-mentioned problems in the old system are given by

$$u_j - u_k = \tfrac{1}{2}(\sigma_j - \sigma_k)(x^2 - y^2), \quad v_j - v_k = (\sigma_j - \sigma_k)xy \tag{I}$$

for the problem (146.1a),

$$u_j - u_k = (\sigma_j - \sigma_k)xy, \quad v_j - v_k = (\sigma_j - \sigma_k)(y^2 - x^2) \tag{II}$$

for the problem (146.2a). Correspondingly, one has for the new system

$$u'_j - u'_k = \tfrac{1}{2}(\sigma_j - \sigma_k)(x'^2 - y'^2), \quad v'_j - v'_k = (\sigma_j - \sigma_k)x'y' \tag{I'}$$

and

$$u'_j - u'_k = \tfrac{1}{2}(\sigma_j - \sigma_k)x'y', \quad v'_j - v'_k = (\sigma_j - \sigma_k)(y'^2 - x'^2). \tag{II'}$$

In order to compare these problems, the boundary conditions (I′) and (II′) will be expressed in terms of the old coordinates x, y. In fact, by (148.7), one obtains instead of (I′)

$$\begin{aligned} u'_j - u'_k &= \tfrac{1}{2}(\sigma_k - \sigma_j)(x^2 - y^2)\cos 2\alpha + (\sigma_j - \sigma_k)xy \sin 2\alpha, \\ v'_j - v'_k &= (\sigma_j - \sigma_k)xy \cos 2\alpha - \tfrac{1}{2}(\sigma_j - \sigma_k)(x^2 - y^2)\sin 2\alpha. \end{aligned}$$

Introducing now instead of $u'_j - u'_k$, $v'_j - v'_k$ the quantities $u_j - u_k$, $v_j - v_k$, expressing the same discontinuities in terms of the old coordinates,

and taking into consideration that

$$u_j - u_k = (u_j' - u_k') \cos \alpha - (v_j' - v_k') \sin \alpha,$$
$$v_j - v_k = (u_j' - u_k') \sin \alpha + (v_j' - v_k') \cos \alpha,$$

one obtains

$$\begin{aligned} u_j - u_k &= \tfrac{1}{2}(\sigma_j - \sigma_k)(x^2 - y^2) \cos \alpha + (\sigma_j - \sigma_k)xy \sin \alpha, \\ v_j - v_k &= (\sigma_j - \sigma_k)xy \cos \alpha + \tfrac{1}{2}(\sigma_j - \sigma_k)(y^2 - x^2) \sin \alpha. \end{aligned} \tag{148.8}$$

The deduction of (148.8) may be simplified by considering, instead of x, y and u, v, the variables $z = x + iy$ and $u + iv$.

Thus it is seen that the solution of the problem, corresponding to (I′), may be obtained by adding the solutions of the problems, corresponding to (I) and (II), which must be multiplied beforehand by $\cos \alpha$ and $\sin \alpha$ respectively. Hence, if $Z_z'^{(1)}$ denotes the component Z_z, corresponding to the problem (I′), and $Z_z^{(1)}$, $Z_z^{(2)}$, as before, the stress components Z_z, corresponding to the problems (I), (II), then

$$Z_z'^{(1)} = Z_z^{(1)} \cos \alpha + Z_z^{(2)} \sin \alpha. \tag{148.9}$$

Similarly, one obtains for the problem (II′)

$$Z_z'^{(2)} = - Z_z^{(1)} \sin \alpha + Z_z^{(2)} \cos \alpha. \tag{148.10}$$

Using (148.9) and (148.10), the quantities K_{11}', K_{12}', K_{22}' are easily expressed in terms of K_{11}, K_{12}, K_{22}. For example,

$$\begin{aligned} K_{12}' = K_{21}' &= \iint_S y' Z_z'^{(1)} dx'\, dy' \\ &= \iint_S (- x \sin \alpha + y \cos \alpha)(Z_z^{(1)} \cos \alpha + Z_z^{(2)} \sin \alpha) dx\, dy, \end{aligned}$$

whence

$$K_{12}' = K_{12} \cos 2\alpha - \tfrac{1}{2}(K_{11} - K_{22}) \sin 2\alpha. \tag{148.11}$$

The expressions for K_{11}' and K_{22}' may be deduced by the reader (cf. Note at the end of this section).

Now the expression will be deduced for the reduced product of inertia I_{12}' in the new system. One has

$$I_{12}' = \iint E x' y' dx'\, dy' = \iint E(x \cos \alpha + y \sin \alpha)(- x \sin \alpha + y \cos \alpha) dx\, dy,$$

whence

$$I'_{12} = I_{12}\cos 2\alpha - \tfrac{1}{2}(I_{11} - I_{22})\sin 2\alpha; \tag{148.12}$$

the complete analogy with (148.11) is obvious (see the Note at the end of this section).

The angle α will now be chosen in such a manner that

$$I'_{12} + K'_{12} = I'_{21} + K'_{21} = 0. \tag{148.13}$$

By (148.11) and (148.12), one obtains

$$(I_{12} + K_{12})\cos 2\alpha - \tfrac{1}{2}(I_{11} + K_{11} - I_{22} - K_{22})\sin 2\alpha = 0,$$

whence

$$\tan 2\alpha = \frac{2(I_{12} + K_{12})}{I_{11} + K_{11} - I_{22} - K_{22}}. \tag{148.14}$$

Giving α one of the values, satisfying this condition (the other values differing by integral multiples of a right angle), one arrives at a system of axes $Ox'y'$ for which (148.4) assumes the very simple form, mentioned at the beginning of the present section,

$$-M_{y'} = (I'_{11} + K'_{11})a_1, \quad M_{x'} = (I'_{22} + K'_{22})a_2, \quad F = (S_E + K_{33})a_3,$$

because, as is readily seen, one has in the new system $I'_{13} + K'_{13} = S_E x'_0 + K'_{13} = 0$, $I'_{23} + K'_{23} = S_E y'_0 + K'_{23} = 0$, and, in addition, $K'_{33} = K_{33}$.

The planes $Ox'z$ and $Oy'z$ will be called *principal planes of bending.*

It is seen that, if Oz is the principal axis of extension and if $Ox'z$, $Oy'z$ are the principal planes of bending, the problems of extension by forces with the line of action Oz and of bending by couples with planes parallel to $Ox'z$, $Oy'z$ may be solved independently of each other.

Omitting the accents, the last equations may be rewritten as

$$-M_y = (I_{11} + K_{11})a_1, \quad M_x = (I_{22} + K_{22})a_2, \quad F = (S_E + K_{33})a_3. \tag{148.15}$$

It is seen that the law of Bernoulli-Euler is valid for bending by couples with the planes Oxz, Oyz and that the respective flexural rigidities are given by

$$I_{11} + K_{11}, \quad I_{22} + K_{22}; \tag{148.16}$$

the rigidity of extension is again equal to

$$S_E + K_{33}.$$

NOTE. It will be left to the reader to verify that for transition from one system of axes Oxy to another $O'x'y'$ *the quantities* $K_{\alpha\beta}$ *are transformed in accordance with the same formulae as the quantities* $I_{\alpha\beta}$.

Instead of a simple verification, this property may be deduced by investigating the general expression for the strain energy of the deformed bar.

For example, for a translation of the origin O to a new position $O'(a, b)$

$$I_{11} = \iint\limits_{S} Ex'^2dx'\,dy' = \iint\limits_{S'} E(x-a)^2dx\,dy = I_{11} - 2aI_{13} + a^2I_{33};$$

in correspondence with this one has

$$K'_{11} = K_{11} - 2aK_{13} + a^2K_{33}.$$

It follows from this result that, from the point of view of simplifying the notation, it would have been expedient not to consider the quantities $K_{\alpha\beta}$, $I_{\alpha\beta}$separately, but to consider their sums $I^*_{\alpha\beta} = I_{\alpha\beta} + K_{\alpha\beta}$ which alone occur in (146.4). This has not been done (cf. § 146, Note 1), because it was desired to distinguish clearly the terms $K_{\alpha\beta}$ which only occur in the case where $\sigma_j - \sigma_k \neq 0$.

§ 149. Application of complex representation. Examples. In order to find the solutions of the auxiliary problems of plane deformation, it is convenient, as in many other cases, to use functions of the complex variable

$$z = x + iy.$$

The general solution of the equations of plane elasticity for a homogeneous isotropic body (§ 32) will now be written in the form

$$u + iv = \alpha\varphi(z) - \beta z\overline{\varphi'(z)} - \beta\overline{\psi(z)}, \tag{149.1}$$

$$X_x + Y_y \quad \Re\varphi'(z), \quad Y_y - X_x + 2iX_y = 2[\bar{z}\varphi''(z) + \psi'(z)], \tag{149.2}$$

where $\varphi(z)$, $\psi(z)$ are analytic functions of z in the region under consideration and

$$\alpha = \frac{\varkappa}{2\mu} = \frac{\lambda + 3\mu}{2\mu(\lambda+\mu)} = \frac{(3-4\sigma)(1+\sigma)}{E}, \quad \beta = \frac{1}{2\mu} = \frac{1+\sigma}{E}. \tag{149.3}$$

In the case of the auxiliary problem of § 145, the constants α, β have different values α_j, β_j in the regions S_j, constituting the cross-section

of the bar, and the functions $\varphi(z)$, $\psi(z)$ are holomorphic in each of these regions (the multi-valued terms in the functions φ, ψ drop out in the present case, because the resultant vectors of the forces, applied to the boundaries of the regions S_j, are all equal to zero).

It will be recalled that the components X_n, Y_n of the stress vector, applied to the element ds of any contour from the positive direction of the normal n, are given by

$$(X_n + iY_n)ds = -id[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}], \tag{149.4}$$

where it has been assumed that the positive directions of the normal n and of the element ds are orientated with respect to each other as the axes Ox, Oy.

In correspondence with this, the condition (145.1) may now be written

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = \text{const.} \tag{149.5}$$

on the boundary of the region S, while (145.2) becomes

$$[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}]_j = [\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}]_k + \text{const.} \tag{149.6}$$

on the dividing lines between the parts S_j, S_k.

Further, the condition (145.3) takes the form

$$[\alpha\varphi(z) - \beta z\varphi'(z) - \beta\psi(z)]_j - [\alpha\varphi(z) - \beta z\varphi'(z) - \beta\psi(z)]_k = f \tag{149.7}$$

on the dividing lines between S_j, S_k, where f denotes functions given on these lines. In the cases (146.1*a*), (146.2*a*), (146.3*a*) respectively one will have

$$f = g_1 + ih_1 = \tfrac{1}{2}(\sigma_j - \sigma_k)z^2, \tag{149.8$_1$}$$

$$f = g_2 + ih_2 = -\frac{i}{2}(\sigma_j - \sigma_k)z^2, \tag{149.8$_2$}$$

$$f = g_3 + ih_3 = (\sigma_j - \sigma_k)z. \tag{149.8$_3$}$$

As an example, the case will be considered where the free surface is a circular cylinder and the dividing surface between the two materials is likewise a circular cylinder with the same axis. Let the region S_1 be bounded by a circle with radius R_1 and the region S_2 by the same circle and a circle with radius R_2; the origin O will be placed at the centre of these circles.

As a consequence of the symmetry, it is obvious that the axis Oz will be the principal axis of extension and that the planes Oxz, Oyz will be principal planes of bending.

In the present case, the solutions of the auxiliary problems are easily found by expanding the functions φ and ψ in the regions S_1 and S_2 in positive and in negative and positive powers of z respectively. Substituting in (149.5), (149.6) and (149.7), the coefficients are immediately determined; any arbitrary constants which may occur do not influence the stress distribution (because of the uniqueness of the solutions of the problems).

However, the case to be considered here is somewhat simpler, as the form of the solutions may be guessed immediately and, instead of infinite series, only a few terms need be retained (see later).

1°. The *problem of extension* will be solved first. It is easily guessed that it will be sufficient to write in this case

$$\varphi_1(z) = A_1 z, \quad \psi_1(z) = 0 \quad \text{in } S_1,$$
$$\varphi_2(z) = A_2 z, \quad \psi_2(z) = \frac{B_2}{z} \quad \text{in } S_2,$$

where A_1, A_2, B_2 are real constants and the subscripts 1 and 2 with the functions φ and ψ indicate the relationship of the functions to the regions S_1, S_2.

For $f = (\sigma_1 - \sigma_2)z$, the conditions (149.5), (149.6) and (149.7) respectively give, omitting arbitrary constants,

$$2A_2 z + B_2 \bar{z}^{-1} = 0 \qquad \text{for } |z| = R_2,$$
$$2A_1 z = 2A_2 z + B_2 \bar{z}^{-1} \quad \text{for } |z| = R_1, \tag{149.9}$$
$$(\alpha_1 - \beta_1)A_1 z = (\alpha_2 - \beta_2)A_2 z - \beta_2 B_2 \bar{z}^{-1} + (\sigma_1 - \sigma_2)z \quad \text{for } |z| = R_1.$$

Further, for $z = re^{i\vartheta}$, $\bar{z} = re^{-i\vartheta}$, one obtains, after division by $e^{i\vartheta}$,

$$2A_2 R_2 + \frac{B_2}{R_2} = 0, \quad 2A_1 R_1 = 2A_2 R_1 + \frac{B_2}{R_1},$$
$$(\alpha_1 - \beta_1)A_1 R_1 = (\alpha_2 - \beta_2)A_2 R_1 - \frac{\beta_2 R_2}{R_1} + (\sigma_1 - \sigma_2)R_1.$$

Hence

$$A_1 = \frac{(\sigma_1 - \sigma_2)(R_2^2 - R_1^2)}{(\alpha_1 - \beta_1)(R_2^2 - R_1^2) + (\alpha_2 - \beta_2)R_1^2 + 2\beta_2 R_2^2},$$
$$A_2 = \frac{(\sigma_1 - \sigma_2)R_1^2}{(\alpha_1 - \beta_1)(R_2^2 - R_1^2) + (\alpha_2 - \beta_2)R_1^2 + 2\beta_2 R_2^2}, \tag{149.10}$$
$$A_2 = \frac{2(\sigma_1 - \sigma_2)R_1^2 R_2^2}{(\sigma_1 - \beta_1)(R_2^2 - R_1^2) + (\alpha_2 - \beta_2)R_1^2 + 2\beta_2 R_2^2};$$

the quantities α_1, β_1, α_2, β_2 are given by (149.3), where E and σ must be given the corresponding subscripts; in fact,

$$\alpha_1 - \beta_1 = \frac{2(1+\sigma_1)(1-2\sigma_1)}{E_1}, \quad \alpha_2 - \beta_2 = \frac{2(1+\sigma_2)(1-2\sigma_2)}{E_2},$$

$$\beta_2 = \frac{1+\sigma_2}{E_2}.$$

Since always $\sigma_j < \frac{1}{2}$, these expressions are all positive.

Superimposing the above solution of the auxiliary problem, after multiplication by a_3, on the solution (146.3) which must also be multiplied by a_3, one obtains the solution of the original problem, provided a_3 is given the value

$$a_3 = \frac{F}{S_E + K_{33}}, \tag{149.11}$$

where F is the magnitude of the tensile force,

$$S_E = S_1 E_1 + S_2 E_2 = \pi[R_1^2 E_1 + (R_2^2 - R_1^2)E_2], \tag{149.12}$$

$$K_{33} \qquad Z_z^{(3)} dx\, dy$$

and

$$Z_z^{(3)} = \sigma_j(X_x^{(3)} + Y_y^{(3)}) \text{ in } S_j \ (j = 1,2) \text{ in the notation of § 146.}$$

In the present case

$$\sigma_j(X_x^{(3)} + Y_y^{(3)}) = 4\sigma_j \Re\varphi_j'(z) = 4\sigma_j A_j \text{ in } S_j \quad (j = 1, 2).$$

Hence

$$K_{33} = 4(S_1\sigma_1 A_1 + S_2\sigma_2 A_2) =$$
$$= \frac{4\pi(\sigma_1 - \sigma_2)^2 (R_2^2 - R_1^2) R_1^2}{(\alpha_1 - \beta_1)(R_2^2 - R_1^2) + (\alpha_2 - \beta_2) R_1^2 + 2\beta_2 R_2^2}; \tag{148.13}$$

as was to be expected, for $\sigma_1 \neq \sigma_2$, $K_{33} > 0$, as it only contains the factor $(\sigma_1 - \sigma_2)^2$.

2°. Next consider the *problem of bending by a couple*, assuming its plane to be parallel to Oxz. In this case the conditions of the auxiliary problem of plane deformation, corresponding to (149.8_1), may be satisfied

by writing

$$\begin{aligned} &\varphi_1 = A_1 z^2, \quad \psi_1 = 0 && \text{in } S_1, \\ &\varphi_1 = A_2 z^2, \quad \psi_2 = \frac{B_2}{z^2} + C_2 && \text{in } S_2, \end{aligned} \tag{149.14}$$

where A_1, A_2, B_2, C_2 are real constants.

Substituting these expressions in (149.5), (149.6) and (149.17) with $f = \frac{1}{2}(\sigma_1 - \sigma_2)z^2$ gives, as in the preceding example, four equations for the determination of the constants A_1, A_2, B_2, C_2 which are easily solved and render the values of these constants. Only the expressions for the first three of these constants will be given here, since C_2 does not influence the stress distribution:

$$\begin{aligned} A_1 &= \frac{(\sigma_1 - \sigma_2)(R_2^4 - R_1^4)}{\alpha_1(R_2^4 - R_1^4) + \alpha_2 R_1^4 + \beta_2 R_2^4}, \\ A_2 &= -\frac{(\sigma_1 - \sigma_2)R_1^4}{\alpha_1(R_2^4 - R_1^4) + \alpha_2 R_1^4 + \beta_2 R_2^4}, \\ B_2 &= \frac{1}{2}\cdot\frac{(\sigma_1 - \sigma_2)R_1^4 R_2^4}{\alpha_1(R_2^4 - R_1^4) + \alpha_2 R_1^4 + \beta_2 R_2^4}. \end{aligned} \tag{149.15}$$

The stress $Z_z^{(1)}$, corresponding to this auxiliary problem, is given by

$$Z_z^{(1)} = \sigma_j(X_x^{(1)} + Y_y^{(1)}) = 4\sigma_j \,\Re\varphi'(z) = 8\sigma_j A_j x \text{ in } S_j \quad (j = 1, 2).$$

Hence, in the notation of § 146,

$$K_{11} = \iint xZ_z^{(1)} dx\,dy = 8\sigma_1 A_1 \iint x^2\,dx\,dy + 8\sigma_2 A_2 \iint_{S_2} x^2\,dx\,dy =$$

$$= 2\pi\sigma_1 A_1 R_1^4 + 2\pi\sigma_2 A_2 (R_2^4 - R_1^4)$$

or, by (149.15),

$$K_{11} = \frac{\pi(\sigma_1 - \sigma_2)^2(R_2^4 - R_1^4)R_1^4}{\alpha_1(R_2^4 - R_1^4) + \alpha_2 R_1^4 + \beta_2 R_2^4}. \tag{149.16}$$

Thus, the flexural rigidity is equal to

$$I_E + K_{11} \tag{149.17}$$

(where I_E has been written instead of I_{11}); one has for I_E the formula

$$I_E = \frac{\pi}{4}[E_1 R_1^4 + E_2(R_2^4 - R_1^4)]. \tag{149.18}$$

As was to be expected, for $\sigma_1 \neq \sigma_2$, $K_{11} > 0$, as it only contains the factor $(\sigma_1 - \sigma_2)^2$.

§ 150. Problem of bending by a transverse force.

The solution, presented in this section, was given by A. K. Rukhadze [3]; however, not all statements in that paper are correct. They will only be so, if by the system of axes $Oxyz$ is understood the system which will be used below and not that used by A. K. Rukhadze and if one (inessential) modification is introduced into his reasoning.

Let the axis Oz be the principal axis of extension and the planes Oxz, Oyz the principal planes of bending (§ 148). For such a system one has, in the notation of § 146,

$$I_{13} + K_{13} = S_E x_0 + K_{13} = 0, \quad I_{23} + K_{23} = S_E y_0 + K_{23} = 0,$$
$$I_{12} + K_{12} = 0, \tag{150.1}$$

where x_0, y_0 denote the coordinates of the reduced centre of gravity of the "left" end.

It will be assumed that the bending force of magnitude W is applied at the point, where the axis Oz intersects the "right" ("upper") end, and that it is directed parallel to Ox.

The solution for the general case will be obtained by combining the solution of the above problem with the analogous solution, obtained by interchanging the roles of Ox, Oy.

Guided by the form of the solution, obtained in § 144 for the case of constant Poisson's ratio, the solution of the present problem will be sought in the form

$$\begin{aligned} u^{(0)} &= -\tau yz + A[\tfrac{1}{2}\sigma_j(l - z)(x^2 - y^2) + \tfrac{1}{2}lz^2 - \tfrac{1}{6}z^3], \\ v^{(0)} &= \tau xz + A\sigma_j(l - z)xy, \\ w^{(0)} &= \tau\varphi(x, y) - A[\chi(x, y) + x(lz - \tfrac{1}{2}z^2) + xy^2] \end{aligned} \tag{150.2}$$

in the regions S_j; in these formulae τ, A are constants, subject to definition, $\varphi(x, y)$ is the torsion function, defined as in § 139, and $\chi(x, y)$ is some functions, *continuous throughout* S and subject to definition.

The stress components, corresponding to (150.2), are given in the regions S_j by the formulae

$$X_x^{(0)} = Y_y^{(0)} = X_y^{(0)} = 0, \tag{150.3}$$

$$\begin{aligned} X_z^{(0)} &= \tau\mu_j\left(\frac{\partial\varphi}{\partial x} - y\right) - B_j\left[\frac{\partial\chi}{\partial x} + \tfrac{1}{2}\sigma_j x^2 + (1 - \tfrac{1}{2}\sigma_j)y^2\right], \\ Y_z^{(0)} &= \tau\mu_j\left(\frac{\partial\varphi}{\partial y} + x\right) - B_j\left[\frac{\partial\chi}{\partial y} + (2 + \sigma_j)xy\right], \\ Z_z^{(0)} &= -K_j(l - z)x, \end{aligned} \tag{150.4}$$

where

$$B_j = A\mu_j, \quad K_j = AE_j. \tag{150.5}$$

The displacements (150.2) cannot satisfy the conditions of the problem, because u, v are not continuous for a passage through the dividing line of the regions S_j, S_k; in fact, on these lines

$$\begin{aligned} u_j^{(0)} - u_k^{(0)} &= \tfrac{1}{2}A(\sigma_j - \sigma_k)(l - z)(x^2 - y^2), \\ v_j^{(0)} - v_k^{(0)} &= A(\sigma_j - \sigma_k)(l - z)xy. \end{aligned} \tag{150.6}$$

These discontinuities cannot be removed by finding a solution of the problem of plane deformation, since they depend also on z.

However, a beginning will be made with the solution of the auxiliary problem of plane deformation, formulated in § 145, for the following discontinuities in the displacement components on the dividing lines:

$$u_j - u_k = g = \tfrac{1}{2}(\sigma_j - \sigma_k)(x^2 - y^2), \quad v_j - v_k = h = (\sigma_j - \sigma_k)xy; \tag{150.7}$$

this is the problem (146.1a)

As in § 146, denote the stress and displacement components, corresponding to this problem, by the relevant symbols with the superscript $^{(1)}$ and assume the auxiliary problem to have been solved; consider now the spatial deformation, characterized by the following displacement components:

$$u^* = (l - z)u^{(1)}, \quad v^* = (l - z)v^{(1)}, \quad w^* = 0. \tag{150.8}$$

The corresponding stress components are given by

$$X_x^* = (l - z)X_x^{(1)}, \quad Y_y^* = (l - z)Y_y^{(1)}, \quad X_y^* = (l - z)X_y^{(1)}, \tag{150.9}$$

$$Z_z^* = (l - z)Z_z^{(1)} \quad \sigma_j(l - z)(X_x^{(1)} + Y_y^{(1)}), \tag{150.10}$$

$$X_z^* = -\mu_j u^{(1)}, \quad Y_z^* = \quad \mu_j v^{(1)} \tag{150.11}$$

in the regions S_j.

Finally, the deformation will be written down which is obtained by superposition of the deformations, corresponding to (150.2) and (150.8), where the last is to be multiplied by $-A$, i.e., the deformation, corresponding to the displacements

$$u \quad u^{(0)} - Au^*, \quad v = v^{(0)} - Av^*, \quad w = w^{(0)}. \tag{150.12}$$

The corresponding stress components are given by

$$
\begin{aligned}
X_x &= X_x^{(0)} - AX_x^*, \quad Y_y = Y_y^{(0)} - AY_y^*, \quad Z_z = Z_z^{(0)} - AZ_z^*, \\
Y_z &= Y_z^{(0)} - AY_z^*, \quad Z_x = Z_x^{(0)} - AZ_x^*, \quad X_y = X_y^{(0)} - AX_y^*.
\end{aligned}
\tag{150.13}
$$

Substituting these values in the equilibrium equations, it is readily verified that they will be satisfied, provided the function $\chi(x, y)$ satisfies the equation

$$\Delta\chi(x, y) = \rho(x, y) \tag{150.14}$$

in each of the regions S_j, where

$$\rho(x, y) = \frac{\lambda_j + \mu_j}{\mu_j}\theta^{(1)}, \quad \theta^{(1)} = \frac{\partial u^{(1)}}{\partial x} + \frac{\partial u^{(1)}}{\partial y} \text{ in } S_j; \tag{150.15}$$

it may be assumed that this function is given, since the auxiliary problem of plane deformation has been assumed to be solved.

Further, supposing for definiteness that one is dealing with the *basic case* of a compound bar (§ 139,1°) and formulating the boundary conditions on the free surface and on the dividing surfaces, one easily obtains, in the former notation,

$$\mu_0\left(\frac{d\chi}{dn}\right)_0 - \mu_j\left(\frac{d\chi}{dn}\right)_j = f_j \text{ on } L_j \tag{150.16}$$

($j = 1, 2, \ldots, m+1,\ \mu_{m+1} = 0$), where the functions

$$
\begin{aligned}
f_j = &-\{\tfrac{1}{2}(\mu_0\sigma_0 - \mu_j\sigma_j)x^2 + \\
&+ \left[\mu_0\left(1 - \frac{\sigma_0}{2}\right) - \mu_j\left(1 - \frac{\sigma_j}{2}\right)\right]y^2 - \mu_0 u_0^{(1)} + \mu_j u_j^{(1)}\}\cos(n, x) - \\
&- \{[\mu_0(2 + \sigma_0) - \mu_j(2 + \sigma_j)]xy - \mu_0 v_0^{(1)} + \mu_j v_j^{(1)}\}\cos(n, y)
\end{aligned}
\tag{150.17}
$$

are known on L_j.

One has thus arrived at the familiar boundary problem (150.16), except that the unknown function $\chi(x, y)$ does not this time satisfy the Laplace equation $\Delta\chi = 0$, but the somewhat more general Poisson equation (150.14).

However, this problem is easily reduced to the case where the unknown function satisfies Laplace's equation. In fact, let $\chi_0(x, y)$ be any particular solution of (150.14); such a particular solution is always easily found.

* For example, it is known that the logarithmic potential

$$\chi_0(x, y) = \frac{1}{2\pi} \iint_S \rho(\xi, \eta) \log r \, d\xi \, d\eta$$

is such a particular solution, where $r^2 = (x - \xi)^2 + (y - \eta)^2$. In practice, however, it is usually more convenient to find a particular solution by different elementary methods.

Writing

$$\chi(x, y) = \chi_0(x, y) + \chi^*(x, y), \tag{150.18}$$

where $\chi^*(x, y)$ is a new unknown function which obviously satisfies the equation $\Delta\chi^* = 0$, one arrives at the boundary conditions

$$\mu_0 \left(\frac{d\chi^*}{dn}\right)_0 - \mu_j \left(\frac{d\chi^*}{dn}\right)_j = f_j^* \text{ on } L_j (j = 1, 2, \ldots, m+1 \,;\, \mu_{m+1} = 0), \tag{150.19}$$

where

$$f_j^* = f_j - \mu_0 \left(\frac{d\chi_0}{dn}\right)_0 + \mu_j \left(\frac{d\chi_0}{dn}\right)_j \text{ on } L_j. \tag{150.20}$$

It is known that the condition of solubility of the problem (150.19) is given by

$$\sum_{j=1}^{m+1} \int_{L_j} f_j^* ds = 0. \tag{150.21}$$

This formula will be somewhat simplified. Substituting for f_j^* its expression (150.20), one obtains

$$\sum_{j=1}^{m+1} \int_{L_j} f_j ds - \mu_0 \sum_{j=1}^{m+1} \int_{L_j} \left(\frac{d\chi_0}{dn}\right)_0 ds + \sum_{j=1}^{m} \mu_j \int_{L_j} \left(\frac{d\chi_0}{dn}\right)_j ds = 0$$

or, transforming the last integrals by use of Green's formula,

$$\sum_{j=1}^{m+1} \int_{L_j} f_j ds - \mu_0 \iint_{S_0} \Delta\chi_0 dx\, dy - \sum_{j=1}^{m} \mu_j \iint_{S_j} \Delta\chi_0 dx\, dy =$$

$$= \sum_{j=1}^{m+1} \int f_j ds - \iint \mu \Delta\chi_0 dx\, dy = 0$$

or, remembering that $\Delta\chi_0 = \rho(x, y)$,

$$\sum_{j=1}^{m+1} \int_{L_j} f_j ds - \iint_S \mu\rho dx\, dy = 0. \qquad (150.22)$$

It will now be verified as to whether in the present case this condition is satisfied. Substituting in (150.22) for f_j from (150.17) and transforming the integrals by means of Green's formula, one readily finds that (150.22) reduces in the present case to the following condition:

$$-\iint_S 2\mu(1 + \sigma)x\, dx\, dy + \iint_S \mu\theta^{(1)}\, dx\, dy - \iint_S \mu\rho\, dx\, dy = 0$$

or, since

$$2\mu(1 + \sigma) = E, \quad \mu\rho = (\lambda + \mu)\theta^{(1)}, \quad \lambda\theta^{(1)} = Z_z^{(1)},$$

to the condition

$$-\iint_S Ex dx\, dy - \iint Z_z^{(1)} dx\, dy = 0$$

or, finally, to

$$S_E x_0 + K_{13} = 0;$$

however, this last condition is imposed by (150.1).

Thus, in the present case, the boundary problem (150.16) is soluble; its solution is determined, apart from an arbitrary constant term which does not influence the stress distribution.

If one substitutes this solution for $\chi(x, y)$ in (150.2), the formulae (150.12), (150.13) then determine the solution of the original problem which satisfies all the required conditions on the side surface and on the dividing surfaces.

It will now be shown that the constants A and τ may always be chosen in such a manner that the forces, applied to the "right" ("upper") end, likewise satisfy the required conditions. For this purpose the resultant vector (X, Y, Z) and the resultant moment of these forces will be calculated. Since for $z = l$: $Z_z^{(0)} = Z_z^* = 0$, one obviously has $Z = 0$. Further,

$$X = \iint_S X_z\, dx\, dy. \qquad (150.23)$$

This formula will now be transformed. By the equilibrium equation,

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} = 0;$$

substituting for $Z_z = Z_z^{(0)} - AZ_z^*$ its value, given by (150.4), (150.5) and (150.10), one finds

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + A(E_j x + Z_z^{(1)}) = 0 \text{ in } S_j. \tag{150.24}$$

Thus, one may write

$$X_z = \frac{\partial(xX_z)}{\partial x} + \frac{\partial(xY_z)}{\partial y} + A(E_j x + Z_z^{(1)})\, x \text{ in } S_j.$$

Consequently (150.23) may be rewritten

$$X = \iint\limits_S \left\{\frac{\partial(xX_z)}{\partial x} + \frac{\partial(xY_z)}{\partial y}\right\} dx\, dy + A\left\{\iint\limits_S Ex^2\, dx\, dy + \iint\limits_S xZ_z^{(1)} dx\, dy\right\}.$$

Transforming the first integral by use of Green's formula, it is readily verified that it is equal to zero. Hence, in the notation of § 146,

$$X = A(I_{11} + K_{11}). \tag{150.25}$$

Further, one finds by the same method

$$Y = \iint Y_z\, dx\, dy = A(I_{12} + K_{12}),$$

whence it follows, by (150.1), that $Y = 0$.

Thus the resultant vector of the external forces, applied to the "right" end, is parallel to the axis Ox.

It is also easily seen that the resultant moment of these forces about the point of intersection between the axis Oz and the "right" end is parallel to Oz and that its magnitude is given by

$$M = \tau D + A \sum_j \mu_j \iint\limits_{S_j} \left\{ y\frac{\partial \chi}{\partial x} - x\frac{\partial \chi}{\partial y} + (1 - \tfrac{1}{2}\sigma_j)y^3 - \right.$$
$$\left. - (2 + \tfrac{1}{2}\sigma_j)x^2 y \right\} dx\, dy + A \sum_j \mu_j \iint\limits_{S_j} (xv^{(1)} - yu^{(1)}) dx\, dy, \tag{150.26}$$

where D is the torsional rigidity which is known to be always larger than zero.

Consequently, all the conditions of the problem will be satisfied, provided the constants A and τ are chosen in such a manner that $X = W$ and $M = 0$ respectively. The first condition, taking into consideration

(150.25), gives

$$A = \frac{W}{I_{11} + K_{11}} \tag{150.27}$$

on the basis of (150.26) and (150.27), the second condition determines τ, since $D \neq 0$.

Thus the problem is solved. It is readily seen that also in the present case the Bernoulli-Euler Law remains valid and that the flexural rigidity is given by

$$I_{11} + K_{11}, \tag{150.28}$$

as in the case of bending by a couple.

When the dividing line and the external boundary of the region S are concentric circles, as in the example of the preceding section, the problem is readily solved in closed form.

APPENDIX 1

ON THE CONCEPT OF A TENSOR

1. Tensor calculus has rapidly achieved recognition in pure as well as in applied contemporary mathematics and is beginning to enter into technical literature, in particular, into the literature dealing with the theory of elasticity. For this reason it is considered necessary to give here at least an elementary introduction to the concept of a *tensor* which, for the sake of simplicity, will be confined exclusively to orthogonal coordinates. It should, however, be noted that the principal advantage of tensor calculus arises in its application to curvilinear coordinates of the general type. In order to give the subsequent definition of a tensor a more natural background, certain remarks will first be made with regard to the concept of a vector (since a vector is a particular type of a tensor, in fact, it is a first order tensor).

It will be assumed that the ordinary geometrical definition of a vector as a straight segment which has direction is known. Further, coordinate axes will not be denoted by Ox, Oy, Oz, as in elementary analytical geometry, but by Ox_1, Ox_2, Ox_3. Correspondingly, the components of a vector $\vec{P}$ will not be denoted by ξ, η, ζ, as in the main part of this book, but by ξ_1, ξ_2, ξ_3.

Only the length and direction of the vector, and not the position of its starting point, will be considered; thus, a vector will be considered completely known, if its components ξ_1, ξ_2, ξ_3 (i.e., its projections on the coordinate axes) are given. The vector $\vec{P}$ with components ξ_1, ξ_2, ξ_3 will be denoted by (ξ_1, ξ_2, ξ_3) or, still more briefly, by (ξ_i); the index i takes then the values 1, 2, 3.

Thus, a vector in space is characterized by *three* scalar quantities.

Many physical and geometrical quantities exist which for a *given choice of coordinate axes* are likewise characterized by three scalars, for example: velocity, force (applied to a given point), etc. However, not every such quantity can be represented as a vector, as may, for example be done with a velocity or a force. In fact, let ξ_1, ξ_2, ξ_3 be scalars character-

izing a given physical quantity *for a given choice of coördinate axes.* One may, of course, always construct a vector

$$\vec{P} = (\xi_1, \xi_2, \xi_3)$$

with components ξ_1, ξ_2, ξ_3 and claim that it represents the given physical quantity *for the given choice of coordinate axes.* However, this relation between the given quantity and the vector may be disturbed, if the system of coordinate axes is replaced by another one. In fact, it may happen that the scalars ξ_1', ξ_2', ξ_3', characterizing the original physical quantity in the new coordinate system, do not coincide with the components of the vector $\vec{P}$ in the new system, i.e., the vector $\vec{P}'$, having in the new coordinate system the components ξ_1', ξ_2', ξ_3', may differ from $\vec{P}$. In order that the representation of a physical quantity be independent of the choice of the coordinate system, it is obviously necessary that the scalars ξ_1, ξ_2, ξ_3, characterizing it, transform in the transition from one coordinate system to another according to the same law as the components of a vector. It may only then be said that the given physical quantity is represented by a vector, or that it is a vectorial quantity. In future, vectorial quantities will often simply be called vectors, i.e., they will be identified with the vectors, representing them.

The law by which the components of vectors change during transition from one coordinate system to another will now be recalled. The notation, used in the main part of this book, will be somewhat modified. In fact, the cosines of the angles between the old and new axes will now be denoted by

	x_1	x_2	x_3
x_1'	l_{11}	l_{12}	l_{13}
x_2'	l_{21}	l_{22}	l_{23}
x_3'	l_{31}	l_{32}	l_{33}

(A)

The relations between the new components ξ_1', ξ_2', ξ_3' of a vector $\vec{P}$ and its old components ξ_1, ξ_2, ξ_3 may then be written

$$\xi_k = \sum_{i=1}^{3} l_{ik}\xi_i', \quad \xi_i' = \sum_{k=1}^{3} l_{ik}\xi_k. \tag{1.1.1}$$

The following well known relations hold between the elements of

table (A):

$$\sum_{i=1}^{3} l_{ki} l_{mi} = \delta_{km}, \quad \sum_{i=1}^{3} l_{ik} l_{im} = \delta_{km}, \tag{1.1.2}$$

where

$$\delta_{km} = \begin{cases} 1 \text{ for } k = m, \\ 0 \text{ for } k \neq m. \end{cases}$$

Consider now the two vectors

$$\vec{A} = (a_1, a_2, a_3)$$

and

$$\vec{P} = (\xi_1, \xi_2, \xi_3).$$

Their scalar product is given by

$$\vec{A} . \vec{P} = a_1\xi_1 + a_2\xi_2 + a_3\xi_3 = \sum_{i=1}^{3} a_i\xi_i.$$

By definition,

$$\vec{A} . \vec{P} = AP \cos(\vec{A}, \vec{P}),$$

where A, P denote the lengths of the vectors $\vec{A}$, $\vec{P}$; it is thus seen that the scalar product does not depend on the choice of coordinate axes, i.e., that

$$a_1'\xi_1' + a_2'\xi_2' + a_3'\xi_3' = a_1\xi_1 + a_2\xi_2 + a_3\xi_3. \tag{1.1.3}$$

The reader will easily verify this formula directly from (1.1.1) and (1.1.2)

Conversely, it will now be shown that, if a_1, a_2, a_3 are three scalars which are related to the coordinate axes in such a manner that the *linear form*

$$F = a_1\xi_1 + a_2\xi_2 + a_3\xi_3 = \sum_{i=1}^{3} a_i\xi_i, \tag{1.1.4}$$

where ξ_1, ξ_2, ξ_3 are the components of *an arbitrary vector*, is invariant for the transition from one coordinate system to another, then the triad of numbers (a_1, a_2, a_3) represents a vectorial quantity (i.e., a vector). In order to prove this statement, it is sufficient to verify that the quantities a_1, a_2, a_3 transform for the passage from one coordinate system to another by the same law (1.1.1) as the components of a vector. In fact, one has by supposition

$$\sum_{i=1}^{3} a_i'\xi_i' = \sum_{k=1}^{3} a_k\xi_k;$$

substituting on the right-hand side for ξ_k the expression (1.1.1), one obtains

$$\sum_{i=1}^{3} a_i'\xi_i' = \sum_{k=1}^{3}\sum_{i=1}^{3} a_k l_{ik}\xi_i' = \sum_{i=1}^{3}\xi_i'\sum_{i=1}^{3} l_{ik}a_k.$$

Since this equality must hold true for any values of ξ_1', ξ_2', ξ_3', the coefficients of ξ_i' must be equal; hence

$$a_i' = \sum_{k=1} l_{ik}a_k, \tag{1.1.J}$$

and this formula agrees with the second formula of (1.1.1), if a is replaced by ξ. The proposition is thus proved. Therefore:

If the linear form

$$\sum_{i=1}^{3} a_i\xi_i$$

is invariant for coordinate transformation and ξ_i are the components of an arbitrary vector, then a_i are likewise the components of a vector.

2. Generalizing the concept of a vector, based solely on the above stated property, one arrives by a natural process at the concept of a tensor. In fact, instead of the linear form (1.1.4), consider *the bilinear form*

$$\begin{aligned} F = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\xi_i\eta_j = \sum_{i,j=1}^{3} a_{ij}\xi_i\eta_j = {} & a_{11}\xi_1\eta_1 + a_{12}\xi_1\eta_2 + a_{13}\xi_1\eta_3 + \\ & + a_{21}\xi_2\eta_1 + a_{22}\xi_2\eta_2 + a_{23}\xi_2\eta_3 + \\ & + a_{31}\xi_3\eta_1 + a_{32}\xi_3\eta_2 + a_{33}\xi_3\eta_3 \end{aligned} \tag{1.2.1}$$

which depends linearly on the components of *two* vectors

$$\vec{P} = (\xi_1, \xi_2, \xi_3)$$

and

$$Q = (\eta_1, \eta_2, \eta_3).$$

It will now be postulated that the coefficients a_{ij} of this form vary for transformation of coordinates in such a way that the form F remains invariant. Under this condition it will be said that *the set of quantities* a_{ij}, depending on the two indices i, j, *represents a tensor of second order* (since there are two indices); a_{ij} are called the *components* of this tensor (with respect to a given system of axes). This tensor will be denoted by the symbol (a_{ij}).

On the basis of this definition, the transformation law of the tensor components is easily found. Let a_{ij}', ξ_i', η_j' be the components of the tensor

a_{ij} and of the vectors $\vec{P}$, $\vec{Q}$ in the new coordinate system. By definition,

$$\sum_{i,j=1}^{3} a'_{ij}\xi'_i\eta'_j = \sum_{k,m=1}^{3} a_{km}\xi_k\eta_m.$$

Substituting on the right-hand side the expressions

$$\xi_k = \sum_{i=1}^{3} l_{ik}\xi'_i, \quad \eta_m = \sum_{j=1}^{3} l_{jm}\eta'_j,$$

one finds

$$\sum_{i,j=1}^{3} a'_{ij}\xi'_i\eta'_j = \sum_{i,j=1}^{3} \xi'_i\eta'_j \sum_{k,m=1}^{3} l_{ik}l_{jm}a_{km},$$

whence, comparing the coefficients of the products $\xi'_i.\eta'_j$,

$$a'_{ij} = \sum_{k,m=1}^{3} l_{ik}l_{jm}a_{kn} \tag{1.2.2}$$

This is the required transformation formula.

A second order tensor is called *symmetric*, if $a_{ij} = a_{ji}$. It is easily seen from (1.2.2) that this property of symmetry is retained during coordinate transformations.

In the case of a symmetrical tensor, one may use for its definition, instead of the bilinear form (1.2.1), the quadratic form $2\Omega(\xi_1, \xi_2, \xi_3)$ which is obtained from F by putting $\xi_i = \eta_i$. In this way one obtains the definition, given in § 5 of the main part of this book. The transformation formulae for the components of the stress tensor, given in § 5, coincide with the formulae (1.2.2), if the last are rewritten in the notation of that section.

The simplest symmetrical tensor is the tensor (δ_{ij}), defined by

$$\delta_{ij} = \begin{cases} 1 \text{ for } i = j, \\ 0 \text{ for } i \neq j. \end{cases} \tag{1.2.3}$$

It is easily seen that (δ_{ij}) is a tensor, since

$$\sum_{i,j=1}^{3} \delta_{ij}\xi_i\eta_j = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$$

is obviously invariant (as it is the scalar product of the vectors $\vec{P}$ and $\vec{Q}$). The tensor (δ_{ij}) is called the unit tensor.

A tensor is called anti-symmetrical, if $a_{ij} = - a_{ji}$. Since, in particular, one must then have $a_{ii} = - a_{ii}$, in an anti-symmetrical tensor $a_{11} = a_{22} = a_{33} = 0$. Thus, an anti-symmetrical tensor is characterized

by three quantities p_1, p_2, p_3, such that

$$p_1 = a_{32} = -a_{23}, \quad p_2 = a_{13} = -a_{31}, \quad p_3 = a_{21} = -a_{12}.$$

It is easily seen from (1.2.2) that the property of anti-symmetry is retained during transformation of coordinates.

3. Two tensors (a_{ij}) and (b_{ij}) are said to be *equal*, if $a_{ij} = b_{ij}$. The tensor (c_{ij}) whose components are equal to the sums of the components of two given tensors

$$c_{ij} = a_{ij} + b_{ij}$$

is called the *sum* of the tensors (a_{ij}) and (b_{ij}). It follows from

$$\sum_{i,j=1}^{3} c_{ij}\xi_i\eta_j = \sum_{i,j=1}^{3} a_{ij}\xi_i\eta_j + \sum_{i,j=1}^{3} b_{ij}\xi_i\eta_j$$

that (c_{ij}) is a tensor. Since the terms on the right-hand side are invariant, the left-hand side is also invariant, and this proves the tensorial character of the set of quantities c_{ij}. The difference of two tensors may be defined in an analogous manner.

If (a_{ij}) is a tensor, the quantities $a^*_{ij} = a_{ji}$ likewise determine some tensor (a^*_{ij}); this result follows likewise directly from the definition of a tensor.

Every tensor (a_{ij}) may (in a unique manner) be decomposed into the sum of a symmetric tensor (e_{ij}) and of an anti-symmetric tensor (p_{ij}). In fact, let $a_{ij} = e_{ij} + p_{ij}$. Interchanging the indices i, j and noting that, by supposition, $e_{ij} = e_{ji}$, $p_{ji} = -p_{ij}$, one finds $a_{ji} = e_{ij} - p_{ij}$. In combination with the preceding equation, one finds

$$e_{ij} = \tfrac{1}{2}(a_{ij} + a_{ji}), \quad p_{ij} = \tfrac{1}{2}(e_{ij} - e_{ji}). \tag{1.3.1}$$

It is readily verified that the tensors (e_{ij}) and (p_{ij}) satisfy the imposed conditions.

Several examples of tensors will now be presented.

Let (a_i) and (b_i) be two vectors. Write $c_{ij} = a_ib_j$. The set of the quantities (c_{ij}) is a tensor. In fact, let (ξ_i), (η_i) be two arbitrary vectors. One has

$$\sum_{i,j=1}^{3} c_{ij}\xi_i\eta_j = \sum_{i,j=1}^{3} a_ib_j\xi_i\eta_j = \sum_{i=1}^{3} a_i\xi_i \sum_{j=1}^{3} b_j\eta_j.$$

The right-hand side is invariant (as product of two invariant quantities). Hence also the left-hand side is invariant, and this proves the assertion.

It is known that (c^*_{ij}), where $c^*_{ij} = c_{ji} = a_jb_i$, is likewise a tensor.

Hence, if one writes

$$p_{ij} = c^*_{ij} - c_{ij} = a_j b_i - a_i b_j, \tag{1.3.2}$$

then (p_{ij}) is also a tensor which is obviously anti-symmetrical.

This tensor is called *the vector product of the two given vectors*. In vector analysis, the vector product is considered as a vector, and not as a tensor. In order to elucidate this, consider the following. Let

$$p_1 = \quad = a_2 b_3 \quad a_3 b_2, \; p_2 = p_{13} \quad a_3 b_1 - a_1 b_3,$$
$$p_3 = p_{21} \quad a_1 b_2 - a_2 b_1. \tag{1.3.3}$$

It will be investigated whether the set of quantities (p_1, p_2, p_3) is a vector. For this purpose the criterion, formulated earlier, will be applied, i.e., an arbitrary vector (ξ_1, ξ_2, ξ_3) will be introduced and it will be verified whether

$$p_1 \xi_1 + p_2 \xi_2 + p_3 \xi_3$$

is invariant. Obviously, one has

$$p_1 \xi_1 + p_2 \xi_2 + p_3 \xi_3 \qquad \begin{array}{ccc} \xi_1 & \xi_2 & \xi_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{array} \tag{1.3.4}$$

However, it is known from analytic geometry that this determinant represents the volume of the parallelepiped, constructed on the vectors (ξ_i), (a_i), (b_i); the sign of this volume depends on the orientation of the coordinate axes: the volume changes sign as one changes from a left-handed system of axes to a right-handed one. The sign will not vary otherwise. Thus, the expression (1.3.4) will be invariant, if one uses consistently a right-handed or a left-handed system, and in that case the quantities p_1, p_2, p_3 may be considered as representing a vector which does not depend on the choice of coordinate axes.

It is easily shown that for transition from right-handed to left-handed systems or conversely the vector (p_1, p_2, p_3), defined by (1.3.2) and (1.3.3), inverts its direction.

Finally, it will be shown that, when the above-mentioned restriction of the choice of coordinate axes is imposed, every anti-symmetrical second order tensor may be represented as a vector (having all the time in mind three-dimensional space, since otherwise this assertion is not true). In fact, let (p_{ij}) be any anti-symmetric second order tensor.

Construct the sum

$$\sum_{i,j=1} p_{ij}\xi_i\eta_j = -(p_1\zeta_1 + p_2\zeta_2 + p_3\zeta_3), \tag{a}$$

where (ξ_i), (η_i) are two arbitrary vectors and where

$$p_1 = p_{32} = -p_{23}, \quad p_2 = p_{13} = -p_{31}, \quad p_3 = p_{21} = -p_{12},$$
$$\zeta_1 = \xi_2\eta_3 - \xi_3\eta_2, \quad \zeta_2 = \xi_3\eta_1 - \xi_1\eta_3, \quad \zeta_3 = \xi_1\eta_2 - \xi_2\eta_1. \tag{1.3.5}$$

However, by the statements above, $(\zeta_1, \zeta_2, \zeta_3)$ is a vector. On the other hand, the left-hand side of (a) is invariant. Hence, also the right-hand side is invariant and $(\zeta_1, \zeta_2, \zeta_3)$ is an arbitrary vector. This means that (p_1, p_2, p_3) is a vector, and the proposition is proved.

4. The concept of a tensor of any order n may be introduced in an analogous manner. For this purpose it is sufficient to consider, instead of a bi-linear form, an n-linear form depending linearly on the components of n arbitrary vectors.

For example, the set of coefficients a_{ijk} of the tri-linear form

$$F = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a_{ijk}\xi_i\eta_j\zeta_k,$$

where ξ_i, η_j, ζ_k are the components of three arbitrary vectors, determines the *third order tensor* (a_{ijk}) with components a_{ijk}. In the same manner one may define a tensor of order n. From this point of view, a vector must be interpreted as a first order tensor, defined by the help of the linear form

$$\sum_{i=1}^{3} a_i\xi_i.$$

5. Consider again the second order tensor (a_{ij}). Let (ξ_i) be some vector and construct the expression

$$\xi_i^* = \sum_{j=1} a_{ij}\xi_j. \tag{1.5.1}$$

It is easily shown that $(\xi_i^*) = (\xi_1^*, \xi_2^*, \xi_3^*)$ is a vector. In fact, let (η_j) be an arbitrary vector. Then

$$\sum_{i=1}^{3} \xi_i^*\eta_i = \sum_{i,j=1}^{3} a_{ij}\eta_i\xi_j$$

is invariant, because the right-hand side is invariant on the basis of the definition of a tensor.

It is obvious that also conversely, if ξ_1^*, ξ_2^*, ξ_3^*, defined by (1.5.1),

where ξ_1, ξ_2, ξ_3 are the components of an arbitrary vector, are the components of a vector, then a_{ij} are the components of a tensor.

Thus the relation (1.5.1) relates to every vector (ξ_i) a completely defined vector (ξ_i^*). For this reason the vector (ξ_i^*) is called the *linear vector function* of the vector (ξ_i), determined by the tensor (a_{ij}).

An example of such a vector function has been encountered in the main part of this book. In fact, the relations (3.2) show that the stress vector (X_n, Y_n, Z_n), acting on the plane with normal n, is a linear vector function of the vector $\vec{n}$, determined by the stress tensor. In this case, $\vec{n}$ denotes a vector of unit length which has the direction of the normal n.

The case where the tensor (a_{ij}) is symmetrical, i.e., where $a_{ij} = a_{ji}$, is of particular interest. It will now be studied in detail. For this purpose introduce the quadratic form

$$2\Omega(\xi_1, \xi_2, \xi_3) = \sum_{i,j=1} a_{ij}\xi_i\xi_j =$$
$$= a_{11}\xi_1^2 + a_{22}\xi_2^2 + a_{33}\xi_3^2 + 2a_{23}\xi_2\xi_3 + 2a_{31}\xi_3\xi_1 + 2a_{12}\xi_1\xi_2. \quad (1.5.2)$$

In this case one may rewrite (1.5.1)

$$\xi_i^* = \frac{\partial\Omega}{\partial\xi_i}. \quad (1.5.3)$$

The following important proposition will now be proved: *By a suitable choice of new coordinate axes* (i.e., rectilinear, orthogonal) Ox_1', Ox_2', Ox_3', *every quadratic form* 2Ω *may be reduced to its canonical form*

$$2\Omega = \lambda_1\xi_1'^2 + \lambda_2\xi_2'^2 + \lambda_3\xi_3'^2, \quad (1.5.4)$$

where $\lambda_1, \lambda_2, \lambda_3$ *are real constants* (*where it has been assumed that* a_{ij} *are real*). This proposition is equivalent to the following one. By a suitable choice of coordinate axes, it may be ensured that the new components a_{ij}' of any symmetrical tensor (a_{ij}), having different indices, vanish, i.e., that

$$a_{23} = a_{31} = a_{12} = 0$$

(while the remaining, i.e., the "diagonal", components

$$a_{11}' = \lambda_1, \quad a_{22}' = \lambda_2, \quad a_{33}' = \lambda_3$$

will, in general, be different from zero).

If the form 2Ω has the stated canonical form, the relations (1.5.3) in the new coordinate system will reduce to the following:

$$\xi_1'^* = \lambda_1\xi_1', \quad \xi_2'^* = \lambda_2\xi_2', \quad \xi_3'^* = \lambda_3\xi_3'. \quad (1.5.5)$$

These relations show that, if the vector (ξ_i') is directed along one of the new coordinate axes, the corresponding vector $(\xi_i'^*)$ will be parallel to it. For example, the vector (ξ_i'), parallel to the axis Ox_1', has the components ξ_1', 0, 0 for $\xi_1' \neq 0$. Its corresponding vector $(\xi_i'^*)$ has the components $\lambda_1\xi_1'$, 0, 0.

Hence, in order to reduce the form 2Ω to the required type, one has first of all to find the directions with the above stated property. Thus there arises the following problem concerning the relations (1.5.1): To what direction of the finite vector (ξ_i) corresponds a vector (ξ_i^*) with the same direction? In order that the vectors (ξ_i) and (ξ_i^*) will be parallel, it is known to be necessary and sufficient that

$$\xi_1^* \quad \lambda\xi_1, \quad \xi_2^* \quad \lambda\xi_2, \quad \xi_3^* = \lambda\xi_3,$$

where λ is some parameter. Introducing here the expressions (1.5.1) for ξ_i^*, one obtains the system of equations

$$\begin{aligned} &(a_{11} - \lambda)\xi_1 + a_{12}\xi_2 + a_{13}\xi_3 = 0, \\ &a_{21}\xi_1 + (a_{22} - \lambda)\xi_2 + a_{23}\xi_3 = 0, \\ &a_{31}\xi_1 + a_{32}\xi_2 + (a_{33} - \lambda)\xi_3 = 0. \end{aligned} \tag{1.5.6}$$

This system of linear homogeneous equations in ξ_1, ξ_2, ξ_3 admits n-zero solutions if, and only if, its determinant is equal to zero, i.e.,

$$\begin{array}{ccc} & & a_{13} \\ & a_{22} - \lambda & a_{23} \\ a_{31} & & \end{array} \qquad 0. \tag{1.5.7}$$

This equation is a third order polynomial in λ. It will be shown below that all its roots are real. At the moment it will only be noted that this equation has at least one real root, since it is a polynomial of odd degree. This root will be denoted by λ_3.

If one substitutes in (1.5.6) for λ the value λ_3, this system will give a solution for which not all ξ_i are simultaneously zero. Let ξ_1^0, ξ_2^0, ξ_3^0 be one such solution. The vector (ξ_i^0) determines such a direction that for any vector (ξ_i), parallel to it, its corresponding vector (ξ_i^*) will likewise be in the same direction.

Every such direction is called a *principal direction*, corresponding to the tensor (a_{ij}).

The new system of axes Ox_1'', Ox_2'', Ox_3'' will now be chosen such that the axis Ox_3'' has the principal direction, found above. The two other axes (perpendicular to this direction and between themselves) remain for the time being arbitrary.

The components of tensors and vectors in the new system will be denoted by the same symbols as before, but with two accents. In the new system the equations (1.5.6), which may be written

$$\xi_i^* = \lambda \xi_i,$$

take then the form

$$\xi_i''^* = \lambda \xi_i'',$$

where

$$\xi_i''^* = \sum_{j=1}^{3} a_{ij}'' \xi_j'';$$

thus, written explicitly, one finds

$$\begin{aligned}
(a_{11}'' - \lambda)\xi_1'' + \quad a_{12}''\xi_2'' \quad + \quad a_{13}''\xi_3'' \quad &= 0,\\
a_{21}''\xi_1'' \quad + (a_{22}'' - \lambda)\xi_2'' + \quad a_{23}''\xi_3'' \quad &= 0,\\
a_{31}''\xi_1'' \quad + \quad a_{32}''\xi_2'' \quad + (a_{33}'' - \lambda)\xi_3'' &= 0.
\end{aligned}$$

When $\lambda = \lambda_3$, these equations must have the solution $(0, 0, \xi_3'')$ for $\xi_3'' \neq 0$. Hence

$$a_{13}'' = 0, \quad a_{23}'' = 0, \quad a_{33}'' = \lambda_3,$$

so that the quadratic form 2Ω becomes in the new system

$$2\Omega = a_{11}''\xi_1''^2 + 2a_{12}''\xi_1''\xi_2'' + a_{22}''\xi_2''^2 + \lambda_3\xi_3''^2. \tag{1.5.8}$$

In order to reduce 2Ω to the required type, it is sufficient to rotate the axes Ox_1'', Ox_2'' in their plane, leaving Ox_3'' unchanged, so that the term with the product $\xi_1''\xi_2''$ in (1.5.8) vanishes. This may always be done. In fact, let the new axes be Ox_1', Ox_2', Ox_3' and let Ox_1' make an angle α with Ox_1''. Then

$$\xi_1'' = \xi_1' \cos\alpha - \xi_2' \sin\alpha, \quad \xi_2'' = \xi_1' \sin\alpha + \xi_2' \cos\alpha, \quad \xi_3'' = \xi_3'.$$

Substituting these expressions in (1.5.8.), one obtains

$$2\Omega = a_{11}'\xi_1'^2 + 2a_{12}'\xi_1'\xi_2' + a_{22}''\xi_2'^2 + \lambda_3\xi_3'^2, \tag{1.5.9}$$

where, in particular,

$$\begin{aligned}
a_{12}' = -(a_{11}'' - a_{22}'') \sin\alpha\cos\alpha + a_{12}''(\cos^2\alpha - \sin^2\alpha) &= \\
= -\tfrac{1}{2}(a_{11}'' - a_{22}'') \sin 2\alpha + a_{12}'' \cos 2\alpha.&
\end{aligned}$$

Hence $a'_{12} = 0$ for

$$\tan 2\alpha = \frac{2a''_{12}}{a''_{11} - a''_{22}} \tag{1.5.10}$$

If α_0 is an angle which satisfies this condition, then

$$\alpha_0 \quad \frac{\pi}{2}$$

as well as all angles

$$\alpha_0 + \frac{k\pi}{2}$$

will satisfy it, where k is an integer. Thus, two mutually perpendicular directions have been found which satisfy the required conditions (both of these being perpendicular to Ox'_3, which coincides with Ox''_3); only for $a''_{12} = 0$ and $a''_{11} = a''_{22}$ will there be an infinite number of such directions, in which case $\alpha'_{12} = 0$ for all values of α. Choosing one of the axes for Ox'_1 (and its perpendicular for Ox'_2), the form 2Ω is reduced to the required type (1.5.4), where λ_1, λ_2, λ_3 are real numbers. One has thus not only proved the possibility of the stated reduction, but also deduced an effective method for its execution and for the determination of the directions of the corresponding new axes.

It is known that λ_3 is one of the roots of (1.5.7). It will now be shown that λ_1, λ_2 are the two other roots of the same equation. For this purpose it will first be noted that the determinant

$$D_0 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{1.5.11}$$

is invariant, i.e., that it does not change for transformation of coordinates (this determinant is called the *discriminant* of the quadratic form 2Ω). In fact, for transition to new axes Ox'_1, Ox'_2, Ox'_3, this determinant becomes

$$D'_0 \quad \begin{matrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{matrix}$$

where, by (1.2.2),

$$a'_{ij} = \sum_{k=1}^{3} \sum_{m=1}^{3} l_{ik} l_{jm} a_{km} = \sum_{k=1}^{3} l_{ik} b_{kj}$$

with

$$b_{kj} = \sum_{m=1}^{3} l_{jm} a_{km}.$$

On the basis of the well known theorem on multiplication of determinants, one has

$$D_0' = \Delta . \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}, \text{ where } \Delta = \begin{vmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{vmatrix}.$$

On the basis of the same theorem

$$\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \Delta . \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \Delta . D_0.$$

Hence $D_0' = \Delta^2 D_0$. However, on the basis of well known properties of the direction cosines l_{ij}, $\Delta = \pm 1$, whence it follows that $D_0' = D_0$, as was to be proved.

Next consider the tensor with the components $A_{ik} = a_{ik} - \lambda\delta_{ik}$, where λ is an arbitrary number and (δ_{ik}) the unit tensor. The determinant of the components of the tensor (A_{ik})

$$D_\lambda = \begin{matrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{matrix}$$

will, on the basis of the earlier statements, not depend on the choice of coordinates, i.e., D_λ is invariant.

Now choose new axes Ox_1', Ox_2', Ox_3' in such a way that the new components a_{ik}' of the tensor (a_{ik}), which have different indices, are equal to zero, so that the quadratic form 2Ω is of the type

$$\lambda_1\xi_1'^2 + \lambda_2\xi_2'^2 + \lambda_3\xi_3'^2.$$

The determinant D_λ for this new system of axes will be

$$\begin{matrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & \lambda_2 - \lambda & 0 \\ 0 & 0 & \lambda_3 - \lambda \end{matrix} \qquad (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda).$$

Hence one has the identity

$$D_\lambda = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda),$$

and it is seen that the real numbers $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation

$$D_\lambda = 0.$$

Thus, in passing, the important theorem of algebra has been proved by which all the roots of (1.5.7), which is called the characteristic equation, are real (under the essential supposition that a_{ij} are real and, in addition, $a_{ij} = a_{ji}$).

Now the linear vector function, defined by (1.5.1), will be considered, retaining the supposition $a_{ij} = a_{ji}$. It has been seen that one may always find at least one triad of mutually perpendicular principal directions and that, if the coordinate axes are given these directions, the form 2Ω becomes

$$\lambda_1\xi_1^2 + \lambda_2\xi_2^2 + \lambda_3\xi_3^2,$$

while the relations (1.5.1) have the form

$$\xi_1^* = \lambda_1\xi_1, \quad \xi_2^* = \lambda_2\xi_2, \quad \xi_3^* = \lambda_3\xi_3 \tag{1.5.12}$$

(where now the accents have been omitted).

Next, the question will be discussed whether there are any principal directions other than the three, found above. If (ξ_1, ξ_2, ξ_3) is a vector, parallel to any principal direction, then, by definition, the vector (ξ_i^*) must be parallel to the vector (ξ_i), i.e.,

$$\xi_1^* = \lambda\xi_1, \quad \xi_2^* = \lambda\xi_2, \quad \xi_3^* = \lambda\xi_3.$$

Substituting from (1.5.12), one finds

$$(\lambda_1-\lambda)\xi_1 = 0, \quad (\lambda_2-\lambda)\xi_2 = 0, \quad (\lambda_3-\lambda)\xi_3 = 0, \tag{1.5.13}$$

whence it follows that λ can only have one of the three values $\lambda_1, \lambda_2, \lambda_3$ (otherwise one would have to have $\xi_1 = \xi_2 = \xi_3 = 0$).

First suppose that $\lambda_1, \lambda_2, \lambda_3$ are all different. Substituting $\lambda = \lambda_1$ in (1.5.13), it is seen that these equations are only satisfied by the following values: $\xi_1 =$ an arbitrary quantity, $\xi_2 = \xi_3 = 0$. Thus the vector, corresponding to $\lambda = \lambda_1$, is parallel to the axis Ox_1; this gives one of the possible principal directions (which is already known). In an analogous

manner it is verified that the values $\lambda = \lambda_2$, $\lambda = \lambda_3$ correspond to the directions of the axes Ox_2, Ox_3.

Thus, if the three roots of (1.5.7) are different, there are only three principal directions which are mutually perpendicular.

Now let $\lambda_1 = \lambda_2 \neq \lambda_3$. In that case one obtains again for $\lambda = \lambda_3$ only one direction, namely the direction of Ox_3. However, for $\lambda = \lambda_1 = \lambda_2$, the solution of (1.5.13) will be: ξ_1 arbitrary, ξ_2 arbitrary, $\xi_3 = 0$. Thus all directions perpendicular to the axis Ox_3 (and only these directions) will be principal directions, corresponding to this value of λ. One may always select among these directions an infinite number of pairs of mutually perpendicular directions (which will also be perpendicular to the axis Ox_3).

Finally, it is obvious that, if $\lambda_1 = \lambda_2 = \lambda_3$, then the equations (1.5.13) will be satisfied for $\lambda = \lambda_1 = \lambda_2 = \lambda_3$ by any values of ξ_1, ξ_2, ξ_3. In other words, in this case any direction is a principal direction.

ON THE DETERMINATION OF FUNCTIONS FROM THEIR PERFECT DIFFERENTIALS IN MULTIPLY CONNECTED REGIONS

1. The case of *two dimensions* will be considered first. Let S denote some region of the Oxy plane. Only such *connected* regions which are bounded by one or several simple contours will be studied. Such regions may also be infinite (infinite plane with holes), but, for the present, consideration will be restricted to finite regions.

A region is called connected, if any two points in it may be joined by a simple line which does not leave the region. A contour is called simple, if it does not intersect itself.

A region S is said to be simply connected, if any cut joining any two points of its boundary disturbs its connectivity, i.e., divides it into separate regions.

A region is said to be multiply connected, if cuts linking points of the boundary may be introduced without dividing it into individual parts.

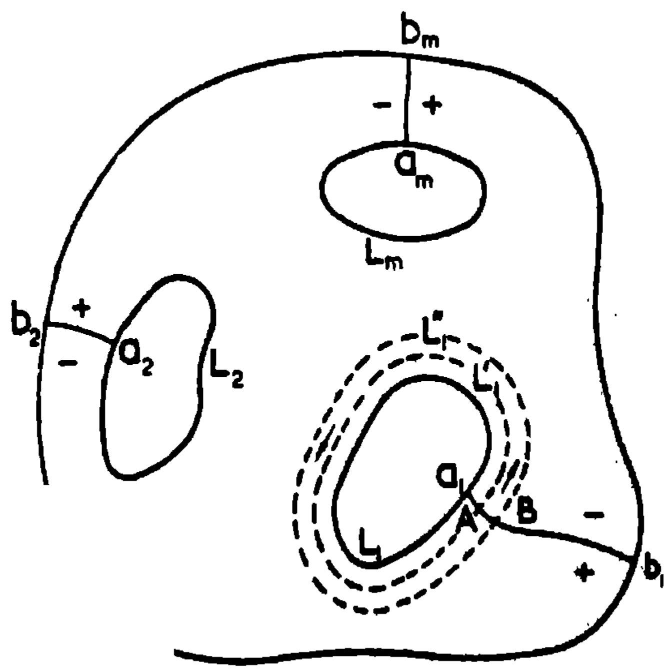


Fig. 64.

It is readily seen that a region, bounded by one simple contour, is simply connected. In contrast, a region, bounded by several simple contours, is multiply connected. In fact, let the boundary of a region consist of the contours $L_1, L_2, \ldots L_m, L_{m+1}$, the last of which contains all the others inside it (Fig. 64). If the region is cut along any line a_1b_1, connecting a point a_1 of L_1 with a point

b_1 of the outer contour L_{m+1}, such a cut does not affect the connectivity of the region.

If, in addition to a_1b_1, similar cuts $a_2b_2, \ldots, a_mb_m$ are introduced, which do not intersect one another, the connectivity is still not disturbed; however, as is easily seen, any further cut will affect the connectivity of the region. Thus, the m cuts $a_1b_1, \ldots, a_mb_m$ make the region under consideration simply connected.

If m cuts are required to convert a given region into a simply connected one, it will be said that the region is $(m + 1)$-ply connected or that its connectivity is equal to $(m + 1)$.

It is seen that in this way the connectivity of a region is equal to the number of contours bounding it. For example, the region between two concentric circles is doubly connected.

A simply connected region differs from a doubly connected one in that it has the following property. If one draws inside the simply connected region a simple contour, the region inside this contour belongs entirely to S; this contour may be shrunk into a point by means of continuous deformations during which it remains always in S.

In the case of multiply connected regions, there exist contours which do not have this property. For example, in Fig. 64, L_1' is one such contour; it is impossible to contract it into one point without cutting it or without the contour leaving S.

2. Let there be given the differential

$$P(x, y)dx + Q(x, y)dy, \tag{2.2.1}$$

where $P(x, y)$ and $Q(x, y)$ are single-valued and continuous functions with continuous first order derivatives in some region S. The following question will be asked: What conditions must be satisfied by the functions P, Q, in order that (2.2.1) should be a perfect differential of some single-valued function $F(x, y)$, i.e., in order that there should exist a function $F(x, y)$ such that

$$dF = Pdx + Qdy \tag{2.2.2}$$

or, what amounts to the same thing, that

$$\frac{\partial F}{\partial x} = P(x, y), \quad \frac{\partial F}{\partial y} = Q(x, y)? \tag{2.2.2'}$$

Although this problem is studied in all, even elementary textbooks on calculus, it has nevertheless been considered necessary to dwell on

it here, in order to draw the reader's attention to certain circumstances which are very essential for the purpose of this book.

First, suppose that the region S is simply connected. Inside S select some fixed point $M_0(x_0, y_0)$ and connect it with the variable point $M(x, y)$ by an arbitrary line M_0M which does not leave S. If the function $F(x, y)$, satisfying (2.2.2), exists, one obtains by integrating both sides along M_0M_1

$$F(x, y) = \int\limits_{M_0M} (Pdx + Qdy) + C, \tag{2.2.3}$$

where $C = F(x_0, y_0)$ is a constant.

By supposition, $F(x, y)$ is a single-valued function of x, y; hence its value at the point $M(x, y)$ must only depend on the position of M and not on the path of integration M_0M. Thus, if $F(x, y)$ exists, the *line integral*

$$\int\limits_{M_0M} (Pdx + Qdy)$$

cannot depend on the path of integration (which lies, of course, in the region S). This condition may be formulated as follows: The integral

$$(Pdx + Qdy)$$

taken along any contour L (entirely inside S) must be equal to zero. In fact, by linking any two points A and B of some contour L with the points M_0 and M respectively (Fig. 65), one has, by supposition,

$$\int\limits_{M_0ADBM} - \int\limits_{M_0AD'BM} = 0;$$

however, since the integrals along M_0A and BM in both terms are equal, one must have

$$0 = \int\limits_{ADB} - \int\limits_{AD'B} = \int\limits_{L},$$

as was to be proved.

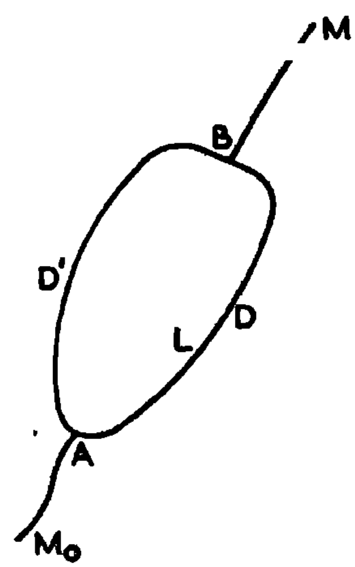


Fig. 65.

On the basis of Green's formula,

$$\int (Pdx + Qdy) = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy, \tag{2.2.4}$$

where σ denotes the region, bounded by the contour L. If follows from the above that the integral on the right-hand side must vanish for every part σ of the region S. Thus the integrand must be zero at each point of S, i.e., one must have

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \tag{2.2.5}$$

throughout S. As has just been seen, (2.2.5) is the necessary condition for the existence of the function $F(x, y)$. It may also be proved that it is *sufficient*.

In fact, if this condition is satisfied, the line integral

$$\int (Pdx + Qdy)$$

does not depend on the path of integration, but only on the starting and end points of this path. This follows directly from the above: If A and B are any points of S, and ADB, $AD'B$ are any two paths connecting these two points, then

$$\int\limits_{ADB} = \int\limits_{AD'B},$$

because (Fig. 65).

$$\int\limits_{ADB} - \int\limits_{AD'B} = \int,$$

and the last integral is zero by (2.2.4) and (2.2.5). For this purpose it has been assumed that the lines ADB and $AD'B$ do not intersect each other, so that they form a simple contour. However, it is readily verified that this condition is not essential; if these lines intersect at one or several points, the difference of the integrals along these paths may be reduced to the sum of integrals along two or several contours.

In particular, the integral on the right-hand side of (2.2.3) represents, for a fixed point $M_0(x_0, y_0)$, a single-valued function of x and y, and hence (2.2.3) determines the single-valued function $F(x, y)$, provided C is

given an arbitrary (constant) value. Further, it is readily verified that (2.2.2′) actually holds true.

In fact, extending the path of integration in (2.2.3) by the straight segment MM', parallel to Ox, to the point $M'(x + \Delta x, y)$, one obviously finds

$$F(x + \Delta x, y) = F(x, y) + \int^{x+\Delta x} P(x, y)dx,$$

whence

$$\frac{\partial F}{\partial x} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int^{x+\Delta x} P(x, y)dx = P(x, y),$$

i.e., one obtains the first relation (2.2.2′); the second relation may be proved in the same manner.

It has thus been seen that the condition (2.2.5) is necessary and sufficient for the existence of the single-valued function $F(x, y)$, satisfying the conditions (2.2.2) or (2.2.2′). If these conditions are satisfied, the function $F(x, y)$ will be determined by (2.2.3), apart from the constant C which is quite arbitrary.

Hitherto, it has been assumed that the region S is simply connected. Next consider what supplementary conditions are required in the case of multiply connected regions.

The condition (2.2.5) is also in this case necessary; its deduction differs in no way from that for the case of a simply connected region. One has only to select for the application of (2.2.4) such a contour L that the region σ, bounded by it, lies entirely in S (this condition will be automatically satisfied in the case of a simply connected region). The question of the sufficiency of this condition will now be investigated. It will be proved that in the present case this condition secures the existence of the function $F(x, y)$, defined by (2.2.3), but that this function will, in general, be multi-valued.

A beginning will be made with the following remark. Let the region S be cut along

$$a_1b_1, \ldots, a_mb_m,$$

as indicated in the preceding section; one thus obtains a simply connected region which will be denoted by S^*.

It should be understood that two edges of the cut region adjoin to

each line $a_k b_k$, so that every point of this line must be considered as two points belonging to the one or the other of the corresponding edges. Hence a distinction must be made between the two edges at each cut which will be denoted by (+) and (−).

Since the region S^* is simply connected, the function $F(x, y)$, defined by (2.2.3) where the path of integration must not go outside S^*, i.e., must not intersect a cut, will be single-valued in S^* on the basis of the earlier results.

However, this does not mean that at points, lying on different edges of the same cut, the values of the function F will be identical (because these points must be conceived as different points of S^*). For example, select any point A of the cut $a_1 b_1$ and denote by F^+, F^- the values of F at the points A^+, A^- of the edges (+) and (−) which coincide at the geometrical point A. By (2.2.3),

$$F^- = \int_{M_0 A^-} (Pdx + Qdy) + C, \quad F^+ = \int_{M_0 A^+} (Pdx + Qdy) + C,$$

where the first integral is taken along any line $M_0 A^-$, lying in S^* and going from M_0 to the point A, approaching it from the side (−); the second integral is taken along a path $M_0 A^+$, likewise beginning from M_0, but approaching A from the positive side (+) [Fig. 64, where the point M_0 and the paths of integration are not shown]. As path of integration for the second integral one may take the path of integration $M_0 A^-$ of the first integral supplementing it by the line L_1' which surrounds the contour L_1 once and leads from the edge (−) to the edge (+) without leaving the cut region S^*. Thus

$$F^+ = \int_{M_0 A^-} + \int_{L_1'} + C = F^- + J_1,$$

where

$$J_1 = \int_{L_1'} (Pdx + Qdy)$$

and L_1' is a simple contour, going in S^* from the edge (−) to the edge (+) of the cut $a_1 b_1$ without intersecting another cut (Fig. 64). This contour intersects the cut $a_1 b_1$, crossing from the side (+) to the side (−). It is readily seen that J_1 does not depend on the choice of the contour L_1' which is to surround the contour L_1 only once, going in S^* from the edge (−) to the edge (+) of $a_1 b_1$.

In fact, let L_1'' be another such contour which intersects a_1b_1 at some point B. Consider the contour which does not leave S^* and which consists of the segment AB of the positive edge of the cut, of the path L_1'' taken in the *negative* direction [i.e., in S^* from the edge (+) to the edge (—)], of the segment BA of the negative side of the cut and, finally, of the contour L_1'. On has

$$\int (Pdx + Qdy) = 0,$$

where the integral is taken along the above-stated closed path in S^*. Further, since the integrals along AB and BA cancel each other, one has

$$-\int_{L_1''} (Pdx + Qdy) + \int_{L_1'} (Pdx + Qdy) = 0,$$

and this proves the assertion (the first integral has here been given a minus sign, assuming L_1'' to denote the path in the positive direction).

Similarly, one obtains for any cut a_kb_k that

$$F^+ = F^- + J_k,$$

where

$$J_k = \int_{L_k'} (Pdz + Qdy) \tag{2.2.6}$$

and L_k' is any contour, surrounding L_k and only intersecting the one cut a_kb_k in the direction from the side (+) to the side (—).

The integral (2.2.6) may, in particular, be taken along the boundary L_k itself, provided the functions P, Q are continuous up to the boundaries.

The nature of the function $F(x, y)$, defined by (2.2.3), is easily seen, if one considers it as defined in the *uncut* region, i.e., if one allows the path of integration to intersect the cuts.

Let $F_0(x, y)$ denote the value given by (2.2.3) in the cut region, i.e., when the path of integration does not intersect a cut. Consider the arbitrary path of integration M_0M (Fig. 66); let it intersect the cuts in, say, n points. Follow the path of integration from the point M_0 to the first intersection with one of the cuts a_kb_k. On the subsequent part of the path M_0M, which lies between the first and second encounter with a cut, select two consecutive points A and B and replace the segment AB by the line AM_0B which goes from A to M_0 and returns to B without

intersecting a cut. This procedure does not, of course, alter the value of the integral (because the new segments belong to the cut region). The original path from M_0 to M is thus replaced by the contour M_0AM_0, which surrounds the contour L_k once, and the path M_0BM, which now intersects the cuts in only $n - 1$, and not n points.

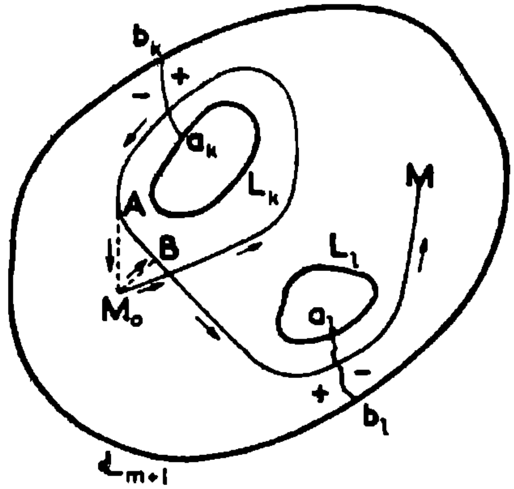


Fig. 66.

The integral taken along the contour M_0AM_0 is, by (2.2.6), equal to $+ J_k$ or $- J_k$, depending on whether its path of integration intersects the cut a_kb_k from the edge $(+)$ to the edge $(-)$ or in the opposite direction. Thus, one may omit from the (modified) path of integration the closed part M_0AM_0 under the condition that a term $\pm J_k$ be added to the final result.

Proceeding in the same manner, one may reduce the path of integration to one which does not intersect any cuts. The integral, taken along this path, must then be combined with the quantities $\pm J_k$; each of these terms must be added according to the number of times which the original path of integration intersects the corresponding cut; the sign $(+)$ will apply, if the cut is crossed from the side $(+)$ to the side $(-)$, while the sign $(-)$ must be taken in the opposite case.

Since the path of integration which does not intersect cuts gives the value $F_0(x, y)$, the final result may be written in the following form:

$$F(x, y) = F_0(x, y) + n_1J_1 + n_2J_2 + \dots + n_mJ_m, \qquad (2.2.7)$$

where $n_1, \dots, n_m$ are integers which are positive or negative and which are easily calculated, on the basis of the above results, as the number of intersections of the path M_0M with the cuts (where account must be taken of the directions of intersection). For example, in the case of Fig. 66.,

$$F(x, y) = F_0(x, y) + J_k + J_l.$$

In order that the function $F(x, y)$ will be single-valued, it is necessary and sufficient that together with (2.2.5) the following condition be satisfied:

$$J_1 = J_2 = \dots = J_m = 0.$$

The above results will likewise apply to the case, where the contour

L_{m+1} is entirely at infinity, so that the region S becomes the infinite plane with holes.

3. Quite analogous results will apply in the case of *three dimensions*. In this case a distinction must also be made between simply and multiply connected three-dimensional regions (bodies). A region will be said to be *simply connected*, if it has the property that every closed line inside it may be shrunk into one point by means of continuous deformation during which it does not leave the region (e.g. sphere, cube). Otherwise a region will be multiply connected. As examples of multiply connected regions, one may quote the torus (i.e., the body, obtained by rotating a circle about an axis in its plane, but not intersecting it) or a cube with one or more holes, drilled through it, etc.

The torus is doubly connected, because it may be made simply connected by the help of a single cut; however, in contrast to the case of two dimensions, the cut will now not be a line, but a surface.

In general, a body will be $(m+1)$-ply connected, if it may be made simply connected by the help of m cuts. Attention is drawn to the fact that a body, bounded by one closed surface, is not necessarily simply connected (e.g.: torus); in contrast, a body may be bounded by several closed surfaces and it may still be simply connected (e.g. the region between two concentric spheres).

Let there be given the differential expression

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz, \tag{2.3.1}$$

where P, Q, R are single-valued and continuous functions which have continuous first order derivatives in some *simply connected* region V.

As in the case of two dimensions, it may be shown that for the existence of a single-valued function $F(x, y, z)$, satisfying the condition

$$dF = Pdx + Qdy + Rdz, \tag{2.3.2}$$

it is neccesary and sufficient that

$$\int_{L} (Pdx + Qdy + Rdz) = 0, \tag{2.3.3}$$

where L is any contour in the region V. Under this condition, the function $F(x, y, z)$ will be determined by the formula

$$F(x, y, z) = \int_{M_0M} (Pdx + Qdy + Rdz) + C, \tag{2.3.4}$$

where C is an arbitrary constant and the integral is taken along any path (in V) which links the fixed point M_0 with the variable point $M(x, y, z)$.

The condition (2.3.3) will now be transformed. For this purpose the well known Stokes formula will be recalled

$$\int_L (Pdx + Qdy + Rdz) = \iint_\sigma \left\{ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \cos (n, x) + \right.$$

$$\left. + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \cos (n, y) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cos (n, z) \right\} d\sigma, \quad (2.3.5)$$

where σ is any (open) surface (inside V) with the boundary L and n is the normal to σ in a definite direction. Applying (2.3.5) to (2.3.3), one finds

$$\iint \left\{ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \cos (n, x) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \cos (n, y) + \right.$$

$$\left. + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cos (n, z) \right\} d\sigma = 0; \quad (2.3.6)$$

this condition must be fulfilled for any surface σ (in V). Selecting for σ any plane, normal to the axis Ox, one obtains, in particular,

$$\iint \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy\, dz = 0,$$

whence (in view of the arbitrariness of the plane σ) one arrives at the first of the following formulae:

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \quad (2.3.7)$$

(the remaining two being obtained by cyclic interchange of symbols).

Conversely, the conditions (2.3.7) are obviously sufficient for the truth of (2.3.3) and, hence, for the existence of the single-valued function $F(x, y, z)$, determined by (2.3.4).

In the case of *multiply connected* regions, if (2.3.7) is satisfied, the function $F(x, y, z)$ determined by (2.3.4) may be multi-valued. In fact, reasoning as in the case of two dimensions, one may establish the following result. If one has introduced m cuts (partitions) which convert the given $(m + 1)$-ply connected region into a simply connected one and

if $F_0(x, y, x)$ denotes the function, defined by (2.3.4) under the condition that the path of integration does not intersect these partitions, then one will have for an arbitrary path of integration

$$F(x, y, z) = F_0(x, y, z) + n_1 J_1 + n_2 J_2 + \ldots + n_m J_m, \qquad (2.3.8)$$

where $n_1, \ldots, n_m$ are integers and $J_1, J_2, \ldots, J_m$ are constants, corresponding to integrals taken along closed paths. In fact,

$$S_k = \int_{L_k} (Pdx + Qdy + Rdz), \qquad (2.3.9)$$

where L_k is the simple contour which intersects only the cut k from its side $(+)$ to the side $(-)$. The numbers n_k are defined in the same way as in the case of two dimensions.

In order that the function F will be single-valued, it is necessary and sufficient that, in addition to (2.3.7), the following conditions be satisfied:

$$J_1 = J_2 = \ldots = J_m = 0. \qquad (2.3.10)$$

DETERMINATION OF A FUNCTION OF A COMPLEX VARIABLE FROM ITS REAL PART. INDEFINITE INTEGRALS OF HOLOMORPHIC FUNCTIONS

1. Let

$$p(x, y) + iq(x, y) = f(z) \tag{3.1.1}$$

be a function of the complex variable $z = x + iy$ which is holomorphic in some region S of the z plane. Under these circumstances, the real and imaginary parts p and q are known to be related by the Cauchy-Riemann conditions

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}, \quad \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x}. \tag{3.1.2}$$

Conversely, it is known from complex function theory that, if two single-valued real functions p and q which have continuous first order derivatives are related by the equations (3.1.2), then $p + iq$ is a holomorphic function of the variable z in a given region.

It is known that holomorphic functions have derivatives of any order (and, further, may be developed into Taylor series in the neighbourhood of any point). Hence the functions p, q also possess this property.

The function q, related to a given function p by the equations (3.1.2), is said to be conjugate to p.

Not every real function p can be the real part of a holomorphic function of a complex variable. In fact, differentiating the equations (3.1.2) with respect to x and y respectively and adding, one obtains

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \Delta p = 0; \tag{3.1.3}$$

hence the function p must be harmonic. In the same manner it may be shown that the function q must be harmonic. In what follows, a function will be understood to be harmonic, if it satisfies (in a given

region S) the equation (3.1.3) and if it has continuous second order derivatives. In addition, it will be assumed that the function p is single-valued.

It is easily shown that one may determine for any harmonic function p a function q, conjugate to it. In fact, by (3.1.2), one has for the determination of q

$$dq = -\frac{\partial p}{\partial y}\,dx + \frac{\partial p}{\partial x}\,dy.$$

The condition for the existence of the function q in the present case reduces to the following (cf. Appendix 2):

$$-\frac{\partial^2 p}{\partial y^2} = \frac{\partial^2 p}{\partial x^2},$$

which is satisfied thanks to (3.1.3). Hence the function q will be determined, apart from an arbitrary constant, by the formula

$$q(x, y) = \int\limits_{M_0 M} \left(-\frac{\partial p}{\partial y}\,dx + \frac{\partial p}{\partial x}\,dy\right) + C, \tag{3.1.4}$$

where M_0M is an arbitrary path which connects some (arbitrarily) fixed point M_0 with the variable point $M(x, y)$ and which does not leave the given region S.

The formula (3.1.4) may be written in a somewhat simpler form. Let t denote the tangent to the path of integration (in the direction M_0 to M) and n the normal towards the right of t (see Fig. 13, § 32), then

$$dx = ds.\cos(t, x) = -ds.\cos(n, y), \quad dy = ds.\cos(t, y) = ds.\cos(n, x),$$

where ds is the arc element of the path of integration; hence

$$-\frac{\partial p}{\partial y}\,dx + \frac{\partial p}{\partial x}\,dy = \left\{\frac{\partial p}{\partial y}\cos(n, y) + \frac{\partial p}{\partial x}\cos(n, x)\right\} ds = \frac{dp}{dn}\,ds$$

and (3.1.4) may be rewritten

$$q(x, y) = \int\limits_{M_0 M} \frac{dp}{dn}\,ds + C. \tag{3.1.4'}$$

This formula could have been written down straight away by noting that always

$$\frac{dq}{ds} = \frac{dp}{dn}.$$

This relation follows directly from the Cauchy-Riemann conditions.

When S is simply connected, the function q determined by (3.1.4) or (3.1.4′) will be single-valued and, on the basis of the above statements, the function

$$f(z) = p + iq$$

will be holomorphic in S; *it will be determined for a given p apart from a purely imaginary arbitrary constant Ci.*

In the case of a multiply connected region, the function $f(z) = p + iq$, where q is determined by (3.1.4) or (3.1.4′), will be holomorphic in every simply connected subregion of S (and, in particular, in the cut region S^*; see Appendix 2). However, if the only restriction on the path of integration is that it must remain in S, the function $f(z)$ may turn out to be multi-valued. In fact, for a circuit along a contour surrounding one of the contours L_k (using the same notation as in Appendix 2), the function q will increase by some constant B_k, and hence $f(z)$ undergoes the purely imaginary increase iB_k.

The constants B_k are determined by the formulae

$$B_k = \int_{L'_k} \left(- \frac{\partial p}{\partial y} dx + \frac{\partial p}{\partial x} dy \right) = \int_{L'_k} \frac{dp}{dn} ds \quad (k = 1, \ldots, m); \qquad (3.1.5)$$

the integrals may be taken along the actual lines L_k, provided the partial derivatives of p are continuous up to the boundary.

In order that the function $f(z)$ will be single-valued in the multiply connected region S, it is necessary and sufficient that all the constants B_k are equal to zero.

2. In connection with the above, one remark will be made regarding the indefinite integral of a function, holomorphic in some region S. By the indefinite integral

$$\int f(z) dz$$

will be understood the function

$$F(z) = \int^{z} f(z) dz + \text{const.}, \qquad (3.2.1)$$

where the integral is taken along an arbitrary path which does not leave S and connects the arbitrarily fixed point z_0 with the variable point z and "const." is an arbitrary (in general, complex) constant.

If S is a simply connected region, $F(z)$ will be a single-valued function. This follows from the fact that, by Cauchy's theorem, the integral

$$\int f(z)dz,$$

if taken around a contour, is equal to zero, so that

$$\int^{z} f(z)dz$$

does not depend on the path of integration (cf. the analogous reasoning in Appendix 2).

However, if the region S is multiply connected (assuming that it has the form indicated at the beginning of Appendix 2), the function $F(z)$ may turn out to be multivalued; in fact, for a circuit around a contour L_k' which surrounds L_k once (see Appendix 2), it undergoes an increase

$$\alpha_k + i\beta_k = \int_{L'_k} f(z)dz. \tag{3.2.2}$$

In general, the integral on the right-hand side of (3.2.2) will be different from zero, because the region contained inside L_k' does not entirely belong to S. The quantity $\alpha_k + i\beta_k$ does not depend on the choice of the contour L_k', except that it is to surround L_k once, that it may not intersect any other cut beside $a_k b_k$ and that it must be described in a definite direction. This may be proved by the same method as the analogous result for the function $F(x, y)$, proved in Appendix 2. Reasoning as in Appendix 2, it is easily established that the function $F(z)$, defined by (3.2.1), may be represented in the form

$$F(z) = F_0(z) + n_1(\alpha_1 + i\beta_1) + \ldots + n_m(\alpha_m + i\beta_m), \tag{3.2.3}$$

where F_0 is a single-valued function, defined in the cut region S^*, and $n_1, n_2, \ldots, n_m$ are integers, defined as in Appendix 2.

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